Mathematics Chalmers & GU

MVE455: Partial Differential Equations for Kf3, 2016–04–06, 8:30-12:30

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 4p. Valid bonus poits will be added to the scores.
Breakings from total of 24 points: Exam(20)+Bonus(4). **3**: 10-14p, **4**: 15-19p och **5**: 20p-
For solutions see the couse diary: http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1516/

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1. Show that for a continuously differentiable function v defined on $(0,1)$ we have that

$$||v||^2 \le v(0)^2 + v(1)^2 + ||v'||^2.$$

Hint: Use partial integration for $\int_0^{1/2} v(x)^2\,dx$ and $\int_{1/2}^1 v(x)^2\,dx$ and note that $(x-1/2)$ has the derivative 1.

2. Prove the following error estimate for the linear interpolation for a function $f \in C^2(0,1)$,

$$||\pi_1 f - f||_{L_1(a,b)} \le (b-a)^2 ||f''||_{L_1(a,b)}.$$

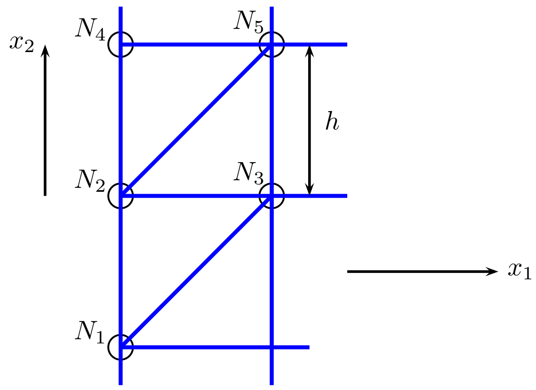
3. Give a variational formulation for the boundary value problem (with periodic boundary conditions).

$$\begin{cases} -u'' + \alpha u = f, & 0 < x < 1, \\ u(0) = u(1), & u'(0) = u'(1), \end{cases}$$

where α is a constant and $f \in L_2(0,1)$. Show that, with an appropriate condition on α, the hypothesis in the Lax-Milgram theorem are fullfild.

4. Let Ω be the domain in the figure below, with the given triangulation and nodes N_i, $i = 1, \dots, 5$. Let U be the cG(1) solution to the problem

(1) $$-\Delta u = 1, \quad \text{in } \Omega \in \mathbf{R}^2, \qquad \text{with} \qquad -\mathbf{n}\cdot\nabla u = 0 \text{ on } \partial\Omega,$$



a) Given the test function φ_2 at node N_2, find the relation between U_1, U_2, U_3, U_4, and U_5.
b) Derive the corresponding relation when the equation is replaced by $-\Delta u + (1,0)\cdot\nabla u = 1$.

5. Consider the Dirichlet boundary value problem:

$$\text{(BVP)} - (a(x)u'(x))' = f(x), \quad \text{for} \quad 0 < x < 1, \qquad u(0) = 0, \quad u(1) = 0.$$

where $a(x) > 0$ (*the modulus of elasticity*). Formulate the corresponding variational formulation (VF), the minimization problem (MP) and prove that $(VF) \Longleftrightarrow (MP)$.

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void!

MVE455: Partial Differential Equations for Kf3, 2016–04–06, 8:30-12:30. Solutions.

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1. We have, using the hint, that

$$
\begin{aligned}
||v||^2 = \int_0^1 v^2\,dx = & \int_0^{1/2} v^2\,dx + \int_{1/2}^1 v^2\,dx \\
=&[(x-1/2)v(x)^2]_0^{1/2} + [(x-1/2)v(x)^2]_{1/2}^1 - \int_0^1 (x-1/2)2v(x)v'(x)\,dx \\
\le& \frac{1}{2}v(0)^2 + \frac{1}{2}v(1)^2 + ||v||||v'|| \le \frac{1}{2}\Big(v(0)^2 + v(1)^2 + ||v'||^2\Big) + \frac{1}{2}||v||^2,
\end{aligned}
$$

and the proof is complete.

2. Let $\lambda_0(x) = \frac{\xi_1 - x}{\xi_1 - x_0}$ and $\lambda_1(x) = \frac{x-\xi_0}{\xi_1 - x_0}$ be two linear base functions, where $\xi_0 \neq \xi_1$, ξ_0, $\xi_1 \in [a,b]$, can be taken as arbitrary interpolation points or just $\xi_0 = a$, $\xi_1 = b$. Then by the integral form of the Taylor formula we may write

$$
\begin{cases}
f(\xi_0) & = f(x) + f'(x)(\xi_0 - x) + \int_x^{\xi_0} (\xi_0 - y) f''(y)\,dy, \\
f(\xi_1) & = f(x) + f'(x)(\xi_1 - x) + \int_x^{\xi_1} (\xi_1 - y) f''(y)\,dy,
\end{cases}
$$

Therefore, the linear function interpolating f in the points ξ_0, $\xi_1 \in [a,b]$, can be written as

$$
\begin{aligned}
\Pi_1 f(x) &= f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) \\
&= f(x) + \lambda_0(x)\int_x^{\xi_0} (\xi_0 - y) f''(y)\,dy + \lambda_1(x)\int_x^{\xi_1} (\xi_1 - y) f''(y)\,dy
\end{aligned}
$$

and by the triangle inequality we get

$$
\begin{aligned}
|f(x) - \Pi_1 f(x)| &= \Big|\lambda_0(x)\int_x^{\xi_0} (\xi_0 - y) f''(y)\,dy + \lambda_1(x)\int_x^{\xi_1} (\xi_1 - y) f''(y)\,dy\Big| \\
&\le |\lambda_0(x)|\Big|\int_x^{\xi_0} (\xi_0 - y) f''(y)\,dy\Big| + |\lambda_1(x)|\Big|\int_x^{\xi_1} (\xi_1 - y) f''(y)\,dy\Big| \\
&\le |\lambda_0(x)|\int_x^{\xi_0} |\xi_0 - y||f''(y)|\,dy + |\lambda_1(x)|\int_x^{\xi_1} |\xi_1 - y||f''(y)|\,dy \\
&\le |\lambda_0(x)|\int_x^{\xi_0} (b-a)|f''(y)|\,dy + |\lambda_1(x)|\int_x^{\xi_1} (b-a)|f''(y)|\,dy \\
&\le (b-a)\Big(|\lambda_0(x)| + |\lambda_1(x)|\Big)\int_a^b |f''(y)|\,dy \\
&= (b-a)\Big(\lambda_0(x) + \lambda_1(x)\Big)\int_a^b |f''(y)|\,dy = (b-a)\int_a^b |f''(y)|\,dy.
\end{aligned}
$$

Integrating over $x \in (a,b)$ we get the desired result.

3. Let $V = \{v \in H^1(0,1) : v(0) = v(1) = 0\}$ with

$$
||v||_V = ||v||_{H^1} = \sqrt{||v||^2 + ||v'||^2}.
$$

Multiplication of the differential equation by $v \in V$, integration by parts taking the boundary conditions into account, leeads to the variational formulation: find $u \in V$ such that

$$
a(u,v) = L(v) \quad \forall v \in V,
$$

where $\quad a(u,v) = (u',v') + \alpha(u,v) \quad$, and $\quad L(v) = (f,v)$.

If $\alpha > 0$ then we have

$$\begin{aligned}
a(u,u) &= ||u'||^2 + \alpha||u||^2 \geq \min(1,\alpha)||u||^2_{H^1},\\
a(u,v) &\leq ||u'||||v'|| + \alpha||u||||v|| \leq (1+\alpha)||u||_{H^1}||v||_{H^1},\\
|L(v)| &\leq ||f||||v|| \leq ||f||||v||_{H^1}.
\end{aligned}$$

4. a) We first compute stiffness matrix for the reference triangle T. The local basis functions are

$$\begin{aligned}
\phi_1(x_1,x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1,x_2) &= -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},\\
\phi_2(x_1,x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1,x_2) &= \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},\\
\phi_3(x_1,x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1,x_2) &= \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$\begin{aligned}
s_{11} &= (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\,dx = \frac{2}{h^2}|T| = 1,\\
s_{12} &= (\nabla\phi_1, \nabla\phi_2) = \int_T |\nabla\phi_1|^2\,dx = -\frac{1}{h^2}|T| = -1/2, & s_{13} &= -1/2\\
s_{22} &= (\nabla\phi_2, \nabla\phi_2) = \int_T |\nabla\phi_2|^2\,dx = \frac{1}{h^2}|T| = 1/2, & s_{23} &= (\nabla\phi_2, \nabla\phi_3) = 0,\\
s_{33} &= (\nabla\phi_3, \nabla\phi_3) = \int_T |\nabla\phi_3|^2\,dx = \frac{1}{h^2}|T| = 1/2,
\end{aligned}$$

Thus using the symmetry we have the local stiffness matrix as

$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrix S from the local s, using the character of our mesh, viz:

$$\begin{aligned}
&S_{11} = 2s_{22} = 1, \quad S_{12} = s_{12} = -1/2 \quad S_{13} = 2s_{23} = 0,\ S_{14} = S_{15} = 0\\
&S_{22} = s_{11} + 2s_{22} = 2, \quad S_{23} = 2s_{12} = -1,\ S_{24} = s_{12} = -1/2,\ S_{25} = 2s_{23} = 0\\
&S_{33} = s_{11} + 2s_{22} = 2,\ S_{34} = 0, S_{35} = s_{12} = -1/2\\
&S_{44} = s_{11} = 1,\ S_{45} = s_{12} = -1/2,\ S_{55} = 2s_{22} = 1
\end{aligned}$$

The remaining matrix elements are obtained by symmetry $S_{ij} = S_{ji}$. Hence,

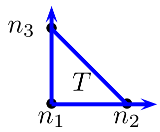
$$S = \frac{1}{2}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 4 & -2 & -1 & 0 \\ 0 & -2 & 4 & 0 & -1 \\ 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

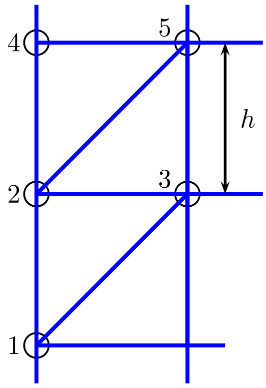
With U expressed in terms of the basis functions φ_j, $j = 1,2,3,4,5$ and with the test function $v = \varphi_2$ in the variational formulation we obtain the relation

$$-\frac{1}{2}U_1 + 2U_2 - U_3 - \frac{1}{2}U_4 = 3 \times \frac{1}{3} \times \frac{1}{2}h^2 = \frac{1}{2}h^2.$$

b) If we change the equation to $-\Delta u + (1,0)\cdot\nabla u = 1$ the relation between the nodal values becomes:

$$-\frac{1}{2}U_1 + 2U_2 - U_3 - \frac{1}{2}U_4 - \frac{h}{3}U_2 + \frac{h}{3}U_3 - \frac{h}{6}U_4 + \frac{h}{6}U_5 = \frac{1}{2}h^2.$$





Finally if, for instance, for $-\nabla \cdot a\nabla u = f$ with $a = 1$ for $x < 0$ and $a = 2$ for $x_2 > 0$, the corresponding relation is:

$$-\frac{1}{2}U_1 + 3U_2 - \frac{3}{2}U_3 - U_4 = \frac{1}{2}h^2.$$

You may work out the details in such a model!

5. See the lecture notes.

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