Mathematics Chalmers & GU

MVE455: Partial Differential Equations for Kf3, 2017–03–13, 14:00-18:00

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 4p. Valid bonus poits will be added to the scores.
Breakings from total of 24 points: Exam(20)+Bonus(4). **3**: 10-14p, **4**: 15-19p och **5**: 20p-
For solutions see couse diary: http://www.math.chalmers.se/Math/Grundutb/CTH/mve455/1617/

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1. Visa att för en lösning till vågekvationen, med homogena Dirichlet randvärden och $f = 0$, konserveras

$$\|\nabla \dot{u}\|^2 + \|\Delta u\|^2.$$

Ledning: Multiplicera ekv med $\Delta \dot{u}$ och integrera över Ω.

2. Prove an a priori and an a posteriori error estimate (in the energy norm: $\|u\|_E^2 := \|u'\|^2 + \|u\|^2$) for the cG(1) finite element method for the problem

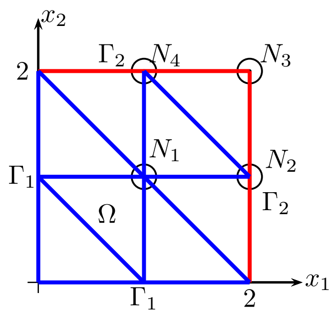
$$-u'' + \alpha u' + u = f, \qquad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

where $\alpha \geq 0$. For which value of α is the a priori error estimate optimal?

3. In the square domain $\Omega := (0,2)^2$, with the boundary $\Gamma := \partial\Omega$, consider the problem of solving

$$(1) \qquad \begin{cases} -\Delta u = 1, & \text{in } \; \Omega = \{x = (x_1, x_2) : 0 < x_1 < 2, \; 0 < x_2 < 2\}, \\ u = 0 & \text{on } \; \Gamma_1 := \Gamma \setminus \Gamma_2, \\ \frac{\partial u}{\partial x_1}|_{x_1=2} = 0 \;\; \frac{\partial u}{\partial x_2}|_{x_2=2} = 0, & \Gamma_2 := \{x_1 = 2\} \cup \{x_2 = 2\}. \end{cases}$$

Determine the stiffness matrix and load vector if the cG(1) finite element method with piecewise linear approximation is applied to the equation (1) above and on the following triangulation:



4. Let Ω be a bounded domain in $\mathbb{R}^d$, $d = 2, 3$. Consider the boundary value problem

$$-\Delta u = 0, \quad \text{in } \; \Omega \qquad u + \partial u/\partial n = g, \; \text{ on } \; \Gamma = \partial\Omega.$$

a) Prove the L_2 stability estimate

$$||\nabla u||_{L_2(\Omega)} + \frac{1}{2}||u||_{L_2(\Gamma)} \leq \frac{1}{2}||g||_{L_2(\Gamma)}.$$

b) Verify the conditions on Riesz/Lax-Milgram theorems for this problem.

5. Consider the initial value problem $\dot{u}(t) + a(t)u(t) = f(t)$, for $0 < t < T$, and $u(0) = u_0$. Prove the stability estimates

$$|u(t)| \leq e^{-\alpha t}|u_0| + \frac{1}{\alpha}(1 - e^{-\alpha t}) \max_{0 \leq s \leq t} |f(s)|, \quad a(t) \geq \alpha > 0, \quad \text{and}$$

$$|u(t)| \leq |u_0| + \int_0^t |f(s)| \, ds, \quad a(t) \geq 0.$$

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MVE455: Partial Differential Equations for Kf3, 2017–03–13, 14:00-18:00. Solutions.

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1. Multiply the equation by $\Delta\dot{u}$ and integrate over Ω to get

$$\begin{aligned}(\ddot{u}\,,\Delta\dot{u})-(\Delta u,\,\Delta\dot{u})&=-(\nabla\ddot{u}\,,\nabla\dot{u})-(\Delta u,\,\Delta\dot{u})\\ &=-\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega|\nabla\dot{u}|^2\,dx+\int_\Omega|\Delta u|^2\,dx\Big)=0,\end{aligned}$$

where in the first equality we used Green's formula and the vanishing boundary data. Relabeling t to s and integrating over $(0,t)$ we get the desired result.

2. <u>The Variational formulation:</u>
Multiply the equation by $v\in V$, integrate by parts over $(0,1)$ and use the boundary conditions to obtain

$$\text{(2)}\qquad \text{Find } u\in V:\ \int_0^1 u'v'\,dx+\int_0^1\alpha u'v\,dx+\int_0^1 uv\,dx=\int_0^1 fv\,dx,\quad \forall v\in V.$$

<u>cG(1):</u>

$$\text{(3)}\qquad \text{Find } U\in V_h:\ \int_0^1 U'v'\,dx+\int_0^1\alpha U'v\,dx+\int_0^1 Uv\,dx=\int_0^1 fv\,dx,\quad \forall v\in V_h.$$

From (1)-(2), we find <u>The Galerkin orthogonality:</u>

$$\text{(4)}\qquad \int_0^1\Big((u-U)'v'+\alpha(u-U)'v+(u-U)v\Big)\,dx=0,\quad \forall v\in V_h.$$

We define the inner product $(\cdot,\cdot)_E$ associated to the energy norm to be

$$(v,w)_E=\int_0^1(v'w'+vw)\,dx,\qquad \forall v,w\in V.$$

Note that

$$\text{(5)}\qquad \int_0^1 e'e\,dx=\frac{1}{2}\int_0^1\frac{d}{dx}\Big(e^2\Big)\,dx=\frac{1}{2}[e^2]_0^1=0.$$

Thus using (5) we have

$$\text{(6)}\qquad ||e||_E^2=\int_0^1(e'e'+ee)\,dx=\int_0^1(e'e'+\alpha e'e+ee)\,dx.$$

We split the second factor e as $e=u-U=u-v+v-U$, with $v\in V_h$ and write

$$\begin{aligned}||e||_E^2&=\int_0^1\Big(e'(u-U)'+\alpha e'(u-U)+e(u-U)\Big)\,dx=\Big\{v\in V_h\Big\}\\ &=\int_0^1\Big(e'(u-v)'+\alpha e'(u-v)+e(u-v)\Big)\,dx\\ &+\int_0^1\Big(e'(v-U)'+\alpha e'(v-U)+e(v-U)\Big)\,dx\\ &=\int_0^1\Big(e'(u-v)'+\alpha e'(u-v)+e(u-v)\Big)\,dx,\end{aligned}$$

where, in the last step, we have used the Galerkin orthogonality to eliminate terms involving U. Now we can write

$$\begin{aligned} ||e||_E^2 &= \int_0^1 \Big(e'(u-v)' + e(u-v) + \alpha e'(u-v) \Big)\, dx \\ &\le ||e||_E \cdot ||u-v||_E + \alpha ||e'||_{L_2} ||u-v||_{L_2} \\ &\le ||e||_E \Big(||u-v||_E + \alpha ||u-v||_{L_2} \Big) \le ||e||_E ||u-v||_E (1+\alpha), \end{aligned}$$

and derive the a priori error estimate:

$$||e||_E \le ||u-v||_E (1+\alpha), \quad \forall v \in V_h.$$

To obtain a posteriori error estimates the idea is to eliminate u-terms, by using the differential equation, and replacing their contributions by the data f. Then this f combined with the remaining U-terms would yield to the residual error:

A posteriori error estimate:

$$\begin{aligned} ||e||_E^2 &= \int_0^1 (e'e' + ee)\, dx = \int_0^1 (e'e' + \alpha e'e + ee)\, dx \\ &= \int_0^1 (u'e' + \alpha u'e + ue)\, dx - \int_0^1 (U'e' + \alpha U'e + Ue)\, dx. \end{aligned} \tag{7}$$

Now using the variational formulation (2) we have that

$$\int_0^1 (u'e' + \alpha u'e + ue)\, dx = \int_0^1 fe\, dx.$$

Inserting in (7) and using (3) with $v = \Pi_k e$ we get

$$\begin{aligned} ||e||_E^2 &= \int_0^1 fe\, dx - \int_0^1 (U'e' + \alpha U'e + Ue)\, dx \\ &+ \int_0^1 (U'\Pi_h e' + \alpha U'\Pi_h e + U\Pi_h e)\, dx - \int_0^1 f\Pi_h e\, dx. \end{aligned} \tag{8}$$

Thus

$$\begin{aligned} ||e||_E^2 &= \int_0^1 f(e-\Pi_h e)\, dx - \int_0^1 \Big(U'(e-\Pi_h e)' + \alpha U'(e-\Pi_h e) + U(e-\Pi_h e) \Big)\, dx \\ &= \int_0^1 f(e-\Pi_h e)\, dx - \int_0^1 (\alpha U' + U)(e-\Pi_h e)\, dx - \sum_{j=1}^{M+1} \int_{I_j} U'(e-\Pi_h e)'\, dx \\ &= \{\text{partial integration}\} \\ &= \int_0^1 f(e-\Pi_h e)\, dx - \int_0^1 (\alpha U' + U)(e-\Pi_h e)\, dx + \sum_{j=1}^{M+1} \int_{I_j} U''(e-\Pi_h e)\, dx \\ &= \int_0^1 (f + U'' - \alpha U' - U)(e-\Pi_h e)\, dx = \int_0^1 R(U)(e-\Pi_h e)\, dx \\ &= \int_0^1 hR(U)h^{-1}(e-\Pi_h e)\, dx \le ||hR(U)||_{L_2} ||h^{-1}(e-\Pi_h e)||_{L_2} \\ &\le C_i ||hR(U)||_{L_2} \cdot ||e'||_{L_2} \le ||hR(U)||_{L_2} \cdot ||e||_E. \end{aligned}$$

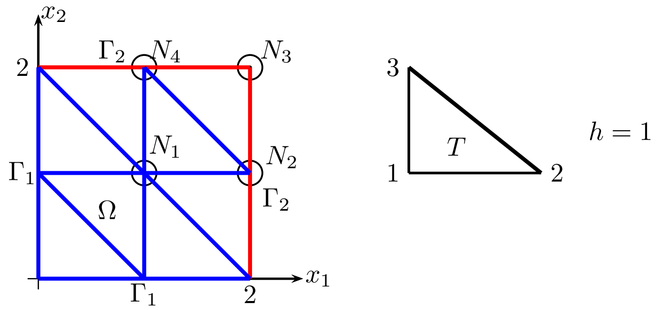
This gives the a posteriori error estimate:

$$||e||_E \le C_i ||hR(U)||_{L_2},$$

with $R(U) = f + U'' - \alpha U' - U = f - \alpha U' - U$ on (x_{i-1}, x_i), $i = 1, \dots, M+1$.

The a priori error estimate is optimal for $\alpha = 0$.

3. Recall that the mesh size is $h = 1$. Further, the first triangle (the triangle with nodes at $(0,0)$, $(1,0)$ and $(0,1)$) is not in the support of the test function of N_1, whereas the last triangle (the triangle with nodes at $(4,4)$, $(2,4)$ and $(4,2)$) is in the support of the test function for all other 3 nodes: $N_2, N3, N4!$. Thus, the nodal basis function φ_1 shares 2 triangles with φ_2 and 2 triangles with φ_4. Likewise, φ_2 and φ_3 are sharing 1 triangle, φ_2 and φ_4, 2 triangle, and finally φ_3 and φ_4 1 triangle. see figure below. We define the test function space



$$V = \{v : v \in H^1(\Omega), \quad v = 0 \ \text{ on } \Gamma_1\}. \tag{9}$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) = (1, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$\begin{aligned} -(\Delta u, v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds \\ &= (\nabla u, \nabla v) - \int_{\Gamma_1} (n \cdot \nabla u) v \, ds - \int_{\Gamma_2} (n \cdot \nabla u) v \, ds \\ &= (\nabla u, \nabla v), \qquad \forall v \in V. \end{aligned}$$

Thus the variational formulation reads as

$$(\nabla u, \nabla v) = (1, v), \qquad \forall v \in V.$$

The corresponding cG(1) finite element is: Find $u_h \in V_h^0$ such that

$$(\nabla u_h, \nabla v) = (1, v), \qquad \forall v \in V_h^0,$$

where

$$V_h^0 := \{v : v \text{ is continuous, piecewise linear on the above partition and } \ v = 0, \ \text{ on } \Gamma_1\}.$$

Making the "Ansatz" $U(x) = \sum_{j=1}^4 \xi_j \varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^{4} \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx \Big) = \int_\Omega \varphi_i \, dx, \quad i = 1, 2, 3,$$

or, in matrix form,

$$S\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix and $F_i = (1, \varphi_i)$ is the load vector. We first compute the stiffness matrix for the reference triangle T. The local basis functions are

$$\begin{aligned} \phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1, x_2) &= -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\, dx = \frac{2}{h^2}|T| = 1.$$

and computing all other s_{ij}'s, we end up with

$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrix S from the local matrix s:

$$\begin{aligned} &S_{11} = 2s_{11} + 4s_{22} = 4, && S_{12} = S_{14} = 2s_{12} = -1, \quad S_{13} = 0, \\ &S_{22} = S_{44} = s_{11} + 2s_{22} = 1 + 1 = 2, && S_{23} = S_{34} = s_{12} = -1/2, \quad S_{24} = 2s_{23} = 0 \\ &S_{33} = 2s_{11} = 2, && S_{44} = 2s_{22} = 1 \end{aligned}$$

The remaining matrix elements are obtained by symmetry $S_{ij} = S_{ji}$. Hence,

$$S = \varepsilon\begin{bmatrix} 4 & -1 & 0 & -1 \\ -1 & 2 & -1/2 & 0 \\ 0 & -1/2 & 1 & -1/2 \\ -1 & 0 & -1/2 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} (1,\varphi_1) \\ (1,\varphi_2) \\ (1,\varphi_3) \\ (1,\varphi_4) \end{bmatrix} = \begin{bmatrix} 6\cdot\frac{1}{3}\cdot\frac{1}{2} = 1 \\ 3\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{2} \\ 1\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{6} \\ 3\cdot\frac{1}{3}\cdot\frac{1}{2} = \frac{1}{2} \end{bmatrix}.$$

4. a) Using Greens formula we have that

$$\int_\Omega |\nabla u|^2 = \int_\Omega \nabla u\cdot\nabla u = -\int_\Omega (\Delta u)u + \int_{\partial\Omega}\frac{\partial u}{\partial n}u = \int_{\partial\Omega}(g-u)u.$$

In other words

$$||\nabla u||^2_{L_2(\Omega)} + ||u||^2_{L_2(\Gamma)} = \int_{\partial\Omega} gu \le ||g||^2_{L_2(\Gamma)}||u||^2_{L_2(\Gamma)} \le \frac{1}{2}||g||^2_{L_2(\Gamma)} + \frac{1}{2}||u||^2_{L_2(\Gamma)},$$

which gives the desired estimate. To show the Riesz/Lax-Milgram conditions we introduce the notation

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v + \int_{\partial\Omega} uv, \quad \text{and} \quad L(v) = \int_{\partial\Omega} gv.$$

Then $a(u,v)$ is a scalar product with the corresponding norm $||v||_a = a(v,v)^{1/2}$. For instance we have that $||v||_a = 0$, only if $v = 0$:

$$0 = ||v||_a^2 = a(u,v) = \int_\Omega |\nabla v|^2 + \int_{\partial\Omega} v^2 \ge \alpha\int_\Omega v^2, \qquad \text{for some } \alpha > 0 \Rightarrow v = 0.$$

Further $L(v)$ is bounded in this norm, e.g. if $||g||_{\partial\Omega} < \infty$, then

$$|L(v)| \le ||g||_{\partial\Omega}||v||_{\partial\Omega} \le ||g||_{\partial\Omega}||v||_a.$$

We can also apply Riesz theorem in the sense that there existes u such that

$$a(u,v) = L(v), \qquad \forall v,$$

and u is uniquely determined by

$$||u||_a = ||g||_{\partial\Omega}.$$

Moreover since

$$a(u,v) = -\int_\Omega \Delta uv + \int_{\partial\Omega}(\frac{\partial u}{\partial n} + u)v,$$

we have that

$$\Delta u = 0, \quad \text{in } \Omega \quad \frac{\partial u}{\partial n} + u = g \quad \text{on } \Gamma.$$

5. See the lecture notes.

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