Mathematics Chalmers & GU

MVE455: Partial Differential Equations for Kf3, 2016–06–10, 8:30-12:30

Telephone: Adam Malik: ankn 5325
Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 4p. Valid bonus poits will be added to the scores.
Breakings from total of 24 points: Exam(20)+Bonus(4). **3**: 10-14p, **4**: 15-19p och **5**: 20p-
For solutions see the couse diary: http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1516/

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1. Show that for a continuously differentiable function v defined on (a,b), with $|I| = b-a = 1$ we have that

$$||v||^2 \le v(a)^2 + v(b)^2 + ||v'||^2.$$

Hint: Let $c = (a+b)/2$. Use partial integration for $\int_a^c v(x)^2\,dx$ and $\int_c^b v(x)^2\,dx$ and note that $(x-c)$ has the derivative 1.

2. Consider a uniform partition $0 = x_0 < x_1 < \ldots < x_N = 1$ of the interval $[0,1]$ and let $\{\varphi_i\}_{i=0}^N$ be a set of piecewise linear continuous basis functions: $\varphi_i(x_j) = 1$ for $i = j$ and $\varphi_i(x_j) = 0$ if $i \ne j$. Given a FEM in form of linear system of equations $(S+C)\xi = \mathbf{b} + \mathbf{d}$, with S and C N-by-N matrices, and ξ, $\mathbf{b}$ and $\mathbf{d}$ vectors of lenght N, where for $i,j = 0,\ldots,N-1$, $S_{ij} = (\varphi_i', \varphi_j')$, $C_{ij} = (\varphi_i, \varphi_j')$, and $\mathbf{b}_i = (\varphi_i, f)$ with f a given function. $d_0 = \alpha$ is the only non-zero element of $\mathbf{d}$. Derive the variational formulation and the strong formulation for the PDE from the above data.

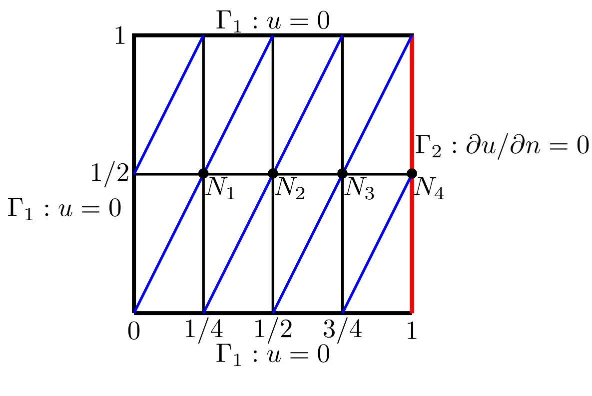
3. Prove an *a posteriori* error estimate for piecewise linear finite element method for the boundary value problem, (the required interpolation estimates can be used without proofs):

$$-u_{xx} + u_x = f, \quad x \in (0,1); \qquad u(0) = u(1) = 0.$$

4. Determine the stiffness matrix and load vector if the $cG(1)$ finite element method approximation is applied to the following Poisson's equation with mixed boundary conditions:

$$\begin{cases} -\Delta u = 1, & \text{on} \quad \Omega = (0,1)\times(0,1), \\ \frac{\partial u}{\partial n} = 0, & \text{for} \quad x_1 = 1, (x \in \Gamma_2) \\ u = 0, & \text{for} \quad x \in \partial\Omega \setminus \{x_1 = 1\} = \partial\Omega \setminus \Gamma_2, \end{cases} \qquad \text{verifying the local stiffness:} \quad s = \begin{pmatrix} 5/4 & -1 & -1/4 \\ -1 & 1 & 0 \\ -1/4 & 0 & 1/4 \end{pmatrix}$$

on a triangulation with triangles of side length 1/4 in the x_1-direction and 1/2 in the x_2-direction.



5. Consider the Dirichlet boundary value problem:

$$\text{(BVP)} \ -(a(x)u'(x))' = f(x), \quad \text{for} \quad 0 < x < 1, \qquad u(0) = 0, \quad u(1) = 0.$$

where $a(x) > 0$ (*the modulus of elasticity*). Formulate the corresponding variational formulation (VF), the minimization problem (MP) and prove that $(VF) \Longleftrightarrow (MP)$.

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MVE455: Partial Differential Equations for Kf3, 2016–06–10, 8:30-12:30. Solutions.

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1. We have, using the hint, $c = (a+b)/2$ that

$$||v||^2 = \int_a^b v^2\,dx = \int_a^c v^2\,dx + \int_c^b v^2\,dx$$
$$= [(x-c)v(x)^2]_a^c + [(x-c)v(x)^2]_c^b - \int_a^b (x-c)2v(x)v'(x)\,dx \le \{x - c \le \frac{b-a}{2}\}$$
$$\le \frac{b-a}{2}\Big(v(a)^2 + v(b)^2\Big) + ||v||||v'|| \le \frac{1}{2}\Big(v(0)^2 + v(1)^2 + ||v'||^2\Big) + \frac{1}{2}||v||^2,$$

and the proof is complete.

2. It is clear that the homogeneous Dirichlet condition is used at $x = 1$ since the basis function φ_N is not present in the matrices and there are no modifivations corresponding the last element of the load vector. Consider now the solution space $V^0 = \{w : \|w\| + \|w'\| < \infty\},\ w(1) = 0\}$ where $\|\cdot\|$ is the usual L_2-norm over I. Then, the variational formulation reads as follows: *find* $u \in V^0$ *s.t.*

(1) $$(v_x, u_x) + (v, u_x) = (v, f) + \alpha v(0), \qquad \forall\ v \in V^0.$$

For the basis functions ginen, $\varphi_0(0) = 1$, which explains the first element of the vector $\mathbf{d}$. Backward integration by parts, together with the Dirichlet data on v yields

(2) $$(v, f) + \alpha v(0) = (v, -u_{xx} + u_x) + v(0)u_x(0).$$

Thus, the strong formulation (PDE) is: find such that

(3) $$-u_{xx} + u_x = f \quad 0 < x < 1 \qquad u_x(0) = \alpha, \quad u(1) = 0.$$

3. We multiply the differential equation by a test function $v \in H_0^1 = \{v : ||v|| + ||v'|| < \infty,\ v(0) = v(1) = 0\}$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

(4) $$\int_I (u'v' + u'v) = \int_I fv, \quad \forall v \in H_0^1(I).$$

Or equivalently, find $u \in H_0^1(I)$ such that

(5) $$(u_x, v_x) + (u_x, v) = (f, v), \quad \forall v \in H_0^1(I),$$

with $(\cdot,\cdot)$ denoting the $L_2(I)$ scalar product: $(u, v) = \int_I u(x)v(x)\,dx$. A *Finite Element Method* with $cG(1)$ reads as follows: Find $u_h \in V_h^0$ such that

(6) $$\int_I (u_h'v' + u_h'v) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I),$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Or equivalently, find $u_h \in V_h^0$ such that

(7) $$(u_{h,x}, v_x) + (u_{h,x}, v) = (f, v), \quad \forall v \in V_h^0.$$

Let now

$$a(u, v) = (u_x, v_x) + (u_x, v).$$

We want to show that $a(\cdot,\cdot)$ is both elliptic and continuous:

ellipticity

$$a(u,u) = (u_x, u_x) + (u_x, u) = ||u_x||^2, \tag{8}$$

where we have used the boundary data, viz,

$$\int_0^1 u_x u\, dx = \Big[\frac{u^2}{2}\Big]_0^1 = 0.$$

continuity

$$a(u,v) = (u_x, v_x) + (u_x, v) \le ||u_x||||v_x|| + ||u_x||||v|| \le 2||u_x||||v_x||, \tag{9}$$

where we used the Poincare inequality $||v|| \le ||v_x||$.

Let now $e = u - u_h$, then (5)- (7) gives that

$$a(u-u_h, v) = (u_x - u_{h,x}, v_x) + (u_x - u_{h,x}, v) = 0, \quad \forall v \in V_h^0, \text{(Galerkin Orthogonality)}. \tag{10}$$

A posteriori error estimate: We use again ellipticity (8), Galerkin orthogonality (10), and the variational formulation (4) to get

$$\begin{aligned} \|e_x\|^2 &= a(e,e) = a(e, e-\pi e) = a(u, e-\pi e) - a(u_h, e-\pi e) \\ &= (f, e-\pi e) - a(u_h, e-\pi e) = (f, e-\pi e) - (u_{h,x}, e_x - (\pi e)_x) - (u_{h,x}, e-\pi e) \\ &= (f - u_{h,x}, e-\pi e) \le C\|h(f-u_{h,x})\|\|e_x\|, \end{aligned} \tag{11}$$

where in the last equality we use the fact that $e(x_j) = (\pi e)(x_j)$, for j:s being the node points, also $u_{h,xx} \equiv 0$ on each $I_j := (x_{j-1}, x_j)$. Thus

$$(u_{h,x}, e_x - (\pi e)_x) = -\sum_j \int_{I_j} u_{h,xx}(e-\pi e) + \sum_j \Big(u_{h,x}(e-\pi e)\Big)\Big|_{I_j} = 0.$$

Hence, (11) yields:

$$\|e_x\| \le C\|h(f-u_{h,x})\|. \tag{12}$$

4. Solution: Let $\Gamma_1 := \partial\Omega \setminus \Gamma_2$ where $\Gamma_2 := \{(1, x_2) : 0 \le x_2 \le 1\}$. Define

$$V = \{v : v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_1\}.$$

Multiply the equation by $v \in V$ and integrate over Ω; using Green's formula

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \frac{\partial u}{\partial n} v = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega v,$$

where we have used $\Gamma = \Gamma_1 \cup \Gamma_2$ and the fact that $v = 0$ on Γ_1 and $\frac{\partial u}{\partial n} = 0$ on Γ_2.

Variational formulation:

Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega v, \qquad \forall v \in V.$$

FEM: cG(1):

Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v = \int_\Omega v, \qquad \forall v \in V_h \subset V, \tag{13}$$

where

$$V_h = \{v : v \text{ is piecewise linear and continuous in } \Omega,\ v = 0 \text{ on } \Gamma_1, \text{ on above mesh } \}.$$

A set of bases functions for the finite dimensional space V_h can be written as $\{\varphi_i\}_{i=1}^4$, where

$$\begin{cases} \varphi_i \in V_h, & i = 1,2,3,4 \\ \varphi_i(N_j) = \delta_{ij}, & i,j = 1,2,3,4. \end{cases}$$

Then the equation (2) is equivalent to: Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla \varphi_i = \int_\Omega \varphi_i, \qquad i = 1,2,3,4. \tag{14}$$

Set $U = \sum_{j=1}^{4} \xi_j \varphi_j$. Invoking in the relation (3) above we get

$$\sum_{j=1}^{4} \xi_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i = \int_\Omega \varphi_i, \qquad i = 1, 2, 3, 4.$$

Now let $a_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i$ and $b_i = \int_\Omega \varphi_i$, then we have that

$$A\xi = b, \quad A \text{ is the stiffness matrix } \; b \text{ is the load vector.}$$

Below we compute a_{ij} and b_i

$$b_i = \int_\Omega \varphi_i = \begin{cases} 6 \cdot \frac{1}{3} \cdot \frac{1/4 \cdot 1/2}{2} \cdot 1 = 1/8, & i = 1, 2, 3 \\ 3 \cdot \frac{1}{3} \cdot \frac{1/4 \cdot 1/2}{2} \cdot 1 = 1/16, & i = 4 \end{cases}$$

and

$$a_{ii} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_i = \begin{cases} 2 \cdot (\frac{5}{4} + 1 + \frac{1}{4}) = 5, & i = 1, 2, 3 \\ \frac{5}{4} + 1 + \frac{1}{4} = 5/2, & i = 4 \end{cases}$$

Further

$$a_{i,i+1} = \int_\Omega \nabla\varphi_{i+1} \cdot \nabla\varphi_i = 2 \cdot (-1) = -2 = a_{i+1,i}, \quad i = 1, 2, 3,$$

and

$$a_{ij} = 0, \quad |i - j| > 1.$$

Thus we have

$$A = \begin{pmatrix} 5 & -2 & 0 & 0 \\ -2 & 5 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 5/2 \end{pmatrix} \qquad b = \frac{1}{16} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

5. See the lecture notes.

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