Mathematics Chalmers & GU

MVE455: Partial Differential Equations for Kf3, 2016–08–26, 8:30-12:30

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 4p. Valid bonus poits will be added to the scores.
Breakings from total of 24 points: Exam(20)+Bonus(4). **3**: 10-14p, **4**: 15-19p och **5**: 20p-
For solutions see the couse diary: http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1516/

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1. Let $\pi_1 f$ be the linear interpolant of a twice continuousely differentiable function f. Prove the *optimal* interpolation error estimate (Note coefficient $\frac{1}{8}$):

$$||f - \pi_1 f||_{L_\infty(a,b)} \leq \frac{1}{8}(b-a)^2 ||f''||_{L_\infty(a,b)}.$$

2. Consider a uniform partition $0 = x_0 < x_1 < \ldots < x_N = 1$ of the interval $[0,1]$ and let $\{\varphi_i\}_{i=0}^N$ be a set of piecewise linear continuous basis functions: $\varphi_i(x_j) = 1$ for $i = j$ and $\varphi_i(x_j) = 0$ if $i \neq j$. Given a FEM in form of linear system of equations $(S + C)\xi = \mathbf{b} + \mathbf{d}$, with S and C N-by-N matrices, and ξ, $\mathbf{b}$ and $\mathbf{d}$ vectors of lenght N, where for $i, j = 0, \ldots, N-1$, $S_{ij} = (\varphi_i', \varphi_j')$, $C_{ij} = (\varphi_i, \varphi_j')$, and $\mathbf{b}_i = (\varphi_i, f)$ with f a given function. $d_0 = \alpha$ is the only non-zero element of $\mathbf{d}$. Derive the variational formulation and the strong formulation for the PDE from the above data.

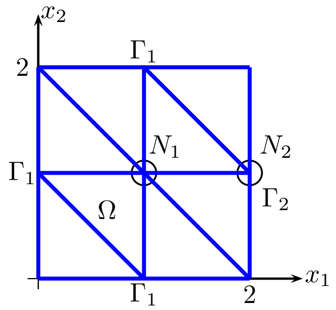
3. Prove an a priori and an a posteriori error estimate, in the H^1-norm: $\|u\|_{H^1} := \|u'\|_{L_2(0,1)}$, for the cG(1) finite element method for the following convection-diffusion-absorption problem

$$-u''(x) + 2xu'(x) + u(x) = f(x), \quad \text{for} \quad x \in (0,1) \qquad \text{and} \quad u(0) = u(1) = 0.$$

4. In the square domain $\Omega := (0,2)^2$, with the boundary $\Gamma := \partial\Omega$, consider the problem of solving

$$(1) \quad \begin{cases} -\frac{\partial^2 u}{\partial x_1^2} - 2\frac{\partial^2 u}{\partial x_2^2} = 1, & \text{in} \quad \Omega = \{x = (x_1, x_2) : 0 < x_1 < 2, \ 0 < x_2 < 2\}, \\ u = 0 \quad \text{on} \quad \Gamma_1 := \Gamma \setminus \Gamma_2, & \frac{\partial u}{\partial x_1} = 0 \quad \text{on} \quad \Gamma_2 = \{x = (x_1, x_2) : \ x_1 = 2, \quad 0 < x_2 < 2\}. \end{cases}$$

Determine the stiffness matrix and load vector if the cG(1) finite element method with piecewise linear approximation is applied to the equation (1) above and on the following triangulation:



5. Formulate and prove the Lax-Milgram theorem in the case of symmetric bilinear form.
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MVE455: Partial Differential Equations for Kf3, 2016–08–26, 8:30-12:30. Solutions.

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1. We have that

$$\pi_1 f(x) = \lambda_a(x)f(a) + \lambda_b(x)f(b) = f(a)\frac{b-x}{b-a} + f(b)\frac{x-a}{b-a}$$

where

$$\lambda_a(x) = \frac{b-x}{b-a},\ \lambda_b(x) = \frac{x-a}{b-a},$$

are the basis functions for linear interpolation in the interval $[a, b]$ with the property:

$$\lambda_a(x) + \lambda_b(x) = 1 \quad \text{and} \quad a\lambda_a(x) + b\lambda_b(x) = x.$$

By Taylor expansions of $f(b)$ and $f(a)$ about x: We have that $\exists, \eta_b \in (x, b)$ and $\eta_a \in (a, x)$:

$$f(b) = f(x) + (b-x)f'(x) + \frac{1}{2}(b-x)^2 f''(\eta_b) \tag{2}$$

$$f(a) = f(x) + (a-x)f'(x) + \frac{1}{2}(a-x)^2 f''(\eta_a) \tag{3}$$

so that

$$\begin{aligned}\pi_1 f(x) &= \lambda_a(x)f(a) + \lambda_b(x)f(b)\\ &= (\lambda_a(x) + \lambda_b(x))f(x) + (\lambda_a(x)(a-x) + \lambda_b(x)(b-x))f'(x)\\ &+ \lambda_a(x)\frac{1}{2}(a-x)^2 f''(\eta_a) + \lambda_b(x)\frac{1}{2}(b-x)^2 f''(\eta_b)\\ &= f(x) + \lambda_a(x)\frac{1}{2}(a-x)^2 f''(\eta_a) + \lambda_b(x)\frac{1}{2}(b-x)^2 f''(\eta_b)\end{aligned} \tag{4}$$

Hence

$$\begin{aligned}|\pi_1 f(x) - f(x)| &\le \frac{1}{2(b-a)}\Big((x-a)(b-x)^2 + (b-x)(a-x)^2\Big)\max_{x\in[a,b]}|f''(x)|\\ &= \frac{1}{2}((x-a)(b-x))\max_{x\in[a,b]}|f''(x)|\end{aligned} \tag{5}$$

Let now $g(x) = (b-x)(x-a)$, then $g'(x) = 0$ yields $x = (a+b)/2$ and $max g(x) = g(\frac{a+b}{2}) = (b-a)^2/4$ which gives the desired result.

2. It is clear that the homogeneous Dirichlet condition is used at $x = 1$ since the basis function φ_N is not present in the matrices and there are no modifivations corresponding the last element of the load vector. Consider now the solution space $V^0 = \{w : \|w\| + \|w'\| < \infty\},\ w(1) = 0\}$ where $\|\cdot\|$ is the usual L_2-norm over I. Then, the variational formulation reads as follows: *find* $u \in V^0$ *s.t.*

$$(v_x, u_x) + (v, u_x) = (v, f) + \alpha v(0), \qquad \forall\ v \in V^0. \tag{6}$$

For the basis functions ginen, $\varphi_0(0) = 1$, which explains the first element of the vector $\mathbf{d}$.

Backward integration by parts, together with the Dirichlet data on v yields

$$(v, f) + \alpha v(0) = (v, -u_{xx} + u_x) + v(0)u_x(0). \tag{7}$$

Thus, the strong formulation (PDE) is: find such that

$$-u_{xx} + u_x = f \quad 0 < x < 1 \qquad u_x(0) = \alpha, \quad u(1) = 0. \tag{8}$$

3. We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + 2xu'v + uv) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{9}$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U \in V_h^0$ such that

$$\int_I (U'v' + 2xU'v + Uv) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{10}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (9)-(10) gives that

$$\int_I (e'v' + 2xe'v + ev) = 0, \quad \forall v \in V_h^0. \tag{11}$$

A posteriori error estimate: We note that using $e(0) = e(1) = 0$, we get

$$\int_I 2xe'e = \int_I x\frac{d}{dx}(e^2) = (xe^2)|_0^1 - \int_I e^2 = -\int_I e^2, \tag{12}$$

so that using variational formulation (9) to replace the terms involving continuous solution u and the finite element method (10) to insert the interpolant $\pi_h e$ of the error we can compute

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I e'e' = \int_I (e'e' + 2xe'e + ee) \\
&= \int_I ((u-U)'e' + 2x(u-U)'e + (u-U)e) = \{v = e \text{ in } (9)\} \\
&= \int_I fe - \int_I (U'e' + 2xU'e + Ue) = \{v = \pi_h e \text{ in } (10)\} \\
&= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + 2xU'(e - \pi_h e) + U(e - \pi_h e)\Big) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),
\end{aligned} \tag{13}$$

where $\mathcal{R}(U) := f - 2xU' - U$, (for approximation with piecewise linears, $U'' \equiv 0$, on each subinterval). Thus (13) implies that

$$\begin{aligned}
\|e\|_{H^1}^2 &\leq \|h\mathcal{R}(U)\| \|h^{-1}(e - \pi_h e)\| \\
&\leq C_i \|h\mathcal{R}(U)\| \|e'\| \leq C_i \|h\mathcal{R}(U)\| \|e\|_{H^1},
\end{aligned}$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \leq C_i \|h\mathcal{R}(U)\|.$$

A priori error estimate: We use (12) and write

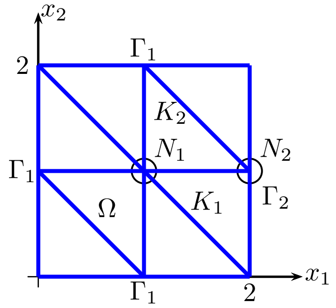
$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I e'e' = \int_I (e'e' + 2xe'e + ee) \\
&= \int_I \Big(e'(u-U)' + 2xe'(u-U) + e(u-U)\Big) = \{v = U - \pi_h u \text{ in}(11)\} \\
&= \int_I \Big(e'(u - \pi_h u)' + 2xe'(u - \pi_h u) + e(u - \pi_h u)\Big) \\
&\leq \|(u - \pi_h u)'\| \|e'\| + 2\|u - \pi_h u\| \|e'\| + \|u - \pi_h u\| \|e\| \\
&\leq \{\|(u - \pi_h u)'\| + 3\|u - \pi_h u\|\} \|e\|_{H^1} \\
&\leq C_i \{\|hu''\| + \|h^2 u''\|\} \|e\|_{H^1},
\end{aligned}$$

where in the last step we used Poincare inequality. This gives that

$$\|e\|_{H^1} \le C_i\{\|hu''\| + \|h^2u''\|\},$$

which is the a priori error estimate.

4. Recall that the mesh size is $h = 1$. Further, the first triangle (the triangle with nodes at $(0,0)$, $(1,0)$ and $(0,1)$) is not in the support of the test function of N_1, whereas the last triangle (the triangle with nodes at $(4,4)$, $(2,4)$ and $(4,2)$) is in the support of the test function for N_2!. Thus, the nodal bases functions φ_1 and φ_2 share the two triangles K_1 and K_2, see figure below. We



$(h = 1)$

define the test function space

$$V = \{v : v \in H^1(\Omega), \quad v = 0 \quad \text{on } \Gamma_1\}. \tag{14}$$

We multiply the differential equation in (1) by $v \in V$ and integrate over Ω. Using Green's formula, the boundary data ($v = 0$ on Γ_1 and $\frac{\partial u}{\partial x_1} = 0$ on Γ_2), and the standard notation $\vec{n} = (n_1, n_2)$ for the outward unit normal on $\Gamma_1 \cup \Gamma_2$, we end up with

$$\begin{aligned}\int_\Omega \Big(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + 2\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\Big)\,dx_1\,dx_2 - \int_\Gamma \Big(\frac{\partial u}{\partial x_1}vn_1 + 2\frac{\partial u}{\partial x_2}vn_2\Big)\,ds \\ = \int_\Omega \Big(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + 2\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\Big)\,dx_1\,dx_2 = \int_\Omega v\,dx_1\,dx_2.\end{aligned} \tag{15}$$

Hence, we have the *variational formulation*: Find $u \in V$ such that

$$\int_\Omega \Big(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + 2\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\Big)\,dx_1\,dx_2 = \int_\Omega v\,dx_1\,dx_2, \qquad \forall v \in V, \tag{16}$$

and the corresponding *finite element method*: Find $U \in V_h$ such that

$$\int_\Omega \Big(\frac{\partial U}{\partial x_1}\frac{\partial v}{\partial x_1} + 2\frac{\partial U}{\partial x_2}\frac{\partial v}{\partial x_2}\Big)\,dx_1\,dx_2 = \int_\Omega v\,dx_1\,dx_2, \qquad \forall v \in V_h \ (\subset V), \tag{17}$$

where

$$V_h := \{v : v \text{ is piecewise linear and continuous on the partition of } \Omega,\ v = 0 \ \text{ on } \Gamma_1\}. \tag{18}$$

A basis for V_h consists of $\{\varphi_i\}_{i=1}^2$, where

$$\begin{cases} \varphi_i \in V_h, & i = 1,2 \\ \varphi_i(N_j) = \delta_{ij}, & i,j = 1,2. \end{cases}$$

Then, (18) is equivalent to: find $U \in V_h$ such that

$$\int_\Omega \Big(\frac{\partial U}{\partial x_1}\frac{\partial \varphi_i}{\partial x_1} + 2\frac{\partial U}{\partial x_2}\frac{\partial \varphi_i}{\partial x_2}\Big)\,dx_1\,dx_2 = \int_\Omega \varphi_i\,dx_1\,dx_2, \qquad i = 1,2. \tag{19}$$

Now, we make the ansatz: $U = \sum_{j=1}^2 \xi_j\varphi_j$. Inserting in (19) gives

$$\sum_{j=1}^2 \xi_j \Big\{\int_\Omega \Big(\frac{\partial \varphi_j}{\partial x_1}\frac{\partial \varphi_i}{\partial x_1} + 2\frac{\varphi_j}{\partial x_2}\frac{\partial \varphi_i}{\partial x_2}\Big)\,dx_1\,dx_2\Big\} = \int_\Omega \varphi_i\,dx_1\,dx_2, \qquad i = 1,2, \tag{20}$$

which can be written in the equivalent form as

$$A\xi = b, \quad a_{ij} = \int_\Omega \Big(\frac{\partial \varphi_j}{\partial x_1}\frac{\partial \varphi_i}{\partial x_1} + 2\frac{\varphi_j}{\partial x_2}\frac{\partial \varphi_i}{\partial x_2}\Big)\, dx_1\, dx_2, \quad b_i = \int_\Omega \varphi_i \, dx_1\, dx_2. \tag{21}$$

We can easily compute that

$$\begin{aligned} a_{11} &= \textstyle\int_\Omega \Big(\frac{\partial \varphi_1}{\partial x_1}\frac{\partial \varphi_1}{\partial x_1} + 2\frac{\partial \varphi_1}{\partial x_2}\frac{\partial \varphi_1}{\partial x_2}\Big)\, dx_1\, dx_2 = 6, & b_1 &= \textstyle\int_\Omega \varphi_1 \, dx_1\, dx_2 = 1 \\ a_{22} &= \textstyle\int_\Omega \Big(\frac{\partial \varphi_2}{\partial x_2}\frac{\partial \varphi_2}{\partial x_2} + 2\frac{\partial \varphi_2}{\partial x_2}\frac{\partial \varphi_2}{\partial x_2}\Big)\, dx_1\, dx_2 = \frac{a_{11}}{2} = 3, & b_1 &= \textstyle\int_\Omega \varphi_2 \, dx_1\, dx_2 = \frac{b_1}{2} = \frac{1}{2}, \end{aligned} \tag{22}$$

and

$$\begin{aligned} a_{12} = a_{21} &= \int_\Omega \Big(\frac{\partial \varphi_2}{\partial x_1}\frac{\partial \varphi_1}{\partial x_2} + 2\frac{\partial \varphi_2}{\partial x_2}\frac{\partial \varphi_1}{\partial x_2}\Big)\, dx_1\, dx_2 = \{ \text{ see Fig. } \} = \int_{K_1} \dots + \int_{K_2} \dots \\ &= \Big(\frac{1}{h}(-\frac{1}{h}) + 2\cdot\frac{1}{h}\cdot 0)\cdot\frac{h^2}{2}\Big) + \Big(\frac{1}{h}(-\frac{1}{h}) + 2\cdot 0\cdot(-\frac{1}{h})\cdot\frac{h^2}{2}\Big) = -\frac{1}{2} - \frac{1}{2} = -1. \end{aligned} \tag{23}$$

So, in summary we have that the stiffness matrix A, and the load vector b are given by

$$A = \begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}.$$

5. See the lecture notes.

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