Mathematics Chalmers & GU

MVE455: Partial Differential Equations, 2018–03–12, 14:00-18:00

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 10-15p, **4**: 16-20p och **5**: 21p-
For solutions and information about gradings see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/mve455/1718/coursediary

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1. $\pi_1 f$ is the linear interpolant of a twice continuously differentiable function f on I. Use Lagrange interpolation error formula and prove that

$$||f - \pi_1 f||_{L_1(I)} \le (b-a)^2 ||f''||_{L_1(I)}, \qquad I = (a, b).$$

2. Consider the heat equation for $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega = \Gamma$,

$$(1) \qquad \begin{cases} u_t(x,t) - \Delta u(x,t) = 0, & \text{for } x \in \Omega, \ \ 0 < t \le T, \\ u(x,t) = 0, & \text{for } x \in \Gamma, \ \ 0 < t \le T, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega, \end{cases}$$

Let $\tilde{u}$ be the solution of (1) with a modified initial data $\tilde{u}_0(x) = u_0(x) + \varepsilon(x)$.

a) Show that $w := \tilde{u} - u$ solves (1) with data $w_0(x) = \varepsilon(x)$ (and $f = 0$). Derive stability estimates for w, i.e. estimate $\|w(T)\|^2 + 2\int_0^T \|\nabla w\|^2 dt$ by $\|w_0\|^2$.

b) Use stability estimate for w to prove that the solution of (1) is unique.

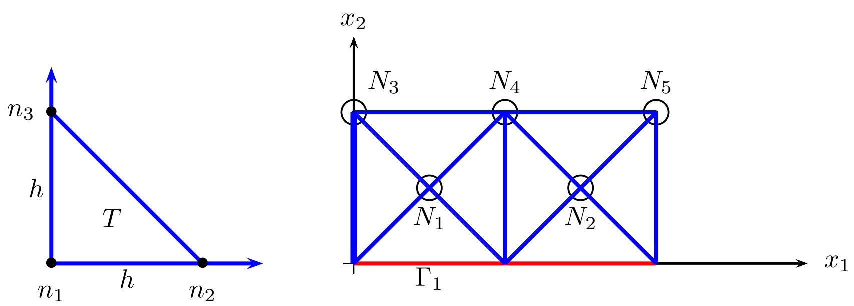
3. Prove an a posteriori error estimate for the cG(1) finite element method for

$$-u''(x) + u'(x) = f, \quad 0 < x < 1; \qquad u(0) = u(1) = 0.$$

4. Formulate the cG(1) piecewise continuous Galerkin method in Ω (see fig. below) for the problem

$$-\Delta u(x) = 1, \quad \text{for } x \in \Omega, \qquad u(x) = 0, \quad \text{for } x \in \Gamma_1, \quad \text{and } \nabla u(x) \cdot \mathbf{n}(x) = 1 \quad \text{for } x \in \partial\Omega \setminus \Gamma_1,$$

where $\mathbf{n}(x)$ is the outward unit normal to $\partial\Omega$ at $x \in \partial\Omega$. Determine the coefficient matrix and load vector for the resulting equation system using the mesh as in the fig. with nodes at $N_1, \ N_2, \ N_3, \ N_4$ and N_5 and a uniform mesh size h. Hint: First compute the matrix for the standard element T.



5. Consider the heat equation (1) in problem 2 above. Prove the following stability estimates

$$\max\left(\|u\|(t)^2 + 2\int_0^t \|\nabla u\|(s)^2 \, ds \right) \le \|u_0\|^2.$$

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1. Let $\lambda_0(x) = \frac{\xi_1 - x}{\xi_1 - x_0}$ and $\lambda_1(x) = \frac{x - \xi_0}{\xi_1 - x_0}$ be two linear base functions. Then by the integral form of the Taylor formula we may write

$$\begin{cases} f(\xi_0) & = f(x) + f'(x)(\xi_0 - x) + \int_x^{\xi_0} (\xi_0 - y) f''(y)\, dy, \\ f(\xi_1) & = f(x) + f'(x)(\xi_1 - x) + \int_x^{\xi_1} (\xi_1 - y) f''(y)\, dy, \end{cases}$$

Therefore

$$\begin{aligned} \Pi_1 f(x) &= f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) \\ &= f(x) + \lambda_0(x) \int_x^{\xi_0} (\xi_0 - y) f''(y)\, dy + \lambda_1(x) \int_x^{\xi_1} (\xi_1 - y) f''(y)\, dy \end{aligned}$$

and by the triangle inequality we get

$$\begin{aligned} |f(x) - \Pi_1 f(x)| &= \Big| \lambda_0(x) \int_x^{\xi_0} (\xi_0 - y) f''(y)\, dy + \lambda_1(x) \int_x^{\xi_1} (\xi_1 - y) f''(y)\, dy \Big| \\ &\le |\lambda_0(x)| \Big| \int_x^{\xi_0} (\xi_0 - y) f''(y)\, dy \Big| + |\lambda_1(x)| \Big| \int_x^{\xi_1} (\xi_1 - y) f''(y)\, dy \Big| \\ &\le |\lambda_0(x)| \int_x^{\xi_0} |\xi_0 - y||f''(y)|\, dy + |\lambda_1(x)| \int_x^{\xi_1} |\xi_1 - y||f''(y)|\, dy \\ &\le |\lambda_0(x)| \int_x^{\xi_0} (b - a)|f''(y)|\, dy + |\lambda_1(x)| \int_x^{\xi_1} (b - a)|f''(y)|\, dy \\ &\le (b - a)\Big(|\lambda_0(x)| + |\lambda_1(x)| \Big) \int_a^b |f''(y)|\, dy \\ &= (b - a)\Big(\lambda_0(x) + \lambda_1(x) \Big) \int_a^b |f''(y)|\, dy = (b - a) \int_a^b |f''(y)|\, dy. \end{aligned}$$

Consequently

$$\int_a^b |f(x) - \Pi_1 f(x)| dx \le \int_a^b (b - a) \Big(\int_a^b |f''(y)|\, dy \Big)\, dx, = (b - a)^2 ||f''||_{L_1(I)}.$$

2. We have that

$$\text{(2)} \qquad \begin{cases} u_t - \Delta u = f, & \text{in } \Omega,\ 0 < t \le T, \\ u(x,t) = 0, & \text{on } \Gamma,\ 0 < t \le T, \\ u(x,0) = u_0(x), & \text{in } \Omega, \end{cases}$$

and

$$\text{(3)} \qquad \begin{cases} \widetilde{u}_t - \Delta \widetilde{u} = f, & \text{in } \Omega,\ 0 < t \le T, \\ \widetilde{u}(x,t) = 0, & \text{on } \Gamma,\ 0 < t \le T, \\ \widetilde{u}(x,0) = u_0(x) + \varepsilon(x), & \text{in } \Omega, \end{cases}$$

Now we study $w = \widetilde{u} - u$. (Propagation of disturbance).

a) Through subtracting (2) from (3) we get the differential equation for w:

$$\text{(4)} \qquad \begin{cases} w_t - \Delta w = f, & \text{in } \Omega,\ 0 < t \le T, \\ w(x,t) = 0, & \text{on } \Gamma,\ 0 < t \le T, \\ w(x,0) = \varepsilon(x), & \text{in } \Omega, \end{cases}$$

By the stability estimates for the heat equation we have that

$$\|w(T)\| + 2\int_0^T \|\nabla w\|^2\,dt \le \|\varepsilon\|^2. \qquad \text{(No growth of disturbance).} \tag{5}$$

b) To prove uniqueness for (2), take $\varepsilon = 0$ in (4) and prove that $w \equiv 0$. This is obvious from (5):

$$\|w(T)\| + 2\int_0^T \|\nabla w\|^2\,dt \le 0,$$

where both $\|w(T)\| \ge 0$ and $\|\nabla w\|^2 \ge 0$. Thus $w \equiv 0$, so the uniqueness is proved.

3. (a) We multiply the differential equation by a test function $v \in H_0^1(I),\ I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + u'v) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{6}$$

A *Finite Element Method* with $cG(1)$ reads as followa: Find $U \in V_h^0$ such that

$$\int_I (U'v' + U'v) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{7}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (1)-(2) gives that

$$\int_I (e'v' + e'v) = 0, \quad \forall v \in V_h^0. \tag{8}$$

We note that using $e(0) = e(1) = 0$, we get

$$\int_I e'e = \int_I \frac{1}{2}\frac{d}{dx}\Big(e^2\Big) = \frac{1}{2}(e^2)|_0^1 = 0. \tag{9}$$

Further, usig Poicare inequality we have

$$\|e\|^2 \le \|e'\|^2.$$

A priori error estimate: We use Poicare inequality and (9) to get

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) \le 2\int_I e'e' = 2\int_I (e'e' + e'e) = 2\int_I \Big(e'(u-U)' + e'(u-U)\Big)\\
&= 2\int_I \Big(e'(u-\pi_h u)' + e'(u-\pi_h u)\Big) + 2\int_I \Big(e'(\pi_h u - U)' + e'(\pi_h u - U)\Big)\\
&= \{v = U - \pi_h u \ \text{ in } (6)\} = 2\int_I \Big(e'(u-\pi_h u)' + e'(u-\pi_h u)\Big)\\
&\le 2\|(u-\pi_h u)'\|\|e'\| + 2\|u-\pi_h u\|\|e'\|\\
&\le 2C_i\{\|hu''\| + \|h^2u''\|\}\|e\|_{H^1},
\end{aligned}$$

this gives that

$$\|e\|_{H^1} \le C_i\{\|hu''\| + \|h^2u''\|\},$$

which is the a priori error estimate.

A posteriori error estimate:

$$
\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) \le 2\int_I e'e' = 2\int_I (e'e' + e'e)\\
&= 2\int_I ((u-U)'e' + (u-U)'e) = \{v = e \ \text{ in } (4)\}\\
&= 2\int_I fe - \int_I (U'e' + U'e) = \{v = \pi_h e \ \text{ in } (5)\} \qquad (10)\\
&= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + U'(e - \pi_h e)\Big)\\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),
\end{aligned}
$$

where $\mathcal{R}(U) := f + U'' - U' = f - U'$, (for approximation with picewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$
\begin{aligned}
\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\| \, \|h^{-1}(e - \pi_h e)\|\\
&\le C_i \|h\mathcal{R}(U)\| \, \|e'\| \le C_i \|h\mathcal{R}(U)\| \, \|e\|_{H^1},
\end{aligned}
$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i \|h\mathcal{R}(U)\|.$$

4. Let V be the linear function space defined by

$$V := \{v : \int_\Omega \Big(v^2 + |\nabla v|^2\Big)\, dx < \infty, \quad v = 0, \ \text{ on } \Gamma_1\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) = (1, v), \qquad \forall v \in V.$$

Now using Green's formula and the boundary conditions we have that

$$-(\Delta u, \nabla v) = (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds = (\nabla u, \nabla v) - \int_{\partial\Omega\setminus\Gamma_1} v\, ds, \qquad \forall v \in V.$$

Thus the variational formulation is:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega v\, dx + \int_{\partial\Omega\setminus\Gamma_1} v\, ds, \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on Γ_1:

$$V_h := \{v : v \ \text{ is continuous piecewise linear in } \Omega, \ v = 0, \ \text{ on } \Gamma_1\}.$$

The $cG(1)$ method is: Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v \, dx = \int_\Omega v\, dx + \int_{\partial\Omega\setminus\Gamma_1} v\, ds, \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^5 \xi_j \varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^4 \xi_j \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx\Big) = \int_\Omega \varphi_i \, dx + \int_{\partial\Omega\setminus\Gamma_1} \varphi_i \, ds, \quad i = 1,2,3,4,5,$$

or, in matrix form,

$$S\xi = \mathbf{b}, \qquad S_{ij} = (\nabla\varphi_i, \nabla\varphi_j),$$

where S is the stiffness matrix, and $\mathbf{b} = \mathbf{b}_1 + \mathbf{b}_2$ is the load vector with components

$$\mathbf{b}_{1,i} = \int_\Omega \varphi_i \, dx, \quad \text{and} \quad \mathbf{b}_{2,i} = \int_{\partial\Omega\setminus\Gamma_1} \varphi_i \, ds.$$

We first compute stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1,x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1,x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$\phi_2(x_1,x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\phi_3(x_1,x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2\,dx = \frac{2}{h^2}|T| = 1,$$
$$s_{12} = (\nabla\phi_1, \nabla\phi_2) = \int_T |\nabla\phi_1|^2\,dx = -\frac{1}{h^2}|T| = -1/2, \qquad s_{13} = -1/2$$
$$s_{22} = (\nabla\phi_2, \nabla\phi_2) = \int_T |\nabla\phi_2|^2\,dx = \frac{1}{h^2}|T| = 1/2, \qquad s_{23} = (\nabla\phi_2, \nabla\phi_3) = 0,$$
$$s_{33} = (\nabla\phi_3, \nabla\phi_3) = \int_T |\nabla\phi_3|^2\,dx = \frac{1}{h^2}|T| = 1/2,$$

Thus using the symmetry we have the local stiffness matrix as

$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrix S from the local s, using the character of our mesh, viz:

$$\begin{array}{llll} S_{11} = 4s_{11} = 4, & S_{12} = 0 & S_{13} = S_{14} = 2s_{12} = -1 & S_{15} = 0 \\ S_{22} = 4s_{11} = 4 & S_{23} = 0 & S_{24} = S_{25} = 2s_{12} = -1 & \\ S_{33} = 2s_{22} = 1, & S_{34} = s_{23} = 0, & S_{35} = 0 & \\ & & S_{44} = 4s_{22} = 2, & S_{45} = s_{23} = 0 \\ & & & S_{55} = 2s_{22} = 1 \end{array}$$

The remaining matrix elements are obtained by symmetry $S_{ij} = S_{ji}$. Hence,

$$S = \begin{bmatrix} 4 & 0 & -1 & -1 & 0 \\ 0 & 4 & 0 & -1 & -1 \\ -1 & 0 & 1 & 0 & 0 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 & 1 \end{bmatrix}.$$

As for the load vector we note that

$$\begin{aligned} \mathbf{b}_{1,1} &= \int_\Omega \varphi_1 = 4 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = 4\frac{h^2}{6}, \\ \mathbf{b}_{1,2} &= 4 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = 4\frac{h^2}{6}, \\ \mathbf{b}_{1,3} &= 2 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = 2\frac{h^2}{6}, \\ \mathbf{b}_{1,4} &= 4 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = 4\frac{h^2}{6}, \\ \mathbf{b}_{1,5} &= 2 \cdot \frac{1}{3} \cdot \frac{h^2}{2} \cdot 1 = 2\frac{h^2}{6}, \end{aligned} \tag{11}$$

$$\mathbf{b}_{2,1} = \mathbf{b}_{2,2} = 0, \mathbf{b}_{2,3} = \mathbf{b}_{2,4} = \mathbf{b}_{2,5} \int_{\partial\Omega} \varphi_i = 2 \cdot \frac{1}{2}(h \cdot 1) = h. \tag{12}$$

Hence the load vector **b** is:

$$\mathbf{b} = \frac{h^2}{6}\begin{bmatrix} 4 \\ 4 \\ 2 \\ 4 \\ 2 \end{bmatrix} + h\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

5. See the Lecture Notes.

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