Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2018–03–14, 14:00-18:00

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-21p, **4**: 22-28p och **5**: 29p- For GU students**G**:15-25p, **VG**: 26p-
For solutions and information about gradings see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1718/index.html

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1. Derive the cG(1)-cG(1), Crank-Nicolson approximation, for the initial boundary value problem

$$\begin{cases} \dot{u} - u'' = f, & 0 < x < 1, \quad t > 0, \\ u'(0,t) = u'(1,t) = 0, \quad u(x,0) = 0, & x \in [0,1], \quad t > 0, \end{cases} \tag{1}$$

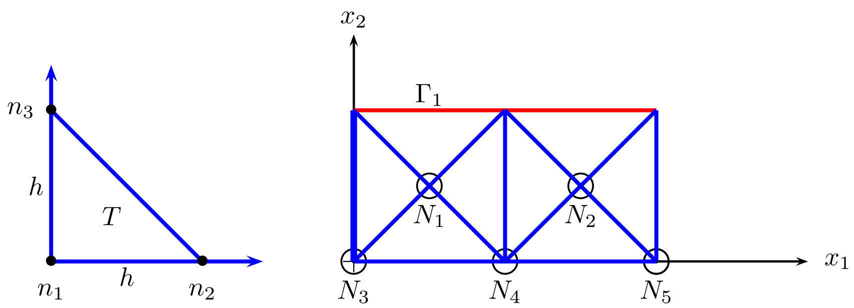
2. $\pi_1 f$ is the linear interpolant of a twice continuously differentiable function f on I. Prove that

$$||f - \pi_1 f||_{L_2(I)} \le (b-a)^2 ||f''||_{L_2(I)}, \qquad I = (a,b).$$

3. Formulate the cG(1) piecewise continuous Galerkin method in Ω (see fig. below) for the problem

$$-\Delta u(x) = \alpha, \quad \text{for } x \in \Omega, \qquad u(x) = 0, \quad \text{for } x \in \Gamma_1, \quad \text{and } \nabla u(x) \cdot \mathbf{n}(x) = \beta \quad \text{for } x \in \partial\Omega \setminus \Gamma_1,$$

where $\mathbf{n}(x)$ is the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. Determine the coefficient matrix and load vector for the resulting equation system using the mesh as in the fig. with nodes at N_1, N_2, N_3, N_4 and N_5 and a uniform mesh size h. Hint: First compute the matrix for the standard element T.



4. a) Let p be a positive constant. Prove an a priori error estimate (in the H^1-norm: $||e||^2_{H^1} = ||e'||^2_{L_2} + ||e||^2_{L_2}$) for a finite element method for problem

$$-u'' + pxu' + (1 + \frac{p}{2})u = f, \quad \text{in } (0,1), \qquad u(0) = u(1) = 0.$$

b) For which value of p the a priori error estimate is optimal?

5. Formulate the Lax-Milgram Theorem. Verify the assumptions of the Lax-Milgram Theorem and determine the constants of the assumptions in the case: $I = (0,1)$, $f \in L_2(I)$, $V = H^1(I)$ and

$$a(v,w) = \int_I (uw + v'w')\,dx + v(0)w(0), \quad L(v) = \int_I fv\,dx. \qquad ||w||^2_V = ||w||^2_{L_2(I)} + ||w'||^2_{L_2(I)}.$$

6. Consider the homogeneous heat equation:

$$u_t - \Delta u = 0, \quad x \in \Omega \subset \mathbb{R}^2, \qquad u(x,t) = 0, \quad x \in \partial\Omega, \qquad u(x,0) = u_0(x).$$

Prove the following stability estimates

$$i) \quad ||\nabla u||(t) \le \frac{1}{\sqrt{2t}} ||u_0|| \qquad \text{and} \qquad ii) \quad \Big(\int_0^t s||\Delta u||^2(s)\,ds\Big)^{1/2} \le \frac{1}{2}||u_0||.$$

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Solutions.

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1. Make the cG(1)-cG(1) ansatz

$$U(x,t)=U_{n-1}(x)\psi_{n-1}(t)+U_n(x)\psi_n(t), \qquad \text{with} \quad U_n(x)=\sum_{j=1}^{M}U_{n,j}\varphi_j(x),$$

in the variational formulation

$$\int_{I_n}\int_0^1 u'v'=\int_{I_n}\int_0^1 f\,v, \qquad I_n=(t_{n-1},t_n).$$

Recall that $v=\varphi_j(x)$, $j=1,\dots,M$ and

$$\psi_{n-1}(t)=\frac{t_n-t}{t_n-t_{n-1}}, \qquad \psi_n(t)=\frac{t-t_{n-1}}{t_n-t_{n-1}}.$$

For a uniform tile partition with $k:=t_n-t_{n-1}$, this yields the equation system

$$(M+\frac{k}{2}S)U_n=(M-\frac{k}{2}S)U_{n-1}+k\mathbf{b}_n.$$

Here U_n is the node-vale vector with entries $U_{n,j}$, M is the mass-matrix with elements $\int_0^1\varphi_i(x)\varphi_j(x)$, S is the stiffness-matrix with elements $\int_0^1\varphi_i'(x)\varphi_j'(x)$, and $\mathbf{b}_n$ is the load vector with elements $\frac{1}{k}\int_{I_n}\int_0^1 f\varphi_i(x)$. The corresponding dG0 ($\approx$ implicit Euler) time-stepping yields

$$(M+kS)U_n=MU_{n-1}+k\mathbf{b}_n.$$

2. Let $\lambda_0(x)=\frac{\xi_1-x}{\xi_1-x_0}$ and $\lambda_1(x)=\frac{x-\xi_0}{\xi_1-x_0}$ be two linear base functions. Then by the integral form of the Taylor formula we may write

$$\begin{cases} f(\xi_0) &= f(x)+f'(x)(\xi_0-x)+\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy,\\ f(\xi_1) &= f(x)+f'(x)(\xi_1-x)+\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy,\end{cases}$$

Therefore

$$\begin{aligned}\Pi_1 f(x)&=f(\xi_0)\lambda_0(x)+f(\xi_1)\lambda_1(x)\\ &=f(x)+\lambda_0(x)\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy+\lambda_1(x)\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy\end{aligned}$$

and by the triangle inequality we get

$$\begin{aligned}|f(x)-\Pi_1 f(x)|&=\Big|\lambda_0(x)\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy+\lambda_1(x)\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy\Big|\\ &\le|\lambda_0(x)|\Big|\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy\Big|+|\lambda_1(x)|\Big|\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy\Big|\\ &\le|\lambda_0(x)|\int_x^{\xi_0}|\xi_0-y||f''(y)|\,dy+|\lambda_1(x)|\int_x^{\xi_1}|\xi_1-y||f''(y)|\,dy\\ &\le|\lambda_0(x)|\int_x^{\xi_0}(b-a)|f''(y)|\,dy+|\lambda_1(x)|\int_x^{\xi_1}(b-a)|f''(y)|\,dy\\ &\le(b-a)\Big(|\lambda_0(x)|+|\lambda_1(x)|\Big)\int_a^b|f''(y)|\,dy\\ &=(b-a)\Big(\lambda_0(x)+\lambda_1(x)\Big)\int_a^b|f''(y)|\,dy=(b-a)\int_a^b|f''(y)|\,dy.\end{aligned}$$

Then, by the Cauchy-Schwarz inequality we get

$$|f(x)-\Pi_1 f(x)|^2 = (b-a)^2\Big(\int_a^b 1\times|f''(y)|\,dy\Big)^2$$

$$\leq (b-a)^2\Big[\Big(\int_a^b 1^2\,dy\Big)^{1/2}\Big(\int_a^b |f''(y)|^2\,dy\Big)^{1/2}\Big]^2 = (b-a)^3||f''||^2_{L_2(I)}.$$

Consequently

$$\int_a^b |f(x)-\Pi_1 f(x)|^2\,dx \leq \int_a^b (b-a)^3\Big(||f''||^2_{L_2(I)}\,dx\Big) = (b-a)^4||f''||^2_{L_2(I)},$$

which, taking the square root, gives the desired L_2 result.

3. Let V be the linear function space defined by

$$V := \{v : \int_\Omega \Big(v^2+|\nabla v|^2\Big)\,dx<\infty, \quad v=0, \text{ on } \Gamma_1\}.$$

Multiplying the differential equation by $v\in V$ and integrating over Ω we get that

$$-(\Delta u, v) = (\alpha, v), \qquad \forall v\in V.$$

Now using Green's formula and the boundary conditions we have that

$$-(\Delta u,\nabla v) = (\nabla u,\nabla v) - \int_{\partial\Omega}(n\cdot\nabla u)v\,ds = (\nabla u,\nabla v) - \int_{\partial\Omega\setminus\Gamma_1} v\,ds, \qquad \forall v\in V.$$

Thus the variational formulation is:

$$\int_\Omega \nabla u\cdot\nabla v\,dx = \alpha\int_\Omega v\,dx + \beta\int_{\partial\Omega\setminus\Gamma_1} v\,ds, \qquad \forall v\in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v=0$ on Γ_1:

$$V_h := \{v : v \text{ is continuous piecewise linear in } \Omega, \ v=0, \text{ on } \Gamma_1\}.$$

The $cG(1)$ method is: Find $U\in V_h$ such that

$$\int_\Omega \nabla U\cdot\nabla v\,dx = \alpha\int_\Omega v\,dx + \beta\int_{\partial\Omega\setminus\Gamma_1} v\,ds, \qquad \forall v\in V_h$$

Making the "Ansatz" $U(x)=\sum_{j=1}^5 \xi_j\varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^5 \xi_j\Big(\int_\Omega \nabla\varphi_i\cdot\nabla\varphi_j\,dx\Big) = \alpha\int_\Omega \varphi_i\,dx + \beta\int_{\partial\Omega\setminus\Gamma_1}\varphi_i\,ds, \quad i=1,2,3,4,5,$$

or, in matrix form,

$$S\xi=\mathbf{b}, \qquad S_{ij}=(\nabla\varphi_i,\nabla\varphi_j),$$

where S is the stiffness matrix, and $\mathbf{b}=\mathbf{b}_1+\mathbf{b}_2$ is the load vector with components

$$\mathbf{b}_{1,i}=\alpha\int_\Omega\varphi_i\,dx, \quad \text{and} \quad \mathbf{b}_{2,i}=\beta\int_{\partial\Omega\setminus\Gamma_1}\varphi_i\,ds.$$

We first compute stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1,x_2) = 1-\frac{x_1}{h}-\frac{x_2}{h}, \qquad \nabla\phi_1(x_1,x_2) = -\frac{1}{h}\begin{bmatrix}1\\1\end{bmatrix},$$

$$\phi_2(x_1,x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1,x_2) = \frac{1}{h}\begin{bmatrix}1\\0\end{bmatrix},$$

$$\phi_3(x_1,x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1,x_2) = \frac{1}{h}\begin{bmatrix}0\\1\end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\,dx = \frac{2}{h^2}|T| = 1,$$
$$s_{12} = (\nabla\phi_1, \nabla\phi_2) = \int_T |\nabla\phi_1|^2\,dx = -\frac{1}{h^2}|T| = -1/2, \qquad s_{13} = -1/2$$
$$s_{22} = (\nabla\phi_2, \nabla\phi_2) = \int_T |\nabla\phi_2|^2\,dx = \frac{1}{h^2}|T| = 1/2, \qquad s_{23} = (\nabla\phi_2, \nabla\phi_3) = 0,$$
$$s_{33} = (\nabla\phi_3, \nabla\phi_3) = \int_T |\nabla\phi_3|^2\,dx = \frac{1}{h^2}|T| = 1/2,$$

Thus using the symmetry we have the local stiffness matrix as

$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrix S from the local s, using the character of our mesh, viz:

$$\begin{array}{llll} S_{11} = 4s_{11} = 4, & S_{12} = 0 & S_{13} = S_{14} = 2s_{12} = -1 & S_{15} = 0 \\ S_{22} = 4s_{11} = 4 & S_{23} = 0 & S_{24} = S_{25} = 2s_{12} = -1 & \\ S_{33} = 2s_{22} = 1, & S_{34} = s_{23} = 0, & S_{35} = 0 & \\ & & S_{44} = 4s_{22} = 2, & S_{45} = s_{23} = 0 \\ & & & S_{55} = 2s_{22} = 1 \end{array}$$

The remaining matrix elements are obtained by symmetry $S_{ij} = S_{ji}$. Hence,

$$S = \begin{bmatrix} 4 & 0 & -1 & -1 & 0 \\ 0 & 4 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

As for the load vector we note that

$$\begin{aligned} \mathbf{b}_{1,1} &= \alpha \int_\Omega \varphi_1 = \alpha\ 4 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = \alpha\ 4\frac{h^2}{6}, \\ \mathbf{b}_{1,2} &= \alpha\ 4 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = \alpha\ 4\frac{h^2}{6}, \\ \mathbf{b}_{1,3} &= \alpha\ 2 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = \alpha\ 2\frac{h^2}{6}, \\ \mathbf{b}_{1,4} &= \alpha\ 4 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = \alpha\ 4\frac{h^2}{6}, \\ \mathbf{b}_{1,5} &= \alpha\ 2 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = \alpha\ 2\frac{h^2}{6}, \end{aligned} \tag{2}$$

$$\mathbf{b}_{2,1} = \mathbf{b}_{2,2} = 0, \qquad \mathbf{b}_{2,3} = \mathbf{b}_{2,4} = \mathbf{b}_{2,5} = \beta \int_{\partial\Omega} \varphi_i = \beta 2 \cdot \frac{1}{2}(\sqrt{2}h \cdot 1) = \sqrt{2}\beta h. \tag{3}$$

Hence the load vector $\mathbf{b}$ is:

$$\mathbf{b} = \alpha\frac{h^2}{3}\begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} + \sqrt{2}\beta h \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

4. We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I \Big(u'v' + pxu'v + (1+\frac{p}{2})uv\Big) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{4}$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U \in V_h^0$ such that

$$\int_I \Big(U'v' + pxU'v + (1+\frac{p}{2})Uv\Big) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{5}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (4)-(5) gives that

$$\int_I \Big(e'v' + pxe'v + (1+\frac{p}{2})ev\Big) = 0, \quad \forall v \in V_h^0. \tag{6}$$

A posteriori error estimate: We note that using $e(0) = e(1) = 0$, we get

$$\int_I pxe'e = \frac{p}{2}\int_I x\frac{d}{dx}(e^2) = \frac{p}{2}(xe^2)|_0^1 - \frac{p}{2}\int_I e^2 = -\frac{p}{2}\int_I e^2, \tag{7}$$

so that

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I \Big(e'e' + pxe'e + (1+\frac{p}{2})ee\Big)\\
&= \int_I \Big((u-U)'e' + px(u-U)'e + (1+\frac{p}{2})(u-U)e\Big) = \{v = e \text{ in}(1)\}\\
&= \int_I fe - \int_I \Big(U'e' + pxU'e + (1+\frac{p}{2})Ue\Big) = \{v = \pi_h e \text{ in}(2)\}\\
&= \int_I f(e-\pi_h e) - \int_I \Big(U'(e-\pi_h e)' + pxU'(e-\pi_h e) + (1+\frac{p}{2})U(e-\pi_h e)\Big)\\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e-\pi_h e),
\end{aligned} \tag{8}$$

where $\mathcal{R}(U) := f + U'' - pxU' - (1+\frac{p}{2})U = f - pxU' - (1+\frac{p}{2})U$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (8) implies that

$$\begin{aligned}
\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\|\|h^{-1}(e-\pi_h e)\|\\
&\le C_i\|h\mathcal{R}(U)\|\|e'\| \le C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},
\end{aligned}$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

A priori error estimate: We use (7) and write

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I (e'e' + pxe'e + (1+\frac{p}{2})ee)\\
&= \int_I \Big(e'(u-U)' + pxe'(u-U) + (1+\frac{p}{2})e(u-U)\Big) = \{v = U - \pi_h u \text{ in}(3)\}\\
&= \int_I \Big(e'(u-\pi_h u)' + pxe'(u-\pi_h u) + (1+\frac{p}{2})e(u-\pi_h u)\Big)\\
&\le \|(u-\pi_h u)'\|\|e'\| + p\|u-\pi_h u\|\|e'\| + (1+\frac{p}{2})\|u-\pi_h u\|\|e\|\\
&\le \{\|(u-\pi_h u)'\| + (1+p)\|u-\pi_h u\|\}\|e\|_{H^1}\\
&\le C_i\{\|hu''\| + (1+p)\|h^2u''\|\}\|e\|_{H^1},
\end{aligned}$$

this gives that

$$\|e\|_{H^1} \leq C_i\{\|hu''\| + (1+p)\|h^2u''\|\},$$

which is the a priori error estimate.

b) As seen $p=0$ (corresponding to zero convection) yields optimal a priori error estimate.

5. For the formulation of the Lax-Milgram theorem see the book, Chapter 21.

As for the given case: $I=(0,1),\ f\in L_2(I),\ V=H^1(I)$ and

$$a(v,w)=\int_I (uw+v'w')\,dx + v(0)w(0), \quad L(v)=\int_I fv\,dx,$$

it is trivial to show that $a(\cdot,\cdot)$ is bilinear and $b(\cdot)$ is linear. We have that

$$a(v,v)=\int_I v^2+(v')^2\,dx+v(0)^2 \geq \int_I (v)^2\,dx + \frac{1}{2}\int_I (v')^2\,dx + \frac{1}{2}v(0)^2 + \frac{1}{2}\int_I (v')^2\,dx. \tag{9}$$

Further

$$v(x)=v(0)+\int_0^x v'(y)\,dy, \quad \forall x\in I$$

implies

$$v^2(x)\leq 2\Big(v(0)^2+(\int_0^x v'(y)\,dy)^2\Big) \leq \{C-S\} \leq 2v(0)^2+2\int_0^1 v'(y)^2\,dy,$$

so that

$$\frac{1}{2}v(0)^2+\frac{1}{2}\int_0^1 v'(y)^2\,dy \geq \frac{1}{4}v^2(x), \quad \forall x\in I.$$

Integrating over x we get

$$\frac{1}{2}v(0)^2+\frac{1}{2}\int_0^1 v'(y)^2\,dy \geq \frac{1}{4}\int_I v^2(x)\,dx. \tag{10}$$

Now combining (9) and (10) we get

$$\begin{aligned} a(v,v) &\geq \frac{5}{4}\int_I v^2(x)\,dx + \frac{1}{2}\int_I (v')^2(x)\,dx \\ &\geq \frac{1}{2}\Big(\int_I v^2(x)\,dx + \int_I (v')^2(x)\,dx\Big) = \frac{1}{2}||v||_V^2, \end{aligned}$$

so that we can take $\kappa_1 = 1/2$. Further

$$\begin{aligned} |a(v,w)| &\leq \Big|\int_I vw\,dx\Big| + \Big|\int_I v'w'\,dx\Big| + |v(0)w(0)| \leq \{C-S\} \\ &\leq ||v||_{L_2(I)}||w||_{L_2(I)} + ||v'||_{L_2(I)}||w'||_{L_2(I)} + |v(0)||w(0)| \\ &\leq \Big(||v||_{L_2(I)}+||v'||_{L_2(I)}\Big)\Big(||w||_{L_2(I)}+||w'||_{L_2(I)}\Big) + |v(0)||w(0)| \\ &\leq \sqrt{2}\Big(||v||^2_{L_2(I)}+||v'||^2_{L_2(I)}\Big)^{1/2}\cdot\sqrt{2}\Big(||w||^2_{L_2(I)}+||w'||^2_{L_2(I)}\Big)^{1/2} + |v(0)||w(0)| \\ &\leq \sqrt{2}||v||_V\sqrt{2}||w||_V + |v(0)||w(0)|. \end{aligned}$$

Now we have that

$$v(0) = -\int_0^x v'(y)\,dy + v(x), \quad \forall x \in I, \tag{11}$$

and by the Mean-value theorem for the integrals: $\exists \xi\in I$ so that $v(\xi)=\int_0^1 v(y)\,dy$. Choose $x=\xi$ in (11) then

$$\begin{aligned} |v(0)| &= \Big|-\int_0^\xi v'(y)\,dy + \int_0^1 v(y)\,dy\Big| \\ &\leq \int_0^1 |v'|\,dy + \int_0^1 |v|\,dy \leq \{C-S\} \leq ||v'||_{L_2(I)} + ||v||_{L_2(I)} \leq 2||v||_V, \end{aligned}$$

implies that

$$|v(0)||w(0)| \leq 4||v||_V||w||_V,$$

and consequently

$$|a(u,w)| \leq 2||v||_V||w||_V + 4||v||_V||w||_V = 6||v||_V||w||_V,$$

so that we can take $\kappa_2 = 6$. Finally

$$|L(v)| = \left| \int_I fv\,dx \right| \leq ||f||_{L_2(I)}||v||_{L_2(I)} \leq ||f||_{L_2(I)}||v||_V,$$

taking $\kappa_3 = ||f||_{L_2(I)}$ all the conditions in the Lax-Milgram theorem are fulfilled.

6. See the Lecture Notes.

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