Mathematics Chalmers & GU

MVE455: Partial Differential Equations, 2017–06–08, 14:00-18:00

Telephone: Felix Held: ankn 5325
Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 4p. Valid bonus poits will be added to the scores.
Breakings from total of 24 points: Exam(20)+Bonus(4). **3**: 10-14p, **4**: 15-19p och **5**: 20p-
For solutions see couse diary: http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1617/

---

1. Consider the Dirichlet problem: $-\nabla\cdot(a(x)\nabla u) = f(x), \quad x\in\Omega\subset\mathbb{R}^2, \qquad u=0, \text{ for } x\in\partial\Omega.$ Assume that c_0 and c_1 are constants such that $c_0 \le a(x) \le c_1$, $\forall x\in\Omega$ and let $U=\sum_{j=1}^N \alpha_j w_j(x)$ be a Galerkin approximation of u in a finite dimensional subspace M of $H_0^1(\Omega)$. Prove the a priori error estimate below and specify C as best you can

$$||u-U||_{H_0^1(\Omega)} \le C \inf_{\chi\in M} ||u-\chi||_{H_0^1(\Omega)}.$$

2. Let n be the outward unit normal to $\Gamma=\partial\Omega$. Consider the Neumann problem

$$-\Delta u + u = f, \quad x\in\Omega\subset\mathbb{R}^d, \qquad n\cdot\nabla u = g, \quad \text{on} \quad \Gamma:=\partial\Omega,$$

(a) Show the following stability estimate: for some constant C,

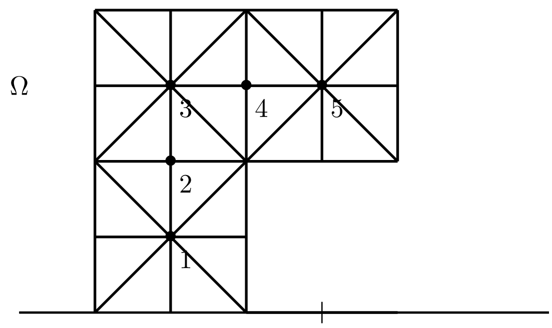
$$||\nabla u||^2_{L_2(\Omega)} + ||u||^2_{L_2(\Omega)} \le C[||f||^2_{L_2(\Omega)} + ||g||^2_{L_2(\Gamma)}].$$

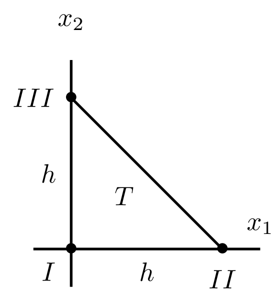
(b) Formulate a finite element method for the $1D$-case and derive the resulting system of equations for $\Omega=[0,1]$, $f(x)=1$, $g(0)=3$ and $g(1)=0$.

3. Formulate the cG(1) Galerkin finite element method for the Dirichlet boundary value problem

$$-\Delta u + u = f, \quad x\in\Omega; \qquad u=0, \quad x\in\partial\Omega,$$

on a smooth domain Ω. Write the matrices for the resulting equation system using the partition below (see fig.) with the nodes at N_1, N_2, N_3, N_4 and N_5 and a uniform mesh size h.





4. Let p be a positive constant. Prove an a priori and an a posteriori error estimate (in the H^1-norm: $\|e\|^2_{H^1} = \|e'\|^2 + \|e\|^2$) for the standard cG(1) finite element method for problem

$$-u'' + pxu' + (1+\frac{p}{2})u = f, \quad \text{in } (0,1), \qquad u(0)=u(1)=0.$$

5. Prove that there exists a unique solution to the abstract minimization problem.
MA

void!

MVE455: Partial Differential Equations, 2017–06–08, 14:00-18:00. Solutions.

---

1. Recall the continuous and approximate weak formulations:

$$(a\nabla u, \nabla v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{1}$$

and

$$(a\nabla U, \nabla v) = (f, v), \qquad \forall v \in M, \tag{2}$$

respectively, so that

$$(a\nabla (u-U), \nabla v) = 0, \qquad \forall v \in M. \tag{3}$$

We may write

$$u - U = u - \chi + \chi - U,$$

where χ is an arbitrary element of M, it follows that

$$\begin{aligned}(a\nabla(u-U), \nabla(u-U)) =& (a\nabla(u-U), \nabla(u-\chi)) \\ & \le ||a\nabla(u-U)|| \cdot ||u-\chi||_{H_0^1(\Omega)} \\ & \le c_1 ||u-U||_{H_0^1(\Omega)} ||u-\chi||_{H_0^1(\Omega)},\end{aligned} \tag{4}$$

on using (3), Schwarz's inequality and the boundedness of a. Also, from the boundedness condition on a, we have that

$$(a\nabla(u-U), \nabla(u-U)) \ge c_0 ||u-U||^2_{H_0^1(\Omega)}. \tag{5}$$

Combining (4) and (5) gives

$$||u-U||_{H_0^1(\Omega)} \le \frac{c_1}{c_0} ||u-\chi||_{H_0^1(\Omega)}.$$

Since χ is an arbitrary element of M, we obtain the result.

2. a) Multiplying the equation by u and performing partial integration we get

$$\int_\Omega \nabla u \cdot \nabla u + uu - \int_\Gamma n \cdot \nabla u u = \int_\Omega fu,$$

i.e.,

$$||\nabla u||^2 + ||u||^2 = \int_\Omega fu + \int_\Gamma gu \le ||f|| ||u|| + ||g||_\Gamma C_\Omega (||\nabla u|| + ||u||) \tag{6}$$

where $||\cdot|| = ||\cdot||_{L_2(\Omega)}$ and we have used the inequality $||u|| \le C_\Omega(||\nabla u|| + ||u||)$. Further using the inequality $ab \le a^2 + b^2/4$ we have

$$||\nabla u||^2 + ||u||^2 \le ||f||^2 + \frac{1}{4}||u||^2 + C||g||^2_\Gamma + \frac{1}{4}||\nabla u||^2 + \frac{1}{4}||u||^2$$

which gives the desired inequality.

b) Consider the variational formulation

$$\int_\Omega \nabla u \cdot \nabla v + uv = \int_\Omega fv + \int_\Gamma gv, \tag{7}$$

set $U(x) = \sum U_j \psi_j(x)$ and $v = \psi_i$ in (7) to obtain

$$\sum_{j=1}^N U_j \int_\Omega \nabla \psi_j \cdot \nabla \psi_i + \psi_j \psi_i = \int_\Omega f\psi_i + \int_\Gamma g\psi_i, \quad i = 1, \dots, N.$$

This gives $AU = b$ where $U = (U_1, \dots, U_N)^T$, $b = (b_i)$ with the elements

$$b_i = h, \; i = 2, \dots, N-1, \quad b(N) = h/2, \quad b(1) = h/2 + 3,$$

and $A = (a_{ij})$ with the elements

$$a_{ij} = \begin{cases} -1/h + h/6, & \text{for } i = j+1 \quad \text{and } i = j-1 \\ 2/h + 2h/3, & \text{for } i = j \qquad\quad \text{and } i = 2, \dots, N-1 \\ 0, & \text{else.} \end{cases}$$

3. Let V be the linear function space defined by

$$V := \{v : v \ \text{ is continuous in } \Omega, \ \ v = 0, \ \text{ on } \partial\Omega\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$-(\Delta u, \nabla v) = (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds = (\nabla u, \nabla v), \qquad \forall v \in V.$$

Thus, since $v = 0$ on $\partial\Omega$, the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let now V_h be the usual finite element space consisting of continuous piecewise linear functions, on the given partition (triangulation), satisfying the boundary condition $v = 0$ on $\partial\Omega$:

$$V_h := \{v : v \ \text{ is continuous piecewise linear in } \Omega, \ \ v = 0, \ \text{ on } \partial\Omega\}.$$

The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^{5} \xi_i \varphi_j(x)$, where φ_j are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^{5} \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx + \int_\Omega \varphi_i \varphi_j \, dx \Big) = \int_\Omega f \varphi_i \, dx, \quad i = 1, 2, 3, 4, 5$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1, x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1, x_2) = -\frac{1}{h} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\phi_2(x_1, x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1, x_2) = \frac{1}{h} \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\phi_3(x_1, x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1, x_2) = \frac{1}{h} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 \, dx_1 dx_2 = \frac{h^2}{12},$$

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2} |T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = \frac{|T|}{3} \sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6} \Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$\begin{aligned}
&M_{11} = M_{33} = M_{55} = 8m_{22} = 8 \times \frac{h^2}{12}, && S_{11} = S_{33} = S_{55} = 8s_{22} = 8 \times \frac{1}{2}8 = 4,\\
&M_{22} = M_{44} = 4m_{11} = 4 \times \frac{h^2}{12} = \frac{h^2}{3}, && S_{22} = S_{44} = 4s_{11} = 4 \times 1 = 4,\\
&M_{12} = M_{23} = M_{34} = M_{45} = 2m_{12} = \frac{1}{12}h^2, && S_{12} = S_{23} = S_{34} = S_{45} = 2s_{12} = -1,\\
&M_{13} = M_{14} = M_{15} = M_{24} = M_{25} = M_{35} = 0, && S_{13} = S_{14} = S_{15} = S_{24} = S_{25} = S_{35} = 0,
\end{aligned}$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 \\ 1 & 1 & 8 & 1 & 0 \\ 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & 1 & 8 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & 0 \\ 0 & -1 & 4 & -1 & 0 \\ 0 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & -1 & 4 \end{bmatrix}.$$

4. We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I \Big(u'v' + pxu'v + (1+\frac{p}{2})uv\Big) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{8}$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U \in V_h^0$ such that

$$\int_I \Big(U'v' + pxU'v + (1+\frac{p}{2})Uv\Big) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{9}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (1)-(2) gives that

$$\int_I \Big(e'v' + pxe'v + (1+\frac{p}{2})ev\Big) = 0, \quad \forall v \in V_h^0. \tag{10}$$

A posteriori error estimate: We note that using $e(0) = e(1) = 0$, we get

$$\int_I pxe'e = \frac{p}{2}\int_I x\frac{d}{dx}(e^2) = \frac{p}{2}(xe^2)|_0^1 - \frac{p}{2}\int_I e^2 = -\frac{p}{2}\int_I e^2, \tag{11}$$

so that

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I \Big(e'e' + pxe'e + (1+\frac{p}{2})ee\Big)\\
&= \int_I \Big((u-U)'e' + px(u-U)'e + (1+\frac{p}{2})(u-U)e\Big) = \{v = e \text{ in } (8)\}\\
&= \int_I fe - \int_I \Big(U'e' + pxU'e + (1+\frac{p}{2})Ue\Big) = \{v = \pi_h e \text{ in } (9)\}\\
&= \int_I f(e-\pi_h e) - \int_I \Big(U'(e-\pi_h e)' + pxU'(e-\pi_h e) + (1+\frac{p}{2})U(e-\pi_h e)\Big)\\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e-\pi_h e),
\end{aligned} \tag{12}$$

where $\mathcal{R}(U) := f + U'' - pxU' - (1+\frac{p}{2})U = f - pxU' - (1+\frac{p}{2})U$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (12) implies that

$$\begin{aligned}\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\|\|h^{-1}(e-\pi_h e)\|\\ &\le C_i\|h\mathcal{R}(U)\|\|e'\| \le C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},\end{aligned}$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

A priori error estimate: We use (11) and write

$$\begin{aligned}\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I (e'e' + pxe'e + (1+\frac{p}{2})ee)\\ &= \int_I \Big(e'(u-U)' + pxe'(u-U) + (1+\frac{p}{2})e(u-U)\Big) = \{v = U - \pi_h u \text{ in } (10)\}\\ &= \int_I \Big(e'(u-\pi_h u)' + pxe'(u-\pi_h u) + (1+\frac{p}{2})e(u-\pi_h u)\Big)\\ &\le \|(u-\pi_h u)'\|\|e'\| + p\|u-\pi_h u\|\|e'\| + (1+\frac{p}{2})\|u-\pi_h u\|\|e\|\\ &\le \{\|(u-\pi_h u)'\| + (1+p)\|u-\pi_h u\|\}\|e\|_{H^1}\\ &\le C_i\{\|hu''\| + (1+p)\|h^2u''\|\}\|e\|_{H^1},\end{aligned}$$

this gives that

$$\|e\|_{H^1} \le C_i\{\|hu''\| + (1+p)\|h^2u''\|\},$$

which is the a priori error estimate.

5. See the lecture notes.

MA