Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2019–03–20, 14:00-18:00

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 5p. Valid bonus poits will be added to the scores.
Breakings for **Chalmers; 3**: 15-21p, **4**: 22-28p, **5**: 29p-, and for **GU; G**: 15-26p, **VG**: 27p-
For solutions and information about gradings see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/TMA372/1819/index.html

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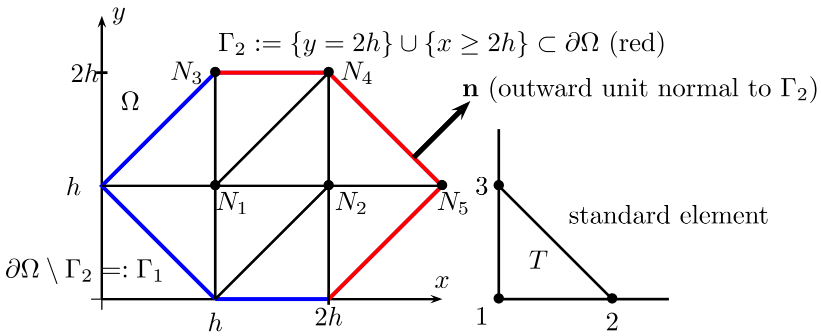
1. Let $\pi_h f$ be the linear interpolant of f in (a, b). Prove the $L_p(a, b)$ interpolation error estimates:

$$||\pi_h f - f||_{L_p(a,b)} \le (b-a)^2 ||f''||_{L_p(a,b)}, \qquad p = 1, 2.$$

2. Let Ω be the hexagonal domain with the uniform triangulation as in the figure below. Compute



the stiffness matrix and the load vector for the cG(1) approximate solution for the problem:

$$(1) \qquad -\Delta u = 1, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \Gamma_1, \quad \partial u/\partial \mathbf{n} = 1, \text{ on } \Gamma_2$$

3. Derive *a posteriori* error estimate, in the energy norm, definied as $||v||_E^2 := ||v'||^2 + ||v||^2$, for the *cG*(1) approximation of the boundary value problem

$$-u''(x) + 2xu'(x) + 2u(x) = f(x), \quad 0 < x < 1, \quad u'(0) = 0, \quad u(1) = 0.$$

4. Formulate dG(0) scheme for $\dot{u}(t) + a(t)u(t) = 0$, $u(0) = u_0$, $a(t) > 0$, and prove the stability

$$|U_N|^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \le |u_0|^2.$$

5. Consider the convection problem

$$\beta \cdot \nabla u + \alpha\, u = f \quad x \in \Omega; \qquad u = g \text{ for } x \in \Gamma_- := \{x \in \partial\Omega : \beta(x) \cdot \mathbf{n}(x) < 0\}.$$

Assume that $\alpha - \frac{1}{2}\nabla \cdot \beta \ge c > 0$. Prove the stability estimate

$$c\|u\|^2 + \int_{\Gamma_+} \mathbf{n} \cdot \beta u^2\, dx \le \frac{1}{c}\|f\|^2 + \int_{\Gamma_-} |\mathbf{n} \cdot \beta| g^2\, dx, \qquad \Gamma_+ := \partial\Omega \setminus \Gamma_-.$$

Hint: Show first $2(\beta \cdot \nabla u, u) = \int_{\Gamma_+} \mathbf{n} \cdot \beta u^2\, dx - \int_{\Gamma_-} |\mathbf{n} \cdot \beta| u^2\, dx - ((\nabla \cdot \beta)u, u)$.

6. Consider the cG(1) approximation u_h for the Poisson equation

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \qquad \Omega \subset \mathbb{R}^d, \;\; d = 2, 3.$$

Let $e := u - u_h$ be the error of approximation and show the following gradient estimate in $L_2(\Omega)$:

$$\|\nabla e\| = \|\nabla(u - u_h)\| \le C\|hD^2 u\|.$$

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TMA372/MMG800: Partial Differential Equations, 2019–03–20, 14:00-18:00. Solutions.

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1. Let $\lambda_a(x) = \frac{b-x}{b-a}$ and $\lambda_b(x) = \frac{x-a}{b-a}$ be two linear base functions. Then by the integral form of the Taylor formula we may write

$$\begin{cases} f(a) &= f(x) + f'(x)(a-x) + \int_x^a (a-y)f''(y)\,dy, \\ f(b) &= f(x) + f'(x)(b-x) + \int_x^b (b-y)f''(y)\,dy, \end{cases}$$

Therefore using the obvious relations $\lambda_a(x) + \lambda_b(x) = 1$ and $a\lambda_a(x) + b\lambda_b(x) = x$,

$$\begin{aligned} \Pi_1 f(x) &= f(a)\lambda_a(x) + f(b)\lambda_b(x) \\ &= f(x) + \lambda_a(x)\int_x^a (a-y)f''(y)\,dy + \lambda_b(x)\int_x^b (b-y)f''(y)\,dy \end{aligned}$$

and by the triangle inequality we get

$$\begin{aligned} |f(x) - \Pi_1 f(x)| &= \Big|\lambda_a(x)\int_x^a (a-y)f''(y)\,dy + \lambda_b(x)\int_x^b (b-y)f''(y)\,dy\Big| \\ &\le |\lambda_a(x)|\Big|\int_x^a (a-y)f''(y)\,dy\Big| + |\lambda_b(x)|\Big|\int_x^b (b-y)f''(y)\,dy\Big| \\ &\le |\lambda_a(x)|\int_x^a |a-y||f''(y)|\,dy + |\lambda_b(x)|\int_x^b |b-y||f''(y)|\,dy \\ &\le |\lambda_a(x)|\int_a^x (b-a)|f''(y)|\,dy + |\lambda_b(x)|\int_x^b (b-a)|f''(y)|\,dy \\ &\le (b-a)\Big(|\lambda_a(x)| + |\lambda_b(x)|\Big)\int_a^b |f''(y)|\,dy \\ &= (b-a)\Big(\lambda_a(x) + \lambda_b(x)\Big)\int_a^b |f''(y)|\,dy = (b-a)\int_a^b |f''(y)|\,dy. \end{aligned} \tag{2}$$

Consequently

$$\int_a^b |f(x) - \Pi_1 f(x)|dx \le \int_a^b (b-a)\Big(\int_a^b |f''(y)|\,dy\Big)\,dx, = (b-a)^2||f''||_{L_1(I)}.$$

To derive $L_2(a,b)$ interpolation error, we square (3) to get

$$\begin{aligned} |f(x) - \Pi_1 f(x)|^2 &\le (b-a)^2\Big(\int_a^b |f''(y)|\,dy\Big)^2 \\ &\le (b-a)^2\Big(\int_a^b 1^2\,dy\Big)\Big(\int_a^b |f''(y)|^2\,dy\Big) = (b-a)^3\int_a^b |f''(y)|^2\,dy. \end{aligned} \tag{3}$$

Then integrating over (a,b) we get

$$||f - \Pi_1 f||^2_{L_2(a,b)} \le (b-a)^4||f''||^2_{L_2(a,b)},$$

which yields the desired result.

2. Let V be the linear function space defined by

$$V := \{v : v \in H^1(\Omega),\ \ v = 0,\ \text{ on } \Gamma_1\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) = (1, v), \qquad \forall v \in V.$$

Now using Green's formula and the fact that $v = 0$ on $\partial\Omega \setminus \Gamma_1$, we have that

$$\begin{aligned}
-(\Delta u, \nabla v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v\, ds \\
&= (\nabla u, \nabla v) - \int_{\Gamma_1} (n \cdot \nabla u) v\, ds - \int_{\Gamma_2} (n \cdot \nabla u) v\, ds \\
&= (\nabla u, \nabla v) - \int_{\Gamma_2} v\, ds, \qquad \forall v \in V,
\end{aligned}$$

Hence, the variational formulation is:

$$(\nabla u, \nabla v) = (1, v) + \langle 1,\ v\rangle_{\Gamma_2}, \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on Γ_1: Then, the $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) = (1, v) + \langle 1,\ v\rangle_{\Gamma_2}, \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^{5} \xi_j \varphi_j(x)$, where φ_j are the standard basis functions (φ_1 is the basis function for the interior node N_1 and φ_2 and φ_3 are corresponding basis functions for the boundary nodes N_1 and N_2, respective) we obtain the system of equations

$$\sum_{j=1}^{3} \xi_j \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\, dx = \int_\Omega \varphi_i\, dx + \int_{Gamma_2} \varphi_i\, ds \quad i = 1, 2, 3, 4, 5.$$

In matrix form this can be written as $S\xi = F$, where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, and $F_i = (1, \varphi_i) + \langle 1,\ \varphi_i\rangle_{\Gamma_2}$ is the load vector.

We first compute the stiffness matrix for the reference triangle T. The local basis functions are

$$\begin{aligned}
\phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, &\qquad \nabla\phi_1(x_1, x_2) &= -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\phi_2(x_1, x_2) &= \frac{x_1}{h}, &\qquad \nabla\phi_2(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\phi_3(x_1, x_2) &= \frac{x_2}{h}, &\qquad \nabla\phi_3(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Hence, with $|T| = \int_T dz = h^2/2$, we can easily compute

$$\begin{aligned}
s_{11} &= (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\, dx = \frac{2}{h^2}|T| = 1, \\
s_{12} &= s_{21} = (\nabla\phi_1, \nabla\phi_2) = \int_T \frac{-1}{h^2}|T| = -1/2, \\
s_{23} &= s_{32} = (\nabla\phi_2, \nabla\phi_3) = 0, \\
s_{22} &= s_{33} = \ldots = \frac{1}{h^2}|T| = 1/2.
\end{aligned}$$

Thus by symmetry we get that the local stiffness matrix for the standard element is:

$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global stiffness matrix S from the local stiffness matrix s:

$$S_{11} = S_{22} = 3s_{11} + 2s_{22} = 3 + 1 = 4, \qquad S_{12} = S_{13} = 2s_{12} = -1$$
$$S_{14} = 2s_{23} = 0, \quad S_{15} = 0, \quad S_{23} = 0, \qquad S_{24} = S_{25} = 2s_{12} = -1,$$
$$S_{33} = s_{11} + s_{22} = 3/2 \qquad S_{34} = s_{12} = -1/2 \qquad S_{35} = 0,$$
$$S_{44} = 3s_{22} = 3/2 \qquad S_{45} = s_{23} = 0 \qquad S_{55} = 2s_{22} = 1$$

The remaining matrix elements are obtained by symmetry $S_{ij} = S_{ji}$. Hence,

$$S = \frac{1}{2}\begin{bmatrix} 8 & -2 & -2 & 0 & 0 \\ -2 & 8 & 0 & -2 & -2 \\ -2 & 0 & 3 & -1 & 0 \\ 0 & -2 & -1 & 3 & 0 \\ 0 & -2 & 0 & 0 & 2 \end{bmatrix}.$$

As for the load vector we have that

$$J := \int_T \pi_i = \frac{1}{3}\frac{h^2}{2}.1 = \frac{h^2}{6}.$$

$$\int_\Omega \varphi_1 = \int_\Omega \varphi_2 = 5J = \frac{5h^2}{6}, \quad \int_\Omega \varphi_3 = J = \frac{h^2}{6}, \quad \int_\Omega \varphi_4 = 3J = \frac{h^2}{2}, \quad \int_\Omega \varphi_5 = 2J = \frac{h^2}{3},$$

and

$$\int_{\Gamma_2} \varphi_1 = \int_{\Gamma_2} \varphi_2 = 0, \quad \int_{\Gamma_3} \varphi_3 = \frac{h}{2}, \quad \int_{\Gamma_2} \varphi_4 = \frac{h}{2} + \frac{\sqrt{2}h}{2}, \quad \int_{\Gamma_2} \varphi_5 = 2\frac{\sqrt{2}h}{2} = \sqrt{2}h$$

Thus the load vector is given by $b = \frac{h^2}{6}(5,5,1,3,2)^t + \frac{h}{2}(0,0,1,1+\sqrt{2},2\sqrt{2})^t$. Observe that, here S becomes independent of h.

3. <u>The Variational formulation:</u>

Let $V := \{v \in H^1(0,1) : v(1) = 0\}$ be the continuous test function space Multiply the equation by $v \in V$, integrate by parts over $(0,1)$ and use the boundary conditions to get

$$\text{(VF) Find } u \in V: \quad \int_0^1 u'v'\,dx + 2\int_0^1 xu'v\,dx + 2\int_0^1 uv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V.$$

<u>cG(1):</u> Let $\mathcal{T}_h$ be a partition of $(0,1)$ and define the discrete test function space

$$V_h := \{v : v \text{ is continuous piecewise linear in } \mathcal{T}_h \text{of}(0,1),\ v(1) = 0\}.$$

(4) $$\text{(FEM) Find } U \in V_h: \quad \int_0^1 U'v'\,dx + 2\int_0^1 xU'v\,dx + 2\int_0^1 Uv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V_h,$$

In VF we restrich $v \in V_h \subset V$. Then (VF)-(FEM) yields

<u>The Galerkin orthogonality:</u>

$$\int_0^1 \Big((u-U)'v' + 2x(u-U)'v + 2(u-U)v\Big)\,dx = 0, \quad \forall v \in V_h.$$

We define the inner product $(\cdot,\cdot)_E$ associated to the energy norm to be

$$(v,w)_E = \int_0^1 (v'w' + vw)\,dx, \qquad \forall v, w \in V.$$

Let $e(x) := u(x) - U(x)$. We use $e(1) = u(1) - U(1) = 0$ to derive the following key identity:

(5) $$\int_0^1 2xe'e = \int_0^1 x\frac{d}{dx}\Big(e^2\Big)\,dx = [PI] = \Big[xe(x)^2\Big]_0^1 - \int_0^1 e^2\,dx = -\int_0^1 e^2\,dx.$$

A posteriori error estimate: Thus using (5)

$$||e||_E^2 = \int_0^1 (e'e' + ee)\,dx = \int_0^1 (e'e' - ee + 2ee)\,dx = \int_0^1 (e'e' + 2xe'e + ee)$$
$$= \int_0^1 \Big(e'(u-U)' + 2xe(u-U)' + 2e(u-U)\Big)\,dx$$
$$= \int_0^1 (e'u + 2xeu' + 2eu)\,dx - \int_0^1 \Big(e'U' + 2xeU' + 2eU\Big)\,dx := T_1 + T_2.$$

Now using the variational formulation (VF) with $v = e \in V$ we have $T_1 = \int_0^1 fe\,dx$. Hence

(6)
$$||e||_E^2 = \int_0^1 fe\,dx - \int_0^1 \Big(e'U' + 2xeU' + 2eU\Big)\,dx = \Big\{ \pm \pi_h e \in V_h \text{ in } e \text{ terms}\Big\}$$
$$\int_0^1 f(e - \pi_h e)\,dx - \int_0^1 \Big(U'(e - \pi_h e)' + 2xU'(e - \pi_h e) + 2U(e - \pi_h e)\Big)\,dx$$

Partial integration gives

(7)
$$-\int_0^1 U'(e - \pi_h e)' = +\sum_{I_k} \Big(\int_{I_k} U''(e - \pi_h e) - [U'(x)(e(x) - \pi_h e(x))]_{x_{k-1}}^{x_k}$$
$$= \sum_{I_k} \Big(\int_{I_k} U''(e - \pi_h e) = \int_0^1 U''(e - \pi_h e).$$

Inserting in (6) and using the interpolation error we end up with

(8)
$$||e||_E^2 = \int_0^1 \Big(f + U'' - 2xU' - 2U)(e - \pi_h e) = \int_0^1 \mathcal{R}(U)(e - \pi_h e)$$
$$\leq \|h\mathcal{R}(U)\|\,\|e'\|$$

This gives the a posteriori error estimate:

$$||e||_E \leq \|h\mathcal{R}(U)\|.$$

4. For dG(0) we have discontinuous, piecewise constant test functions, hence in the variational formulation below

$$(\dot{u}, v) + (au, v) = (f, v),$$

we may take $v \equiv 1$ and hence we have for a single subinterval $I_n = (t_{n-1}, t_n]$ the dG(0) approximation

$$\int_{I_n} (\dot{U} + aU(t)dt + (U_n - U_{n-1})\,dt = \int_{I_n} f\,dt.$$

For $f = 0$ this yields (see als)o Fig below)

(9)
$$aK_nU_n + (U_n - U_{n-1}) = 0.$$

Multiplying by U_n we get

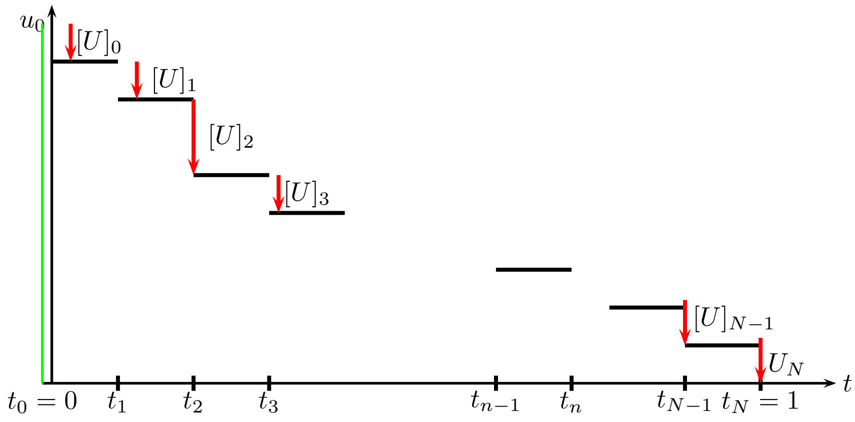
$$ak_nU_n^2 + U_n^2 - U_nU_{n-1} = 0,$$

where $a > 0$, whence

$$U_n^2 - U_nU_{n-1} \leq 0.$$

Now we use, for $n = 1, 2, \ldots, N$,

$$U_n^2 - U_nU_{n-1} = \frac{1}{2}U_n^2 + \frac{1}{2}U_n^2 - U_nU_{n-1},$$



and sum over $n = 1, 2, \dots, N$ to write

$$\begin{aligned}\sum_{n=1}^{N}(U_n^2 - U_nU_{n-1}) &= U_N^2 - U_NU_{N-1} + U_{N-1}^2 - U_{N-1}U_{N-2} + - \dots U_1^2 - U_1U_0 \\ &= U_N^2 - U_NU_{N-1} + U_{N-1}^2 - U_{N-1}U_{N-2} + - \dots U_1^2 - U_1U_0 + \frac{1}{2}U_0^2 2 - \frac{1}{2}U_0^2 \\ &= \frac{1}{2}U_N^2 + \frac{1}{2}(U_N - U_{N-1})^2 + \frac{1}{2}U_{N-1}^2 + \dots + \frac{1}{2}U_1^2 + \frac{1}{2}(U_1 - U_0)^2 - \frac{1}{2}U_0^2 \le 0.\end{aligned}$$

Further by the definition $[U_n] = U_{n+1} - U_n$, hence the above inequality yields the desired result

$$U_N^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \le U_0^2.$$

5. Multiply the equation by u and integrate over Ω

$$(\beta \cdot \nabla u, u) + (\alpha u, u) = (f, u). \tag{10}$$

Green's formula gives

$$\begin{aligned}(\beta \cdot \nabla u, u) &= \int_\Gamma \beta \cdot n u^2\, ds - \int_\Omega u \nabla \cdot (\beta u)\, dx \\ &= \int_\Gamma \beta \cdot n u^2\, ds - \int_\Omega u\Big(\nabla \cdot \beta\, u + \beta \cdot \nabla u\Big)\, dx \\ &= \int_{\Gamma_+} \beta \cdot n\, u^2\, ds + \int_{\Gamma_-} \beta \cdot n\, u^2\, ds - (\nabla \cdot \beta\, u, u) - (\beta \cdot \nabla u, u),\end{aligned}$$

so that, since $\beta \cdot n < 0$ and $u = g$ on Γ_-

$$2(\beta \cdot \nabla u, u) = \int_{\Gamma_+} \beta \cdot n\, u^2\, ds - \int_{\Gamma_-} |\beta \cdot n|\, g^2\, ds - (\nabla \cdot \beta\, u, u).$$

Inserting in (10) we get

$$\int_{\Gamma_+} \beta \cdot n\, u^2\, ds + 2((\alpha - \frac{1}{2}\nabla \cdot \beta)u, u) = 2(u, f) + \int_{\Gamma_-} |\beta \cdot n|\, g^2\, ds. \tag{11}$$

Hence, we have using $\alpha - \frac{1}{2}\nabla \cdot \beta \ge c > 0$ and

$$2(f, u) \le 2\|f\|\|u\| \le 2(c^{-1/2}\|f\|)(c^{1/2}\|u\|) \le \frac{1}{c}\|f\|^2 + c\|u\|^2,$$

so that (11) becomes

$$\int_{\Gamma_+} \beta \cdot n\, u^2\, ds + 2c\|u\|^2 \le \frac{1}{c}\|f\|^2 + c\|u\|^2 + \int_{\Gamma_-} |\beta \cdot n|\, g^2\, ds.$$

This yields the desired result:

$$\int_{\Gamma_+} \beta \cdot n\, u^2\, ds + c\|u\|^2 \le \frac{1}{c}\|f\|^2 + \int_{\Gamma_-} |\beta \cdot n|\, g^2\, ds.$$

6. See the book.

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