Mathematic Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2009–01–13; kl 8.30-13.30.

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Calculators, formula notes and other subject related material are not allowed.

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1. Consider the initial value problem

$$\begin{cases} \dot{u}(t) + au(t) = 0, & t > 0, \\ u(0) = u_0, & a > 0, \quad (a = \text{constant}). \end{cases}$$

Assume a constant time step k and verify the iterative formulas for $dG(0)$ and $cG(1)$ approximations U and $\tilde{U}$ respectively: i.e.

$$U_n = \Big(\frac{1}{1+ak}\Big)^n u_0, \qquad \tilde{U}_n = \Big(\frac{1-ak/2}{1+ak/2}\Big)^n u_0.$$

2. Let $\mathcal{T}_h : x_0 = 0, x_1 = 1/3, x_2 = 2/3, x_3 = 1$ be a partition of the interval [0,1]. Write down the equation system $A\xi = b$ (determine A and b) for the piecewise linear Galerkin approximation on $\mathcal{T}_h$ for the problem:

$$\begin{cases} -u''(x) + 18u(x) = 6, & \text{for} \quad x \in (0,1) \\ u'(0) = -1, \quad u(1) = 0. \end{cases}$$

3. Prove an a priori and an a posteriori error estimate (in the H^1-norm: $||u||^2_{H^1} := ||u'||^2 + 2||u||^2$) for a finite element method for the problem

$$\begin{cases} -u''(x) + 2xu'(x) + 3u(x) = f(x), & \text{for} \quad x \in (0,1) \\ u(0) = u(1) = 0. \end{cases}$$

4. Prove that if $u = 0$ on the boundary of the unit square Ω, then

$$\Big(\int_\Omega |u|^2\, dx\Big)^{1/2} \le \Big(\int_\Omega |\nabla u|^2\, dx\Big)^{1/2}, \qquad \text{(Poincare inequality in 2D)}.$$

5. Consider the boundary value problem

$$\begin{cases} -\Delta u = 0, & \text{in a bounded domain } \Omega \subset \mathbb{R}^d, \ d = 2,3. \\ \frac{\partial u}{\partial n} + u = g, & \text{on } \Gamma = \partial\Omega. \end{cases}$$

a) Prove the L_2 stability estimate

$$||\nabla u||_{L_2(\Omega)} + \frac{1}{2}||u||_{L_2(\Gamma)} \le \frac{1}{2}||g||_{L_2(\Gamma)}.$$

b) Verify the conditions on Riesz/Lax-Milgram theorem for this problem.

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1. See the Book and Lecture notes.

2. Multiply the equation with a test function $v \in H^1_{=} = \{v : ||v|| + ||v'|| < \infty, \quad v(1) = 0\}$. Partial integration over $I = (0, 1)$ yields

$$\int_0^1 u'(x)v'(x)\,dx - [u'(x)v(x)\,dx]_{x=0}^{x=1} + 18\int_0^1 u(x)v(x)\,dx = 6\int_0^1 v(x)\,dx.$$

Using the boundary data we get the *variational formulation*: Find $u \in H^1_0(I)$ such that

$$\int_0^1 u'(x)v'(x)\,dx + 18\int_0^1 u(x)v(x)\,dx = 6\int_0^1 v(x)\,dx + v(0) \tag{1}$$

A *Finite element Method* with piecewise linear approximation is formulated as: Find $U \in V_h^0$ such that

$$\int_0^1 U'(x)v'(x)\,dx + 18\int_0^1 U(x)v(x)\,dx = 6\int_0^1 v(x)\,dx + +v(0), \qquad \forall v \in V_h^0 \subset H^1(I), \tag{2}$$

where

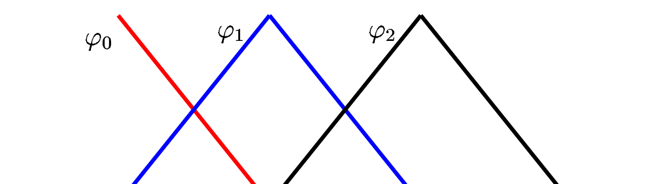
$$V_h^0 = \{v : v \text{ piecewise linear and continuous on the partition } \quad \mathcal{T}_h, \quad v(1) = 0\}.$$

For the partition $\mathcal{T}_h$, and with the Neumann boundary data at $x = 0$: $u'(0) = -1$, and the Dirichlet data at $x = 1$: $u(1) = 0$ our bases functions are as follows:

$$\varphi_0(x) = \begin{cases} 1 - 3x, & 0 \le x \le 1/3 \\ 0, & 1/3 \le x \le 1. \end{cases} \qquad \varphi_1(x) = \begin{cases} 3x, & 0 \le x \le 1/3 \\ 2 - 3x, & 1/3 \le x \le 2/3 \\ 0, & 2/3 \le x \le 1 \end{cases}$$

and

$$\varphi_2(x) = \begin{cases} 0, & 0 \le x \le 1/3 \\ 3x - 1, & 1/3 \le x \le 2/3 \\ 3 - 3x, & 2/3 \le x \le 1 \end{cases}$$



In this base we may use the ansatz $U(x) = \sum_{j=0}^{2} \xi_j \varphi_j(x)$ in (4) and choose $v(x) = \varphi_i(x)$, $i = 0, 1, 2$. Then the finite element method (4) can be rewritten as

$$\sum_{j=0}^{2} \xi_j \int_0^1 \varphi_i'(x)\varphi_j'(x)\,dx + 18\sum_{j=0}^{2} \xi_j \int_0^1 \varphi_i(x)\varphi_j(x)\,dx = 6\int_0^1 \varphi_i(x)\,dx + \varphi_i(0), \qquad i = 0, 1, 2,$$

or as a linear system of equations $A\xi = b$, $\quad A = (a_{ij})$, with

$$a_{ij} = \int_0^1 \varphi_i'(x)\varphi_j'(x)\,dx + 18\sum_{j=0}^{2} \xi_j \int_0^1 \varphi_i(x)\varphi_j(x)\,dx, \qquad i,j = 0,1,2.$$

$$b = (b_i), \quad b_i = 6\int_0^1 \varphi_i(x)\,dx + \varphi_i(0), \qquad i,j = 0,1,2.$$

Observe that the Neumann boundary condition $u'(0) = -1$, would yield a *half-base* function φ_0 at $x = 0$, and therefore compared to the standard case, the values of the first diagonal elements in both mass and stiffness matrices M and S are half of the values in the Dirichlet boundary value case. Thus we end up with the equation system

$$\frac{1}{h}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + 18h\begin{pmatrix} 1/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 1/3 \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = 6\begin{pmatrix} 1/6 \\ 2/6 \\ 2/6 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Finally with $h = 1/3$ we end up with

$$A = \begin{pmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 6 \end{pmatrix} + 18\frac{1}{3}\begin{pmatrix} 1/3 & 1/6 & 0 \\ 1/6 & 2/3 & 1/6 \\ 0 & 1/6 & 2/3 \end{pmatrix} = \begin{pmatrix} 5 & -2 & 0 \\ -2 & 10 & -2 \\ 0 & -2 & 10 \end{pmatrix}, \qquad b = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}.$$

3. We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + 2xu'v + 3uv) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{3}$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U \in V_h^0$ such that

$$\int_I (U'v' + 2xU'v + 3Uv) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{4}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (3)-(4) gives that

$$\int_I (e'v' + 2xe'v + 3ev) = 0, \quad \forall v \in V_h^0. \tag{5}$$

A posteriori error estimate: We note that using $e(0) = e(1) = 0$, we get

$$\int_I 2xe'e = \int_I x\frac{d}{dx}(e^2) = (xe^2)|_0^1 - \int_I e^2 = -\int_I e^2, \tag{6}$$

so that using variational formulation (3) to replace the terms involving continuous solution u and the finite element method (4) to insert the interpolant $\pi_h e$ of the error we can compute

$$\begin{aligned} \|e\|_{H^1}^2 &= \int_I (e'e' + 2ee) = \int_I (e'e' + 2xe'e + 3ee) \\ &= \int_I ((u-U)'e' + 2x(u-U)'e + 3(u-U)e) = \{v = e \text{ in } (3)\} \\ &= \int_I fe - \int_I (U'e' + 2xU'e + 3Ue) = \{v = \pi_h e \text{ in } (4)\} \\ &= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + 2xU'(e-\pi_h e) + 3U(e-\pi_h e)\Big) \\ &= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e), \end{aligned} \tag{7}$$

where $\mathcal{R}(U) := f + U'' - 2xU' - 3U = f - 2xU' - 3U$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (7) implies that

$$\begin{aligned}\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\|\|h^{-1}(e - \pi_h e)\| \\ &\le C_i\|h\mathcal{R}(U)\|\|e'\| \le C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},\end{aligned}$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

A priori error estimate: We use (6) and write

$$\begin{aligned}\|e\|_{H^1}^2 &= \int_I (e'e' + 2ee) = \int_I (e'e' + 2xe'e + 3ee) \\ &= \int_I \Big(e'(u-U)' + 2xe'(u-U) + 3e(u-U)\Big) = \{v = U - \pi_h u \text{ in}(5)\} \\ &= \int_I \Big(e'(u-\pi_h u)' + 2xe'(u-\pi_h u) + 3e(u-\pi_h u)\Big) \\ &\le \|(u-\pi_h u)'\|\|e'\| + 2\|u-\pi_h u\|\|e'\| + 3\|u-\pi_h u\|\|e\| \\ &\le \{\|(u-\pi_h u)'\| + 4\|u-\pi_h u\|\}\|e\|_{H^1} \\ &\le C_i\{\|hu''\| + \|h^2u''\|\}\|e\|_{H^1},\end{aligned}$$

this gives that

$$\|e\|_{H^1} \le C_i\{\|hu''\| + \|h^2u''\|\},$$

which is the a priori error estimate.

4. This is inspired from the proof of the Poincare inequality in the 1D case: We have, due to the vanishing boundary data, that

$$\begin{aligned}|u(x)| = |u(x_1,x_2) - u(0,x_2)| &= \Big|\int_0^{x_1} \frac{\partial}{\partial \bar{x_1}} u(\bar{x_1},x_2)\,d\bar{x_1}\Big| \\ &= \Big|\int_0^{x_1} 1\cdot\frac{\partial}{\partial \bar{x_1}} u(\bar{x_1},x_2)\,d\bar{x_1}\Big| \le \{\text{Cauchy's inequality}\} \\ &\le \Big(\int_0^{x_1} 1^2\,d\bar{x_1}\Big)^{1/2}\cdot\Big(\int_0^{x_1} (\frac{\partial}{\partial \bar{x_1}} u(\bar{x_1},x_2))^2\,d\bar{x_1}\Big)^{1/2} \\ &\le \Big(\int_0^1 (\frac{\partial}{\partial \bar{x_1}} u(\bar{x_1},x_2))^2\,d\bar{x_1}\Big)^{1/2}.\end{aligned}$$

This implies that

$$\begin{aligned}\int_\Omega |u|^2\,dx &\le \int_\Omega \Big(\int_0^1 (\frac{\partial}{\partial \bar{x_1}} u(\bar{x_1},x_2))^2\,d\bar{x_1}\Big)\,dx \\ &= \int_0^1\int_0^1 \Big(\int_0^1 (\frac{\partial}{\partial \bar{x_1}} u(\bar{x_1},x_2))^2\,d\bar{x_1}\Big)\,dx_1\,dx_2 \\ &= \int_0^1 \Big(\int_0^1 (\frac{\partial}{\partial \bar{x_1}} u(\bar{x_1},x_2))^2\,d\bar{x_1}\Big)\,dx_2 = \int_0^1\int_0^1 (\frac{\partial}{\partial x_1} u(x_1,x_2))^2\,dx_1\,dx_2 \\ &= \int_\Omega (\frac{\partial}{\partial x_1} u(x_1,x_2))^2\,dx \le \int_\Omega |\nabla u|^2\,dx,\end{aligned}$$

which gives the desired result:

$$\Big(\int_\Omega |u|^2\,dx\Big)^{1/2} \le \Big(\int_\Omega |\nabla u|^2\,dx\Big)^{1/2}.$$

5. a) Using Greens formula we have that

$$\int_\Omega |\nabla u|^2 = \int_\Omega \nabla u \cdot \nabla u = -\int_\Omega (\Delta u) u + \int_{\partial\Omega} \frac{\partial u}{\partial n} u = \int_{\partial\Omega} (g-u)u,$$

where, in the last equality, we have used both equation and the data. In other words

$$||\nabla u||^2_{L_2(\Omega)} + ||u||^2_{L_2(\Gamma)} = \int_{\partial\Omega} gu \leq ||g||^2_{L_2(\Gamma)} ||u||^2_{L_2(\Gamma)} \leq \frac{1}{2}||g||^2_{L_2(\Gamma)} + \frac{1}{2}||u||^2_{L_2(\Gamma)},$$

which gives the desired estimate.

b) To show the Riesz/Lax-Milgram conditions we introduce the notation

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} uv, \quad \text{and} \quad L(v) = \int_{\partial\Omega} gv.$$

Then $a(u,v)$ is a scalar product with the corresponding norm $||v||_a = a(v,v)^{1/2}$. For instance we have that $||v||_a = 0$, only if $v = 0$:

$$0 = ||v||^2_a = a(v,v) = \int_\Omega |\nabla v|^2 + \int_{\partial\Omega} v^2 \geq \alpha \int_\Omega v^2, \qquad \text{for some } \alpha > 0 \Rightarrow v = 0.$$

Further $L(v)$ is bounded in this norm, e.g. if $||g||_{\partial\Omega} < \infty$, then

$$|L(v)| \leq ||g||_{\partial\Omega} ||v||_{\partial\Omega} \leq ||g||_{\partial\Omega} ||v||_a.$$

We can also apply Riesz theorem in the sense that there exists u such that

$$a(u,v) = L(v), \qquad \forall v,$$

and u is uniquely determined by

$$||u||_a = ||g||_{\partial\Omega}.$$

Moreover since

$$a(u,v) = -\int_\Omega \Delta uv + \int_{\partial\Omega} (\frac{\partial u}{\partial n} + u)v,$$

we have that

$$\Delta u = 0, \quad \text{in } \Omega \quad \text{and} \quad \frac{\partial u}{\partial n} + u = g \quad \text{on } \Gamma.$$

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