Mathematics Chalmers & GU

MVE455: Partial Differential Equations, 2019–03–18, 14:00-18:00

Telephone: Tobias Magnusson: ankn 5325
Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 4p. Valid bonus poits will be added to the scores.
Breakings: **3**: 10-15p, **4**: 16-20p och **5**: 21p-
For solutions and information about gradings see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/mve455/1718/coursediary

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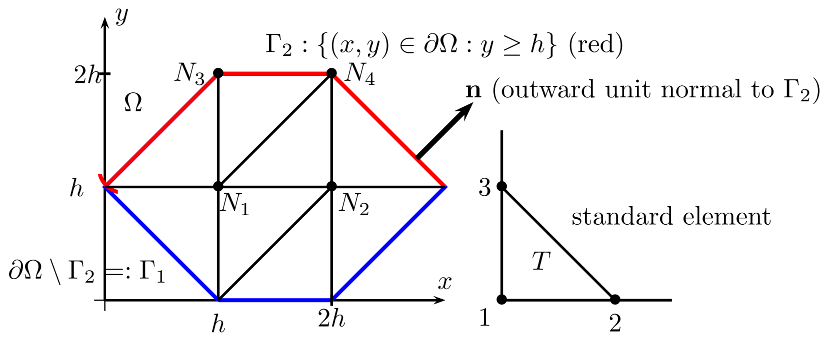
1. Consider a uniform partition $0 = x_0 < x_1 < \ldots < x_N = 1$ of the interval $[0, 1]$ and let $\{\varphi_i\}_{i=0}^N$ be a set of piecewise linear continuous basis functions: $\varphi_i(x_j) = 1$ for $i = j$ and $\varphi_i(x_j) = 0$ if $i \neq j$. Given a FEM in form of linear system of equations $(S + M)\xi = \mathbf{b} + \mathbf{d}$, with S and M, N-by-N matrices, and ξ, $\mathbf{b}$ and $\mathbf{d}$ vectors of lenght N, where for $i, j = 1, \ldots, N$, $S_{ij} = (\varphi_i', \varphi_j')$, $M_{ij} = (\varphi_i, \varphi_j)$, and $\mathbf{b}_i = (\varphi_i, f)$, f is a given function, $d_N = \beta$ is the only non-zero element of $\mathbf{d}$.
a) Derive the variational formulation and the strong formulation for the PDE from the above data.
b) Let now both $u(0) = 0$ and $\beta = 0$ and derive the continuous stability estimate $||u_x|| \leq \frac{1}{2}||f||$.

2. Let Ω be the hexagonal domain with the uniform triangulation as in the figure below. Compute



the stiffness matrix and the load vector for the cG(1) approximate solution for the problem:

$$-\Delta u = 1, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \Gamma_1, \quad \partial u/\partial \mathbf{n} = 0, \quad \text{on } \Gamma_2 \tag{1}$$

3. Derive *a priori* error estimate, in the energy norm $||v||_E^2 = ||v'||^2 + a||v||^2$, for the $cG(1)$ approximation of the boundary value problem

$$-u''(x) + u'(x) + au(x) = f(x), \quad 0 < x < 1, \quad u(0) = u(1) = 0, \quad a \geq 0.$$

4. a)Estimate the stability factor $\int_0^T |\dot{u}|\, dt/|u_0|$ for the initial value problem

$$\dot{u}(t) + a(t)u(t) = 0, \quad u(0) = u_0, \qquad a(t) \geq \lambda > 0.$$

b) Formulate the dG(0)-method for this problem.

5. (Poincare inequalities). Assume that u is the solution of a Dirichlet boundary value problem, and $u, u', |\nabla u| \in L_2(\Omega)$ (they are square integrable in a bounded domian $\Omega \subset \mathbb{R}^d$, $d = 1, 2$). Show that there are positive constants C_L and C_Ω such that

$$\|u\| \leq C_L \|u'\|, \qquad \Omega = [0, L]$$
$$\|u\| \leq C_\Omega \|\nabla u\|, \qquad \Omega \subset \mathbb{R}^2.$$

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MVE455: Partial Differential Equations, 2019–03–18, 14:00-18:00. Solutions.

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1. a) It is clear that the homogeneous Dirichlet condition is used at $x = 0$ since the basis function φ_0 is not present in the matrices and there are no modifications corresponding the first element of the load vector. (The indices start from $i, j = 1, \ldots$. Consider now the solution space $V^0 = \{w : \|w\| + \|w'\| < \infty\},\ w(0) = 0\}$ where $||\cdot||$ is the usual L_2-norm over I. Then, the variational formulation reads as follows: *find* $u \in V^0$ *s.t.*

$$(v_x, u_x) + (v, u) = (v, f) + \beta v(1), \qquad \forall\ v \in V^0. \tag{2}$$

For the basis functions given, $\varphi_N(1) = 1$, which explains the last element of the vector $\mathbf{d}$. Backward integration by parts, together with the Dirichlet data on v yields

$$(v, f) + \beta v(1) = (v, -u_{xx} + u) + v(1)u_x(1). \tag{3}$$

Thus, the strong formulation (PDE) is: find such that

$$-u_{xx} + u = f \quad 0 < x < 1 \qquad u(0) = 0, \quad u_x(1) = \beta. \tag{4}$$

(b) Let in (2) $\beta = 0$ and $v = u$, then

$$(u_x, u_x) + (u, u) = (u, f).$$

Using integration by parts and $u(0) = 0$, and $u_x(1) = 0$, we get

$$||u_x||^2 + ||u||^2 = (f, u) \leq ||u||^2 + \frac{1}{4}||f||^2$$

Hence,

$$||u_x|| \leq \frac{1}{2}||f||,$$

we get the desired result.

2. Let V be the linear function space defined by

$$V := \{v : v \in H^1(\Omega),\ \ v = 0,\ \text{ on } \Gamma_1\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) = (1, v), \qquad \forall v \in V.$$

Now using Green's formula and the fact that $v = 0$ on $\partial\Omega \setminus \Gamma_1$, we have that

$$\begin{aligned} -(\Delta u, \nabla v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds \\ &= (\nabla u, \nabla v) - \int_{\Gamma_1} (n \cdot \nabla u) v \, ds - \int_{\Gamma_2} (n \cdot \nabla u) v \, ds = (\nabla u, \nabla v), \qquad \forall v \in V, \end{aligned}$$

Hence, the variational formulation is:

$$(\nabla u, \nabla v) = (1, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on Γ_1: Then, the $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) = (1, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{j=1}^{4} \xi_j \varphi_j(x)$, where φ_j are the standard basis functions (φ_1 is the basis function for the interior node N_1 and φ_2 and φ_3 are corresponding basis functions for the boundary nodes N_1 and N_2, respective) we obtain the system of equations

$$\sum_{j=1}^{3} \xi_j \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx = \int_\Omega f\varphi_i \, dx, \quad i = 1, 2, 3, 4.$$

In matrix form this can be written as $S\xi = F$, where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, and $F_i = (f, \varphi_i)$ is the load vector.

We first compute the stiffness matrix for the reference triangle T. The local basis functions are

$$\begin{aligned}
\phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1, x_2) &= -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1, x_2) &= \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Hence, with $|T| = \int_T dz = h^2/2$, we can easily compute

$$\begin{aligned}
s_{11} &= (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2}|T| = 1, \\
s_{12} &= s_{21} = (\nabla\phi_1, \nabla\phi_2) = \int_T \frac{-1}{h^2}|T| = -1/2, \\
s_{23} &= s_{32} = (\nabla\phi_2, \nabla\phi_3) = 0, \\
s_{22} &= s_{33} = \ldots = \frac{1}{h^2}|T| = 1/2.
\end{aligned}$$

Thus by symmetry we get that the local stiffness matrix for the standard element is:

$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global stiffness matrix S from the local stiffness matrix s:

$$\begin{aligned}
&S_{11} = S_{22} = 3s_{11} + 2s_{22} = 3 + 1 = 4, && S_{12} = S_{13} = 2s_{12} = -1 && \\
&S_{14} = 2s_{23} = 0, && S_{23} = 0 && S_{24} = 2s_{12} = -1, \\
&S_{33} = s_{11} + s_{22} = 3/2 && S_{34} = s_{12} = -1/2 && S_{44} = 3s_{22} = 3/2
\end{aligned}$$

The remaining matrix elements are obtained by symmetry $S_{ij} = S_{ji}$. Hence,

$$S = \frac{1}{2}\begin{bmatrix} 8 & -2 & -2 & 0 \\ -2 & 8 & 0 & -2 \\ -2 & 0 & 3 & -1 \\ 0 & -2 & -1 & 3 \end{bmatrix}.$$

As for the load vector we have that

$$\int_\Omega \varphi_1 = \int_\Omega \varphi_2 = 5\frac{1}{3}\frac{h^2}{2}.1 = 5h^2/6, \quad \int_\Omega \varphi_3 = 2\frac{1}{3}\frac{h^2}{2}.1 = \frac{h^2}{3}, \quad \int_\Omega \varphi_4 = 3\frac{1}{3}\frac{h^2}{2}.1 = \frac{h^2}{2},$$

Thus the load vector is given by $b = \frac{h^2}{6}(5, 5, 2, 3)^t$. Observe that, here S becomes independent of h.

3. (a) The Variational formulation:
(Multiply the equation by $v \in V$, integrate by parts over $(0,1)$ and use the boundary conditions.)

$$\text{(5)} \qquad \text{Find } u \in V: \int_0^1 u'v'\,dx + \int_0^1 u'v\,dx + \int_0^1 auv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V.$$

cG(1):

$$\text{(6)} \qquad \text{Find } U \in V_h: \int_0^1 U'v'\,dx + \int_0^1 U'v\,dx + \int_0^1 aUv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V_h,$$

where

$$V_h := \{v : v \text{ is continuous piecewise linear in } (0,1),\ v(0) = v(1) = 0\}.$$

From (1)-(2), we find
The Galerkin orthogonality:

$$\text{(7)} \qquad \int_0^1 \Big((u-U)'v' + (u-U)'v + a(u-U)v\Big)\,dx = 0, \quad \forall v \in V_h.$$

We define the inner product $(\cdot,\cdot)_E$ associated to the energy norm to be

$$(v,w)_E = \int_0^1 (v'w' + avw)\,dx, \qquad \forall v,w \in V.$$

A priori error estimate:

$$\begin{aligned}
||e||_E^2 &= \int_0^1 (e'e' + aee)\,dx = \Big\{\int_0^1 e'e\,dx = \frac{1}{2}\int_0^1 \frac{d}{dx}\Big(e^2\Big)\,dx = \frac{1}{2}[e^2]_0^1 = 0\Big\} \\
&= \int_0^1 (e'e' + e'e + aee)\,dx = \int_0^1 \Big(e'(u-U)' + e'(u-U) + ae(u-U)\Big)\,dx \\
&= \Big\{v \in V_h\Big\} = \int_0^1 \Big(e'(u-v)' + e'(u-v) + ae(u-v)\Big)\,dx \\
&\quad + \int_0^1 \Big(e'(v-U)' + e'(v-U) + ae(v-U)\Big)\,dx = \Big\{(3)\Big\} \\
&= \int_0^1 \Big(e'(u-v)' + e'(u-v) + ae(u-v)\Big)\,dx \\
&= \int_0^1 \Big(e'(u-v)' + ae(u-v) + e'(u-v)\Big)\,dx \\
&\quad \le ||e'||_{L_2} \cdot ||(u-v)'||_{L_2} + a||e||_{L_2}||u-v||_{L_2} + ||e'||_{L_2} \cdot ||u-v||_{L_2} \\
&\quad \le ||e'||_{L_2} \cdot ||(u-v)'||_{L_2} + ||e||_E \cdot ||u-v||_{L_2} \\
&\quad \le ||e||_E \cdot ||u-v||_E + ||e||_E \cdot ||u-v||_{L_2}
\end{aligned}$$

This gives the a priori error estimate:

$$||e||_E \le ||u-v||_E + ||u-v||_{L_2}, \quad \forall v \in V_h.$$

4. a) $u(t) = e^{-A(t)}u_0, \qquad A(t) = \int_0^t a(s)\,ds$, so that

$$\begin{aligned}
\int_0^T |\dot{u}(t)|\,dt &= \int_0^T |a(t)e^{-A(t)}u_0|\,dt \\
&= \int_0^T a(t)e^{-A(t)}|u_0|\,dt = \Big(1 - e^{-A(T)}\Big)|u_0| \le \Big(1 - e^{-\lambda t}\Big)|u_0|
\end{aligned}$$

b) See the book.

5. See the book.

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