Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2011–03–14; kl 8.30-13.30.

Telephone: Oskar Hamlet: 0703-088304
Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 5p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p- For GU students**G**:15-27p, **VG**: 28p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1011/index.html

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1. Formulate and prove the Lax-Milgram theorem in the case of symmetric bilinear form.

2. Let $\alpha(x)$ be a bounded positive function on $[0,1]$, i.e. $0 \le \alpha(x) \le M$. Prove an a priori and an a posteriori error estimate for the cG(1) finite element method for the problem

$$-u'' + \alpha u = f, \quad 0 < x < 1, \qquad u(0) = u(1) = 0, \tag{1}$$

in the energy norm defined by $||w||_E^2 := \int_I \Big((w')^2 + \alpha(w)^2\Big)\, dx, \quad I = (0,1).$

3. Let ε be a positive constant, and $f \in L_2(I)$. Consider the problem

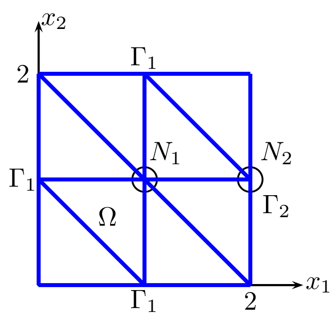
$$-\varepsilon u'' + xu' + u = f \quad \text{in } I = (0,1), \qquad u(0) = u'(1) = 0,$$

Prove that $\qquad \|\varepsilon u''\| \le \|f\|, \qquad (\|\cdot\|$ is the $L_2(I)$- norm).

4. In the square domain $\Omega := (0,1)^2$, with the boundary $\Gamma := \partial\Omega$, consider the problem of solving

$$\begin{cases} -\frac{\partial^2 u}{\partial x_1^2} - 2\frac{\partial^2 u}{\partial x_2^2} = 1, & \text{in} \quad \Omega = \{x = (x_1, x_2) : 0 < x_1 < 2,\ 0 < x_2 < 2\}, \\ u = 0 \quad \text{on} \quad \Gamma_1 := \Gamma \setminus \Gamma_2, \qquad \frac{\partial u}{\partial x_1} = 0 \quad \text{on} \quad \Gamma_2 = \{x = (x_1, x_2) :\ x_1 = 2, \quad 0 < x_2 < 2\}. \end{cases} \tag{2}$$

Determine the stiffness matrix and load vector if the cG(1) finite element method with piecewise linear approximation is applied to the equation (2) above and on the following triangulation:



5. Consider the Poisson equation

$$-\Delta u = f, \quad \text{in } \Omega \in \mathbf{R}^2, \qquad \text{with} \qquad -\mathbf{n}\cdot\nabla u = k\,u, \ \text{on } \partial\Omega, \tag{3}$$

where $k > 0$ and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ ($\partial\Omega$ is the boundary of Ω).
a) Prove the Poincare inequality: $\quad \|u\|_{L_2(\Omega)} \le C_\Omega(\|u\|_{L_2(\partial\Omega)} + \|\nabla u\|_{L_2(\Omega)})$.
b) Use the inequality in a) and the boundary data to show that $\|u\|_{L_2(\partial\Omega)} \to 0$ as $k \to \infty$.

6. a) Formulate a relevant minimization problem for the solution of the Poisson equation

$$-\Delta u = f, \quad \text{in } \Omega \in \mathbf{R}^2, \qquad \text{with} \quad \mathbf{n}\cdot\nabla u = b(g-u), \quad \text{on } \partial\Omega,$$

where $f > 0$, $b > 0$ and g are given functions.
b) Derive an *a priori* error estimate for cG(1) approximation in the corresponding energy-norm.
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Lösningar/Solutions.

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1. See Lecture Notes or text book chapter 21.

2. Multiply (1) by $v \in H_0^1(I)$ and integrate over I. Then, partial integration yields

$$\int_I (u'v' + \alpha uv)\,dx = \int_I fv\,dx. \tag{4}$$

Hence we may formulate the *Variational formulation:* as Find $u \in H_0^1(I)$ such that

$$\text{(VF)} \qquad \int_I (u'v' + \alpha uv)\,dx = \int_I fv\,dx, \qquad \forall v \in H_0^1(I). \tag{5}$$

The corresponding finite element formulation, for the cG(1) method reads: Find $U \in V_h^0$ such that

$$\text{(FEM)} \qquad \int_I (U'v' + \alpha Uv)\,dx = \int_I fv\,dx, \qquad \forall v \in V_h^0. \tag{6}$$

Then, with $e := u - U$ (5)-(6) gives the *Galerkin orthogonality*:

$$(\text{G}^\perp) \qquad \int_I (e'v' + \alpha ev)\,dx = 0, \qquad \forall v \in V_h^0 \subset H_0^1(I). \tag{7}$$

We define the scalar product

$$(v,w)_E := \int_I (v'w' + \alpha vw)\,dx,$$

which, by letting $w = v$ ends up to our choice of the energy norm:

$$||v||_E^2 := (v,v)_E = \int_I \Big((v')^2 + \alpha(v)^2\Big)\,dx.$$

Note also that (7) is written in concise form as

$$(\text{G}^\perp) \qquad (e,v)_E = 0, \qquad \forall v \in V_h^0 \subset H_0^1(I). \tag{8}$$

A priori error estimate: Using (8) with $v = \pi_h u - U$ and Cauchy-Schwarz inequality, we can write

$$||e||_E^2 = (e,e)_E = (e, u-U)_E = (e, u - \pi_h u + \pi_h u - U)_E = (e, u - \pi_h)_E \le ||e||_E ||u - \pi_h||_E. \tag{9}$$

On the other hand by the definition of the energy norm, and interpolation estimates

$$||u-\pi_h||_E^2 \le ||(u-\pi_h)'||_{L_2(I)}^2 + ||\sqrt{\alpha}(u-\pi_h)||_{L_2(I)}^2 \le C_i^2 ||hu''||_{L_2(I)}^2 + C_i^2 M ||h^2 u''||_{L_2(I)}^2. \tag{10}$$

Thus, combining (9) and (10) we get the a priori error estimate, viz

$$||e||_E \le C_i \Big(||hu''||_{L_2(I)} + ||h^2 u''||_{L_2(I)} \Big). \tag{11}$$

A posteriori error estimate: (This time we aim to eliminate u, and keep U and f).

$$\begin{aligned}
||e||_E^2 &= \int_I (e'e' + \alpha ee)\,dx = \int_I (u-U)'e'\,dx + \int_I \alpha(u-U)e\,dx = \{v = e \text{ in } (5)\} \\
&= \int_I fe - \int_I (U'e' + \alpha Ue)\,dx = \{v = \pi_h e \text{ in } (6)\} \\
&\int_I f(e-\pi_h e) - \int_I \Big(U'(e-\pi_h e)' + \alpha U(e - \pi_h e)\Big)\,dx = \{P.I. \text{ over each subinterval}\} \\
&= \int_I (f + U'' - \alpha U)(e - \pi_h e) \equiv \int_I R(U)(e - \pi_h e)\,dx,
\end{aligned} \tag{12}$$

where $R(U) = f + U'' - \alpha U = f - \alpha U$ ($U'' \equiv 0$, since U is linear on each subinterval).

Now using Cauchy-Schwarz and interpolation estimate we get

$$\begin{aligned}||e||_E^2 &\leq ||hR(U)||_{L_2(I)}||h^{-1}(e-\pi_h e)||_{L_2(I)}\\ &\leq C_i||hR(U)||_{L_2(I)}||e'||_{L_2(I)} \leq C_i||hR(U)||_{L_2(I)}||e||_E.\end{aligned} \tag{13}$$

Hence, finally we have the a posteriori error estimate:

$$||e||_E \leq C_i||hR(U)||_{L_2(I)}.$$

3. Multiply the equation $-\varepsilon u'' + xu' + u = f$, by $-\varepsilon u''$ and integrate over $I = (0, 1)$:

$$||\varepsilon u''||_{L_2(I)}^2 - \varepsilon \int_0^1 xu'u''\,dx - \varepsilon \int_0^1 uu''\,dx = \int_0^1 (-\varepsilon u'')f\,dx. \tag{14}$$

Integration by parts, and using the boundary data $u'(1) = 0$ yields

$$\int_0^1 xu'u''\,dx = [xu'^2]_0^1 - \int_0^1 (u' + xu'')u'\,dx = -\int_0^1 u'^2\,dx - \int_0^1 xu'u''\,dx. \tag{15}$$

Thus

$$\int_0^1 xu'u''\,dx = -\frac{1}{2}\int_0^1 u'^2\,dx. \tag{16}$$

Similarly,

$$\int_0^1 uu''\,dx = [uu']_0^1 - \int_0^1 u'^2\,dx. \tag{17}$$

Inserting (16)-(17) in (14) we have

$$||\varepsilon u''||_{L_2(I)}^2 + \frac{\varepsilon}{2}\int_0^1 u'^2\,dx + \varepsilon\int_0^1 u'^2\,dx = \int_0^1 (-\varepsilon u'')f\,dx. \tag{18}$$

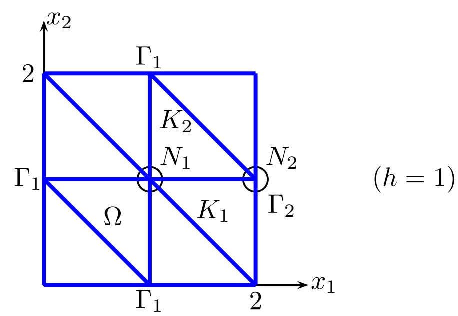
Hence, using Cauchy-Schwarz inequality

$$||\varepsilon u''||_{L_2(I)}^2 \leq int_0^1(-\varepsilon u'')f\,dx \leq \{C-S\} \leq ||\varepsilon u''||_{L_2(I)}||f||_{L_2(I)}, \tag{19}$$

and we have the desired result:

$$||\varepsilon u''||_{L_2(I)} \leq ||f||_{L_2(I)}. \tag{20}$$

4. Recall that the mesh size is $h = 1$. Further, the first triangle (the triangle with nodes at $(0, 0)$, $(1, 0)$ and $(0, 1)$) is not in the support of the test function of N_1, whereas the last triangle (the triangle with nodes at $(4, 4)$, $(2, 4)$ and $(4, 2)$) is in the support of the test function for N_2!. Thus, the nodal bases functions φ_1 and φ_2 share the two triangles K_1 and K_2, see figure below. We



define the test function space

$$V = \{v : v \in H^1(\Omega), \quad v = 0 \;\text{ on } \Gamma_1\}. \tag{21}$$

We multiply the differential equation in (2) by $v \in V$ and integrate over Ω. Using Green's formula, the boundary data ($v = 0$ on Γ_1 and $\frac{\partial u}{\partial x_1} = 0$ on Γ_2), and the standard notation $\vec{n} = (n_1, n_2)$ for the outward unit normal on $\Gamma_1 \cup \Gamma_2$, we end up with

$$\begin{aligned} &\int_\Omega \Big(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + 2\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\Big)\, dx_1\, dx_2 - \int_\Gamma \Big(\frac{\partial u}{\partial x_1} v n_1 + 2\frac{\partial u}{\partial x_2} v n_2\Big)\, ds \\ &\qquad = \int_\Omega \Big(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + 2\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\Big)\, dx_1\, dx_2 = \int_\Omega v\, dx_1\, dx_2. \end{aligned} \tag{22}$$

Hence, we have the *variational formulation*: Find $u \in V$ such that

$$\int_\Omega \Big(\frac{\partial u}{\partial x_1}\frac{\partial v}{\partial x_1} + 2\frac{\partial u}{\partial x_2}\frac{\partial v}{\partial x_2}\Big)\, dx_1\, dx_2 = \int_\Omega v\, dx_1\, dx_2, \qquad \forall v \in V, \tag{23}$$

and the corresponding *finite element method*: Find $U \in V_h$ such that

$$\int_\Omega \Big(\frac{\partial U}{\partial x_1}\frac{\partial v}{\partial x_1} + 2\frac{\partial U}{\partial x_2}\frac{\partial v}{\partial x_2}\Big)\, dx_1\, dx_2 = \int_\Omega v\, dx_1\, dx_2, \qquad \forall v \in V_h \ (\subset V), \tag{24}$$

where

$$V_h := \{v : v \text{ is piecewise linear and continuous on the partition of } \Omega,\ v = 0 \ \text{ on } \Gamma_1\}. \tag{25}$$

A basis for V_h consists of $\{\varphi_i\}_{i=1}^2$, where

$$\begin{cases} \varphi_i \in V_h, & i = 1, 2 \\ \varphi_i(N_j) = \delta_{ij}, & i, j = 1, 2. \end{cases}$$

Then, (25) is equivalent to: find $U \in V_h$ such that

$$\int_\Omega \Big(\frac{\partial U}{\partial x_1}\frac{\partial \varphi_i}{\partial x_1} + 2\frac{\partial U}{\partial x_2}\frac{\partial \varphi_i}{\partial x_2}\Big)\, dx_1\, dx_2 = \int_\Omega \varphi_i\, dx_1\, dx_2, \qquad i = 1, 2. \tag{26}$$

Now, we make the ansatz: $U = \sum_{j=1}^2 \xi_j \varphi_j$. Inserting in (26) gives

$$\sum_{j=1}^2 \xi_j \Big\{\int_\Omega \Big(\frac{\partial \varphi_j}{\partial x_1}\frac{\partial \varphi_i}{\partial x_1} + 2\frac{\varphi_j}{\partial x_2}\frac{\partial \varphi_i}{\partial x_2}\Big)\, dx_1\, dx_2\Big\} = \int_\Omega \varphi_i\, dx_1\, dx_2, \qquad i = 1, 2, \tag{27}$$

which can be written in the equivalent form as

$$A\xi = b, \quad a_{ij} = \int_\Omega \Big(\frac{\partial \varphi_j}{\partial x_1}\frac{\partial \varphi_i}{\partial x_1} + 2\frac{\varphi_j}{\partial x_2}\frac{\partial \varphi_i}{\partial x_2}\Big)\, dx_1\, dx_2, \quad b_i = \int_\Omega \varphi_i\, dx_1\, dx_2. \tag{28}$$

We can easily compute that

$$\begin{aligned} a_{11} &= \textstyle\int_\Omega \Big(\frac{\partial \varphi_1}{\partial x_1}\frac{\partial \varphi_1}{\partial x_1} + 2\frac{\partial \varphi_1}{\partial x_2}\frac{\partial \varphi_1}{\partial x_2}\Big)\, dx_1\, dx_2 = 6, & b_1 &= \textstyle\int_\Omega \varphi_1\, dx_1\, dx_2 = 1 \\ a_{22} &= \textstyle\int_\Omega \Big(\frac{\partial \varphi_2}{\partial x_2}\frac{\partial \varphi_2}{\partial x_2} + 2\frac{\partial \varphi_2}{\partial x_2}\frac{\partial \varphi_2}{\partial x_2}\Big)\, dx_1\, dx_2 = \frac{a_{11}}{2} = 3, & b_1 &= \textstyle\int_\Omega \varphi_2\, dx_1\, dx_2 = \frac{b_1}{2} = \frac{1}{2}, \end{aligned} \tag{29}$$

and

$$\begin{aligned} a_{12} = a_{21} &= \int_\Omega \Big(\frac{\partial \varphi_2}{\partial x_1}\frac{\partial \varphi_1}{\partial x_2} + 2\frac{\partial \varphi_2}{\partial x_2}\frac{\partial \varphi_1}{\partial x_2}\Big)\, dx_1\, dx_2 = \{\text{ see Fig. }\} = \int_{K_1} \dots + \int_{K_2} \dots \\ &= \Big(\frac{1}{h}(-\frac{1}{h}) + 2\cdot\frac{1}{h}\cdot 0)\cdot\frac{h^2}{2}\Big) + \Big(\frac{1}{h}(-\frac{1}{h}) + 2\cdot 0\cdot(-\frac{1}{h})\cdot\frac{h^2}{2}\Big) = -\frac{1}{2} - \frac{1}{2} = -1. \end{aligned} \tag{30}$$

So, in summary we have that the stiffness matrix A, and the load vector b are given by

$$A = \begin{bmatrix} 6 & -1 \\ -1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}.$$

5. a) There is smooth function ϕ such that $\Delta\phi = 1$ so that, using Greens formula

$$\|u\|_\Omega^2 = \int_\Omega u^2\,\Delta\phi = \int_{\partial\Omega} u^2\,\partial_n\phi - \int_\Omega 2u\nabla u\cdot\nabla\phi$$
$$\le C_1\|u\|_{\partial\Omega}^2 + C_2\|u\|\|\nabla u\| \le C_1\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|u\|_\Omega^2 + \frac{1}{2}C_2^2\|\nabla u\|_\Omega^2.$$

This yields

$$\|u\|_\Omega^2 \le 2C_1\|u\|_{\partial\Omega}^2 + C_2^2\|\nabla u\|_\Omega^2 \le C^2(\|u\|_{\partial\Omega}^2 + \|\nabla u\|_\Omega^2),$$

where $C^2 = \max(2C_1, C_2^2)$, $C_1 = \max_{\partial\Omega}|\partial_n\phi|$, and $C_2 = \max_\Omega(2|\nabla\phi|)$.

b) Multiply the equation $-\Delta u = f$ by u and integrate over Ω. Partial integration together with the boundary data $-\partial_{\mathbf{n}}u = ku$ and Cauchy's inequality, yields

$$\|\nabla u\|_\Omega^2 + k\|u\|_{\partial\Omega}^2 = \int_\Omega \nabla u\cdot\nabla u + \int_{\partial\Omega} u(-\partial_{\mathbf{n}}u) = \int_\Omega u(-\Delta u) = \int_\Omega fu$$
$$\le \|u\|_\Omega^{\|}f\|_\Omega \le C_\Omega(\|u\|_{\partial\Omega} + \|\nabla u\|_\Omega)\|f\|_\Omega = \|u\|_{\partial\Omega}C_\Omega\|f\|_\Omega + \|\nabla u\|_\Omega C_\Omega\|f\|_\Omega$$
$$\le \frac{1}{2}\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|\nabla u\|_\Omega^2 + C_\Omega^2\|f\|_\Omega^2.$$

Subtracting $\frac{1}{2}\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|\nabla u\|_\Omega^2$ from the both sides, we end up with

$$(k-\frac{1}{2})\|u\|_{\partial\Omega}^2 \le \frac{1}{2}\|\nabla u\|_\Omega^2 + (k-\frac{1}{2})\|u\|_{\partial\Omega}^2 \le C_\Omega^2\|f\|_\Omega^2,$$

which gives that $\|u\|_{\partial\Omega} \to 0$ as $k \to \infty$.

6. a) Multiply the equation by v, integrate over Ω, partial integrate, and use the boundary data to obtain

$$\int_\Omega fv = -\int_\Omega (\Delta u)v = -\int_\Gamma (\mathbf{n}\cdot\nabla u)v + \int_\Omega \nabla u\cdot\nabla v = \int_\Gamma buv - \int_\Gamma bgv + \int_\Omega \nabla u\nabla v,$$

where $\Gamma := \partial\Omega$. This can be rewritten as

$$\underbrace{\int_\Omega \nabla u\cdot\nabla v + \int_\Gamma buv}_{:=a(u,v)} = \underbrace{\int_\Omega fv + \int_\Gamma bgv}_{:=l(v)}.$$

Let now

$$F(w) = \frac{1}{2} = a(w,w) - l(w) = \frac{1}{2}\int_\Omega \nabla w\cdot\nabla w + \int_\Gamma bww - \int_\Omega fv + \int_\Gamma bgv,$$

and choose $w = u + v$, then

$$F(w) = F(u+v) = F(u) +$$
$$+\underbrace{\int_\Omega \nabla u\cdot\nabla v + \int_\Gamma buv - \int_\Omega fv + \int_\Gamma bgv}_{=0} + \underbrace{\frac{1}{2}\int_\Omega \nabla v\cdot\nabla v + \frac{1}{2}\int_\Gamma bvv}_{\ge 0} \ge F(u).$$

This gives $F(u) \le F(w)$ for arbitrary w.

b) Make the discrete ansatz $U = \sum_{j=1}^M U_j\varphi_j$, and set $v = \varphi_i$, $i = 1, 2, \ldots, M$ in the variational formulation. Then we get the equation system $AU = B$, where U is the column vector with entries U_j, B is the load vector with elements

$$B_i = \int_\Omega f\varphi_i + \int_\Gamma bg\varphi_i,$$

and A is the matrix with elements

$$A_{ij} = \int_\Omega \nabla\varphi_i\cdot\nabla\varphi_j + \int_\Gamma b\varphi_i\varphi_j.$$

Here $\varphi_j = \varphi_j(x)$ is the basis function (*hat-functions*) for the set of all piecewise linear polynomials functions on a triangulation of the domain Ω.

Finally for the energy-norm $\|v\| = a(v,v)^{1/2}$, using the definition for $U = U(x)$, and the Galerkin orthogonality, we estimate the error $e = u - U$ as

$$\begin{aligned}\|e\|^2 &= a(e,e) = a(e,u-U) = a(e,u) - a(e,U) = a(e,u)\\ &= a(e,u) - a(e,v) = a(e,u-v) \leq \|e\|\|u-v\|.\end{aligned}$$

This gives $\|u-U\| = \|e\| \leq \|u-v\|$, for arbitrary piecewise linear function v, due to the fact that for such U and v Galerkin orthogonality gives $a(e,U) = 0$ and $a(e,v) = 0$: Just notice that both U and v are the linear combination of the basis functions φ_j for which according to the definition of U we have that

$$a(e,\varphi_j) = a(u,\varphi_j) - a(U,\varphi_j) = l(\varphi_j) - l(\varphi_j) = 0.$$

In particular, we may chose the piecewise linear function v to be the interpolant u and hence get

$$\|u-U\| \leq \|u-v\| \leq C\|hD^2u\|,$$

where h is the mesh size and C is an interpolation constant independent of h and u.

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