Mathematics Chalmers & GU

MVE455: Partial Differential Equations for Kf3, 2016–03–14, 8:30-12:30

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 4p. Valid bonus poits will be added to the scores.
Breakings from total of 24 points: Exam(20)+Bonus(4). **3**: 10-14p, **4**: 15-19p och **5**: 20p-
For solutions see the couse diary: http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1516/

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1. Consider a uniform partition $0 = x_0 < x_1 < \ldots < x_N = 1$ of the interval $[0, 1]$ and let $\{\varphi_i\}_{i=0}^N$ be a set of piecewise linear continuous basis functions: $\varphi_i(x_j) = 1$ for $i = j$ and $\varphi_i(x_j) = 0$ if $i \neq j$. Given a FEM in form of linear system of equations $(S + C)\xi = \mathbf{b} + \mathbf{d}$, with S and C N-by-N matrices, and ξ, $\mathbf{b}$ and $\mathbf{d}$ vectors of lenght N, where for $i, j = 0, \ldots, N-1$, $S_{ij} = (\varphi_i', \varphi_j')$, $C_{ij} = (\varphi_i, \varphi_j')$, and $\mathbf{b}_i = (\varphi_i, f)$ with f a given function. $d_0 = \alpha$ is the only non-zero element of $\mathbf{d}$.

a) Derive the variational formulation and the strong formulation for the PDE from the above data.

b) Let now both $u(0) = 0$ and $\alpha = 0$ and derive the continuous stability estimate $\|u_x\| \le \|f\|$.

2. Prove an *a priori* error estimate for a finite element method for the boundary value problem, (the required interpolation estimates can be used without proofs):

$$-u_{xx} + u_x = f, \quad x \in (0, 1); \qquad u(0) = u(1) = 0.$$

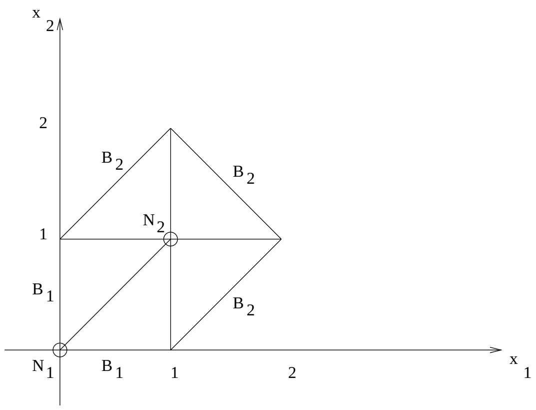
3. The dG(0) solution U for the scalar population dynamics, $\dot{u}(t) + au(t) = f$, $u(0) = u_0$, in the subinterval $I_n = (t_{n-1}, t_n]$ with $k_n = t_n - t_{n-1}$, $n = 1, 2, \ldots N$, and $f \equiv 0$ is given by

$$ak_nU_n + (U_n - U_{n-1}) = 0, \qquad U_n = U|_{I_n} = U_n^- = U_{n-1}^+.$$

Let $a > 0$ and show the discrete stability estimate

$$U_N^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \le U_0^2, \qquad [U_n] := U_n^+ - U_n^- = U_{n+1} - U_n.$$

4. Let Ω be the triangulated domain below. Compute the cG(1) solution of $-\Delta u = 0$ in Ω with the Neumann data: $\partial_n u = 3$ on B_1 and Dirichlet condition: $u = 0$ on B_2.



5. Formulate and prove the Lax-Milgram theorem.

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Solutions.

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1. a) It is clear that the homogeneous Dirichlet condition is used at $x = 1$ since the basis function φ_N is not present in the matrices and there are no modifivations corresponding the last element of the load vector. Consider now the solution space $V^0 = \{w : \|w\| + \|w'\| < \infty\},\ w(1) = 0\}$ where $\|\cdot\|$ is the usual L_2-norm over I. Then, the variational formulation reads as follows: *find* $u \in V^0$ *s.t.*

$$(v_x, u_x) + (v, u_x) = (v, f) + \alpha v(0), \qquad \forall\ v \in V^0. \tag{1}$$

For the basis functions ginen, $\varphi_0(0) = 1$, which explains the first element of the vector $\mathbf{d}$. Backward integration by parts, together with the Dirichlet data on v yields

$$(v, f) + \alpha v(0) = (v, -u_{xx} + u_x) + v(0)u_x(0). \tag{2}$$

Thus, the strong formulation (PDE) is: find such that

$$-u_{xx} + u_x = f \quad 0 < x < 1 \qquad u_x(0) = \alpha, \quad u(1) = 0. \tag{3}$$

(b) Let in (1) $\alpha = 0$ and $v = u$, then

$$(u_x, u_x) + (u, u_x) = (u, f).$$

Using integration by parts and $u(0) = 0$,

$$(u, u_x) = u^2(x)|_{x=0}^{x=1} - (u_x, u) \Longrightarrow (u, u_x) = u^2(1) - u^2(0) = u^2(1) - u^2(0) = 0.$$

Hence, using Cauchy-Schwarz and Poincare inequalities:

$$\|u_x\|^2 = (u, f) \leq \|u\|\|f\| \leq \|u_x\|\|f\|,$$

we get the desired result.

2. We multiply the differential equation by a test function $v \in H_0^1 = \{v : ||v|| + ||v'|| < \infty,\ v(0) = v(1) = 0\}$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + u'v) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{4}$$

Or equivalently, find $u \in H_0^1(I)$ such that

$$(u_x, v_x) + (u_x, v) = (f, v), \quad \forall v \in H_0^1(I), \tag{5}$$

with $(\cdot, \cdot)$ denoting the $L_2(I)$ scalar product: $(u, v) = \int_I u(x)v(x)\,dx$. A *Finite Element Method* with $cG(1)$ reads as follows: Find $u_h \in V_h^0$ such that

$$\int_I (u_h'v' + u_h'v) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{6}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Or equivalently, find $u_h \in V_h^0$ such that

$$(u_{h,x}, v_x) + (u_{h,x}, v) = (f, v), \quad \forall v \in V_h^0. \tag{7}$$

Let now

$$a(u, v) = (u_x, v_x) + (u_x, v).$$

We want to show that $a(\cdot, \cdot)$ is both elliptic and continuous:

ellipticity

$$a(u,u) = (u_x, u_x) + (u_x, u) = ||u_x||^2, \tag{8}$$

where we have used the boundary data, viz,

$$\int_0^1 u_x u\, dx = \Big[\frac{u^2}{2}\Big]_0^1 = 0.$$

continuity

$$a(u,v) = (u_x, v_x) + (u_x, v) \le ||u_x||||v_x|| + ||u_x||||v|| \le 2||u_x||||v_x||, \tag{9}$$

where we used the Poincare inequality $||v|| \le ||v_x||$.

Let now $e = u - u_h$, then (5)- (7) gives that

$$a(u-u_h, v) = (u_x - u_{h,x}, v_x) + (u_x - u_{h,x}, v) = 0, \quad \forall v \in V_h^0, \text{(Galerkin Orthogonality)}. \tag{10}$$

A priori error estimate: We use ellipticity (8), Galerkin orthogonality (10), and the continuity (9) to get

$$\|u_x - u_{h,x}\|^2 = a(u-u_h, u-u_h) = a(u-u_h, u-v) \le 2\|u_x - u_{h,x}\|\|u_x - v_x\|, \quad \forall v \in V_h^0.$$

This gives that

$$\|u_x - u_{h,x}\| \le 2\|u_x - v_x\|, \quad \forall v \in V_h^0, \tag{11}$$

If we choose $v = \pi_h u \in V_h^0$, the interpolant of u, and use the interpolation estimate we get from (11) that

$$\|u_x - u_{h,x}\| \le 2\|u_x - (\pi u)_x\| \le 2C_i\|hu_{xx}\|. \tag{12}$$

3. For dG(0) we have discontinuous, piecewise constant test functions, hence in the variational formulation below

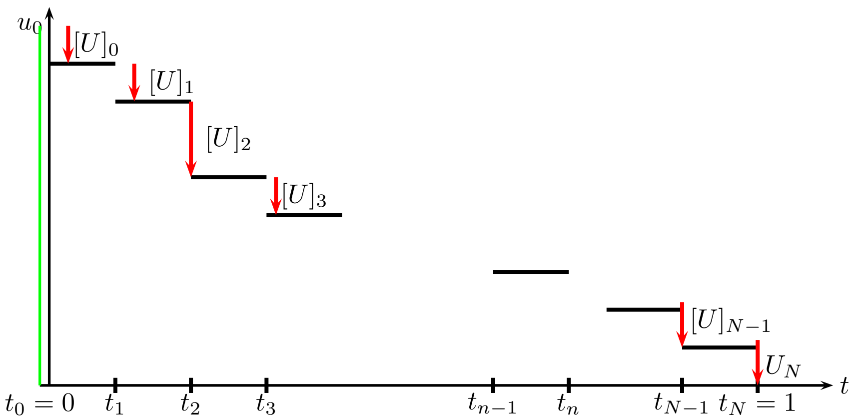
$$(\dot{u}, v) + (au, v) = (f, v),$$

we may take $v \equiv 1$ and hence we have for a single subinterval $I_n = (t_{n-1}, t_n]$ the dG(0) approximation

$$\int_{I_n} (\dot{U} + aU(t)dt + (U_n - U_{n-1})\, dt = \int_{I_n} f\, dt.$$

For $f = 0$ this yields (see als)o Fig below)

$$aK_n U_n + (U_n - U_{n-1}) = 0. \tag{13}$$



Multiplying by U_n we get

$$ak_n U_n^2 + U_n^2 - U_n U_{n-1} = 0,$$

where $a > 0$, whence

$$U_n^2 - U_n U_{n-1} \le 0.$$

Now we use, for $n = 1, 2, \dots, N$,

$$U_n^2 - U_n U_{n-1} = \frac{1}{2}U_n^2 + \frac{1}{2}U_n^2 - U_n U_{n-1},$$

and sum over $n = 1, 2, \dots, N$ to write

$$\begin{aligned}\sum_{n=1}^{N}(U_n^2 - U_n U_{n-1}) &= U_N^2 - U_N U_{N-1} + U_{N-1}^2 - U_{N-1}U_{N-2} + - \dots U_1^2 - U_1 U_0 \\ &= U_N^2 - U_N U_{N-1} + U_{N-1}^2 - U_{N-1}U_{N-2} + - \dots U_1^2 - U_1 U_0 + \frac{1}{2}U_0^2 2 - \frac{1}{2}U_0^2 \\ &= \frac{1}{2}U_N^2 + \frac{1}{2}(U_N - U_{N-1})^2 + \frac{1}{2}U_{N-1}^2 + \dots + \frac{1}{2}U_1^2 + \frac{1}{2}(U_1 - U_0)^2 - \frac{1}{2}U_0^2 \le 0.\end{aligned}$$

Further by the definition $[U_n] = U_{n+1} - U_n$, hence the above inequality yields the desired result

$$U_N^2 + \sum_{n=0}^{N-1} |[U_n]|^2 \le U_0^2.$$

4. Variational Formulation: Using Green's formula we have that

$$\begin{aligned}0 &= \int_\Omega -\Delta u v \, dx = \{\text{Green's}\} = \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma (\partial_n u) v \\ &= \{\Gamma := \partial\Omega := B_1 \cup B_2\} = \{v = 0 \text{ on } B_2, \text{ and } \partial_n u = 3 \text{ on } B_1\} \\ &= \int_\Omega \nabla u \cdot \nabla v \, dx - \int_{B_1} 3v \, ds\end{aligned} \tag{14}$$

Thus we have the finite element formulation: Find piecewise linear function $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v = \int_{B_1} 3v \, ds, \quad \forall v \in V_h. \tag{15}$$

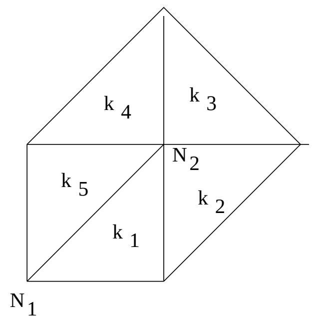
Let now

$$U(x) = U_1\varphi_1(x) + U_2\varphi_2(x), \tag{16}$$

where φ_i are the piecewise linears basis functions for the above discretization of Ω with $\varphi_i(N_j) = \delta_{ij}, \; i, j = 1, 2$. We insert (16) in (15) and let $v = \varphi_i, \; i = 1, 2$ to obtain a 2×2 system viz,

$$\begin{cases} \int_\Omega \nabla\varphi_1 \cdot \nabla\varphi_1 \, dx \, U_1 + \int_\Omega \nabla\varphi_2 \cdot \nabla\varphi_1 \, dx \, U_2 = 3\int_{B_1} \varphi_1 \, ds, \\ \int_\Omega \nabla\varphi_1 \cdot \nabla\varphi_2 \, dx \, U_1 + \int_\Omega \nabla\varphi_2 \cdot \nabla\varphi_2 \, dx \, U_2 = 3\int_{B_1} \varphi_2 \, ds. \end{cases} \tag{17}$$

Note that using the orientation in the figur below we have



$$
\begin{aligned}
\nabla\varphi_1\Big|_{k_1} &= (-1,0) & \nabla\varphi_2\Big|_{k_1} &= (0,1)\\
\nabla\varphi_1\Big|_{k_2} &= (0,0) & \nabla\varphi_2\Big|_{k_2} &= (-1,1)\\
\nabla\varphi_1\Big|_{k_3} &= (0,0) & \nabla\varphi_2\Big|_{k_3} &= (-1,-1)\\
\nabla\varphi_1\Big|_{k_4} &= (0,0) & \nabla\varphi_2\Big|_{k_4} &= (1,-1)\\
\nabla\varphi_1\Big|_{k_5} &= (0,-1) & \nabla\varphi_2\Big|_{k_5} &= (1,0)
\end{aligned}
$$

Thus

$$\int_\Omega \nabla\varphi_1\cdot\nabla\varphi_2\,dx = \int_\Omega \nabla\varphi_2\cdot\nabla\varphi_1\,dx = 0,$$

and

$$
\begin{aligned}
\int_\Omega \nabla\varphi_1\cdot\nabla\varphi_1\,dx =& \sum_{i=1}^{5}|k_i|\Big(\nabla\varphi_1|_{k_i}\cdot\nabla\varphi_1|_{k_i}\Big)\\
&\frac{1}{2}\times(-1,0)\cdot(-1,0)+\frac{1}{2}\times(0,-1)\cdot(0,-1)=\frac{1}{2}+\frac{1}{2}=1.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
\int_\Omega \nabla\varphi_2\cdot\nabla\varphi_2\,dx =& \sum_{i=1}^{5}|k_i|\Big(\nabla\varphi_2|_{k_i}\cdot\nabla\varphi_2|_{k_i}\Big) = \frac{1}{2}\times\Big((0,1)\cdot(0,1)\\
&+(-1,1)\cdot(-1,1)+(-1,-1)\cdot(-1,-1)+(1,-1)\cdot(1,-1)+(1,0)\cdot(1,0)\Big)\\
&=\frac{1}{2}\times\Big(1+2+2+2+1\Big)=4.
\end{aligned}
$$

As for the right hand side we have

$$3\int_{B_1}\varphi_1 = 3\times \text{aread of the side alonge } B_1 = 3\Big(1/2+1/2\Big)=3,$$

while

$$3\int_{B_1}\varphi_2 = 0.$$

Summing up we have a trivial situation as follows:

$$\begin{bmatrix}1 & 0\\ 0 & 4\end{bmatrix}\begin{bmatrix}U_1\\ U_2\end{bmatrix}=\begin{bmatrix}3\\ 0\end{bmatrix}$$

Thus $U(x) = 3\varphi_1(x)$ and actually, with this configuration, we have a trivial one-dimensional problem.

5. See the lecture notes.

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