Partial Differential Equations
Exercise answers

February 6, 2012

Week 3

Exercise 6.1 (CDE)

Check definitions of vector space and subspace.

Exercise 6.2 (CDE)

- $q = 1$ $U_1(t) = 1 + 3t$
- $q = 2$ $U_2(t) = 1 + \frac{8}{11}t + \frac{10}{11}t^2$
- $q = 3$ $U_3(t) = 1 + \frac{30}{29}t + \frac{45}{116}t^2 + \frac{35}{116}t^3$
- $q = 4$ $U_4(t) \approx 1 + 0.9971t + 0.5161t^2 + 0.1371t^3 + 0.0737t^4$

Exercise 6.3 (CDE)

$$(P^3 e^{\cdot})(t) \approx 0.991 + 1.0183t + 0.4212t^2 + 0.2716t^3$$

Exercise 6.11 (CDE)

$$U(x) = \frac{5}{128}\phi_1(x) + \frac{8}{128}\phi_1(x) + \frac{7}{128}\phi_2(x),$$

where

$$\phi_i(x) = \frac{1}{4}\begin{cases} x - (i-1)/4 & x \in ((i-1)/4, i/4] \\ (i+1)/4 - x & x \in (i/4, (i+1)/4] \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, 2, 3$

Exercise 2.5 (Lecture notes)

a $u(x) = (x - x^2)/2$

b $R(x) = \pi^2 A \sin(\pi x) + 4\pi^2 B \sin(2\pi x) - 1$

c $U(x) = 4\sin(\pi x)/\pi^3$

Exercise 2.7 (Lecture notes)

a $u(x) = (\pi^3 - x^3)/6 + (x^2 - \pi^2)/2$

b $R(x) = \xi_0 \cos(x/2)/4 + 9\xi_1 \cos(3x/2)/4 - x + 1$

c $U(x) = (16 - 43/\pi)\cos(x/2) + \frac{16}{81}(1/\pi - 6)\cos(3x/2)$

Week 4

Exercise 5.12 (CDE)

Use Rolle's theorem.

Exercise 5.23 (CDE)

$$\phi_{ij}(x) = \begin{cases} 0 & x \notin [x_{i-1}, x_i) \\ \lambda_{ij}(x) & x \in [x_{i-1}, x_i) \end{cases}$$

for $i = 1, \ldots, m+1$ and $j = 0, 1, 2$, where for $\xi_i \in (x_{i-1}, x_i)$ we have

$$\begin{aligned} \lambda_{i,0} &= \tfrac{(x-\xi_i)(x-x_i)}{(x_{i-1}-\xi_i)(x_{i-1}-x_i)} \\ \lambda_{i,1} &= \tfrac{(x-x_{i-1})(x-x_i)}{(\xi_i-x_{i-1})(\xi_i-x_i)} \\ \lambda_{i,2} &= \tfrac{(x-x_{i-1})(x-\xi_i)}{(x_i-x_{i-1})(x_i-\xi_i)} \end{aligned}$$

Exercise 5.27 (CDE)

$$\phi_{i1}(x) = \begin{cases} 0 & x \notin [x_{i-1}, x_{i+1}) \\ \frac{(x-x_{i-1})(x-\xi_i)}{(x_i-x_{i-1})(x_i-\xi_i)} & x \in [x_{i-1}, x_i) \\ \frac{(x-\xi_{i+1})(x-x_{i+1})}{(x_i-\xi_{i+1})(x_i-x_{i+1})} & x \in [x_i, x_{i+1}) \end{cases}$$

$$\phi_{i2}(x) = \begin{cases} 0 & x \notin [x_{i-1}, x_i) \\ \frac{(x-x_{i-1})(x-x_i)}{(\xi_i-x_{i-1})(\xi_i-x_i)} & x \in [x_{i-1}, x_i) \end{cases}$$

for $i = 1, \ldots, m+1$ and $\xi_i \in (x_{i-1}, x_i)$.

Exercise 5.29 (CDE)

Use mean value theorem.

Exercise 8.1 (CDE)

Consider the balance of forces at equilibrium.

Exercise 8.6 (CDE)

$a_{i,i} = 2i + 2/h$, $a_{i,i+1} = a_{i,i-1} = -1/h - i - 1/2$ all other elements of A are 0
$b_i = -(\sin((i-1)h) - 2\sin(ih) + \sin((i+1)h))/h$

Exercise 8.7 (CDE)

Yes it is still symmetric, positive–definite and tridiagonal.

Exercise 8.8 (CDE)

Use Lemma 4.1

Exercise 8.11 (CDE)

$a_{i,i} = 2/h$ $a_{i,i+1} = a_{i,i-1} - 1/h$, $a_{M+1,M+1} = 1/h$ all other elements of A are 0
$b_{M+1} = 1 + h/2$

Exercise 8.12 (CDE)

$a_{i,i} = 2/h$ $a_{i,i+1} = a_{i,i-1} - 1/h$, $a_{M+1,M+1} = 1/h + \gamma$ all other elements of A are 0
$b_{M+1} = g_1 + \int_{x_M}^{x_{M+1}} f\phi_{M+1}\mathrm{d}x$

Exercise 8.16 (CDE)

$a_{i,i} = (a((i-1)h) + 2a(ih) + a((i+1)h))/2h$, $a_{i,i+1} = a_{i+1,i} = -(a(ih) + a((i+1)h))/2h$
all other elements of A are 0
$b_i = hf(x_i)$

Exercise 8.18 (CDE)

Use Cauchy inequality.

Exercise 8.23 (CDE)

- a priori : $\|u - U\|_E \leq \|u - v\|_E$
- a posteriori : $\|e\|_E \leq C_i \|hR(U)\|_E$ where $R(U) = f - bU' - U$

Week 5

Exercise 9.9 (CDE)

Use that $u(t) = e^{-A(t)}u_0 + \int_0^t e^{-(A(t)-A(s))} f(s)\mathrm{d}s$

Exercise 9.12 (CDE)

Assignment 1

Exercise 9.13 (CDE)

$w_n = (\frac{2}{k_n} \int_{I_n} f\phi_n \mathrm{d}t + w_{n-1}(1/k_n - a/3))/(1/k_n + 2a/3)$

Exercise 9.19 (CDE)

Verify the formulae.

Exercise 9.43 (CDE)

Write U_n in terms of U_{n-1} and use this.

Exercise 9.45 (CDE)

Use the hint.

Exercise 9.46 (CDE)

Use formulae for $S(t_N)$ and $\widetilde{S}(t_N)$

Week 6

Exercise 14.4 (CDE)

Use the hint.

Exercise 14.7 (CDE)

a Relate a point in R^2 to the degrees of freedom of the function.

b No

Exercise 14.10 (CDE)

Use a) from Ex. 14.7

Exercise 14.21 (CDE)

Use relationships inside a triangle.

Week 7

Exercise 15.5 (CDE)

Check if E satisfies the weak formulation as on p. 349.

Exercise 15.13 (CDE)

Use the hint.

Exercise 15.15 (CDE)

($\Leftarrow$) Let $w \in V$ and write it as $w = u + v$ for some $v \in V$.
($\Rightarrow$) Use $g(\epsilon)] = F[u + \epsilon v]$ for some $v \in V$

Exercise 15.20 (CDE)

Draw the picure of ϕ_i and ϕ_{i+1}.

Exercise 15.22 (CDE)

a
- $a_{i,i} = 4$ for both types of basis functions
- $a_{i,i+1} = -1$ for both combinations of basis functions
- $a_{i,i+m} = -1$ for both combinations of basis functions
- 0 otherwise

b
- $a_{i,i} = 4$
- $a_{i,j} = -1$ if nodes i and j are linked vertically or horizontally
- $a_{i,i+m} = 0$ if nodes i and j are linked diagonally

Exercise 15.27 (CDE)

lumped mass $h^2(x_{1k}^3 + x_{2k}^2)$

Exercise 15.39 (CDE)

Find a linear combination of elements from V_g that is not in V_g. For the second part assume that there are solutions and show that they have to be equal.

Exercise 15.44 (CDE)

- $\int_\Omega \phi_i \mathrm{d}x = h^2$ for inside nodes
- $\int_\Omega \phi_i \mathrm{d}x = h^2/2$ for border nodes that are not corners
- $\int_\Omega \phi_i \mathrm{d}x = h^2/3$ for upper left corner node
- $\int_\Omega \phi_i \mathrm{d}x = h^2/6$ for upper right corner node
- $\int_\Omega \bigtriangledown\phi_i \bigtriangledown \phi_j \mathrm{d}x = 2$ i and j lie on the border and node i is not a corner
- $\int_\Omega \bigtriangledown\phi_i \bigtriangledown \phi_j \mathrm{d}x = 1$ i and j lie on the border and node i is a upper left or upper right corner
- $\int_\Omega \bigtriangledown\phi_i \bigtriangledown \phi_j \mathrm{d}x = -1/2$ $|i-j| = 1$, $|i-j| = m$
- $\int_\Gamma \phi_i \mathrm{d}x = h$ node i is on the border

Exercise 15.47 (CDE)

Use the hint.

Week 8

Week 9