Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2011–08–24; kl 8.30-13.30.

Telephone: Ida Säfström: 0703-088304
Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 6p. Valid bonus poits will be added to the scores.
Breakings: **3**: 15-20p, **4**: 21-27p och **5**: 28p- For GU students**G**:15-27p, **VG**: 28p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/1011/index.html

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1. Let $a(x) > 0$, and assume that u and u_h are the solutions of the Dirichlet problem:

$$\text{(1)} \qquad \text{(BVP)} \qquad -\Big(a(x)u'(x)\Big)' = f(x), \quad 0 < x < 1 \qquad u(0) = u(1) = 0,$$

and its $cG(1)$ finite element (FEM) approximation, respectively. Prove that there is a constant C_i, depending only on $a(x)$, such that

$$\text{(2)} \qquad \|u - u_h\|_E \leq C_i \|hu''\|_a.$$

2. Consider the estimate (2). Derive the exact relation that shows how C_i depends on $a(x)$.

3. Consider the two-dimensional Poisson equation with Neumann boundary condition

$$\text{(3)} \qquad -\Delta u = f, \quad \text{in } \Omega \subset \mathbf{R}^2; \qquad -\mathbf{n} \cdot \nabla u = k\, u, \ \text{ on } \partial\Omega,$$

where $k > 0$ and $\mathbf{n}$ is the outward unit normal to $\partial\Omega$ ($\partial\Omega$ is the boundary of Ω).
a) Prove the Poincare inequality:

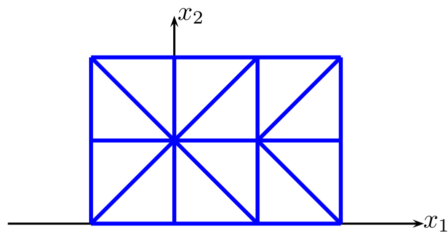
$$\|u\|_{L_2(\Omega)} \leq C_\Omega (\|u\|_{L_2(\partial\Omega)} + \|\nabla u\|_{L_2(\Omega)}).$$

b) Use the inequality in a) and show that $\|u\|_{L_2(\partial\Omega)} \to 0$ as $k \to \infty$.

4. Consider the problem

$$\text{(4)} \qquad \begin{cases} -\Delta u = f, & \text{in } \Omega = \{(x_1, x_2) : -1 < x_1 < 2,\, 0 < x_2 < 2\} \\ u = 0, & \text{0n } \Gamma = \partial\Omega, \end{cases}$$

where $f = 1$ for $x_1 < 0$ and $f = 2$ for $x_1 > 0$.
a) Write down the discrete system $SU = \mathbf{b}$ (S is the stiffness matrix and $\mathbf{b}$ is the load vector) in approximating (4) by cG(1) FEM in the following triangulation:



b) Consider the same problem as in a), replacing the Dirichlet data $u = 0$ (only) on $x_1 = 2$ by the Neumann data: $\partial_n u = 0$ on $x_1 = 2$, $0 < x_2 < 2$.

5. a) Formulate a relevant minimization problem for the solution of the Poisson equation

$$\text{(5)} \qquad -\Delta u = f, \quad \text{in } \Omega \in \mathbf{R}^2, \qquad \text{with} \quad \mathbf{n} \cdot \nabla u = b(g - u), \quad \text{on } \partial\Omega,$$

where $f > 0$, $b > 0$ and g are given functions.
b) Derive an *a priori* error estimate for cG(1) approximation in the corresponding energy-norm.
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Lösningar/Solutions.

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1. See Lecture Notes or text book chapter 8.

2. Note that the interpolation theorem is not in the weighted norm. The $a(x)$ dependence of the interpolation constant C_i can be shown as follows:

$$
\begin{aligned}
\|u' - (\pi_h u)'\|_a &= \Big(\int_0^1 a(x)(u'(x) - (\pi_h u)'(x))^2\,dx\Big)^{1/2}\\
&\le \Big(\max_{x\in[0,1]} a(x)^{1/2}\Big)\cdot \|u' - (\pi_h u)'\|_{L_2} \le c_i\Big(\max_{x\in[0,1]} a(x)^{1/2}\Big)\|hu''\|_{L_2}\\
&= c_i\Big(\max_{x\in[0,1]} a(x)^{1/2}\Big)\Big(\int_0^1 h(x)^2 u''(x)^2\,dx\Big)^{1/2}\\
&\le c_i\frac{(\max_{x\in[0,1]} a(x)^{1/2})}{(\min_{x\in[0,1]} a(x)^{1/2})}\cdot\Big(\int_0^1 a(x)h(x)^2u''(x)^2\,dx\Big)^{1/2}.
\end{aligned}
$$

Thus

$$
C_i = c_i\frac{(\max_{x\in[0,1]} a(x)^{1/2})}{(\min_{x\in[0,1]} a(x)^{1/2})}, \tag{6}
$$

where c_i is the interpolation constant independent of $a(x)$.

3. a) There is smooth function ϕ such that $\Delta\phi = 1$ so that, using Greens formula

$$
\begin{aligned}
\|u\|_\Omega^2 &= \int_\Omega u^2\,\Delta\phi = \int_{\partial\Omega} u^2\,\partial_n\phi - \int_\Omega 2u\nabla u\cdot\nabla\phi\\
&\le C_1\|u\|_{\partial\Omega}^2 + C_2\|u\|\|\nabla u\| \le C_1\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|u\|_\Omega^2 + \frac{1}{2}C_2^2\|\nabla u\|_\Omega^2.
\end{aligned}
$$

This yields

$$
\|u\|_\Omega^2 \le 2C_1\|u\|_{\partial\Omega}^2 + C_2^2\|\nabla u\|_\Omega^2 \le C^2(\|u\|_{\partial\Omega}^2 + \|\nabla u\|_\Omega^2),
$$

where $C^2 = \max(2C_1, C_2^2)$, $C_1 = \max_{\partial\Omega}|\partial_n\phi|$, and $C_2 = \max_\Omega(2|\nabla\phi|)$.

b) Multiply the equation $-\Delta u = f$ by u and integrate over Ω. Partial integration together with the boundary data $-\partial_{\mathbf{n}}u = ku$ and Cauchy's inequality, yields

$$
\begin{aligned}
\|\nabla u\|_\Omega^2 + k\|u\|_{\partial\Omega}^2 &= \int_\Omega \nabla u\cdot\nabla u + \int_{\partial\Omega} u(-\partial_{\mathbf{n}}u) = \int_\Omega u(-\Delta u) = \int_\Omega fu\\
&\le \|u\|_\Omega^{\|}f\|_\Omega \le C_\Omega(\|u\|_{\partial\Omega} + \|\nabla u\|_\Omega)\|f\|_\Omega = \|u\|_{\partial\Omega}C_\Omega\|f\|_\Omega + \|\nabla u\|_\Omega C_\Omega\|f\|_\Omega\\
&\le \frac{1}{2}\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|\nabla u\|_\Omega^2 + C_\Omega^2\|f\|_\Omega^2.
\end{aligned}
$$

Subtracting $\frac{1}{2}\|u\|_{\partial\Omega}^2 + \frac{1}{2}\|\nabla u\|_\Omega^2$ from the both sides, we end up with

$$
(k - \frac{1}{2})\|u\|_{\partial\Omega}^2 \le \frac{1}{2}\|\nabla u\|_\Omega^2 + (k - \frac{1}{2})\|u\|_{\partial\Omega}^2 \le C_\Omega^2\|f\|_\Omega^2,
$$

which gives that $\|u\|_{\partial\Omega} \to 0$ as $k\to\infty$.

4. Let V be the linear function space defined by

$$V_h := \{v : v \text{ is continuous in } \Omega,\ \ v = 0,\ \text{ on } \partial\Omega\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$-(\Delta u, \nabla v) = (\nabla u, \nabla v) - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds = (\nabla u, \nabla v), \qquad \forall v \in V.$$

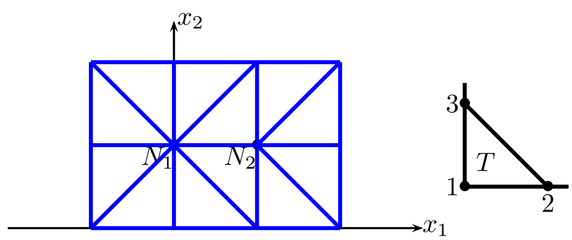
Thus the variational formulation is:

$$(\nabla u, \nabla v) = (f, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega$: The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) = (f, v) \qquad \forall v \in V_h$$

With this boundary conditions we have the internal nodes N_1 and N_2. Making the "Ansatz"



$U(x) = \sum_{j=1}^{2} \xi_j \varphi_j(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{i=1}^{2} \xi_j \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx = \int_\Omega f \varphi_j \, dx, \quad i = 1, 2,$$

or, in matrix form,

$$S\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix and $F_j = (f, \varphi_j)$ is the load vector. We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1, x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1, x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$\phi_2(x_1, x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

$$\phi_3(x_1, x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2}|T| = 1.$$

Similarly we can compute the other elements and obtain

$$s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrix S from the local one s:

$$S_{11} = 8s_{22} = 4, \qquad S_{12} = 2s_{12} = -1,$$

$$S_{21} = 2s_{12} = -1, \qquad S_{22} = 2s_{11} + 4s_{22} = 2 + 2 = 4$$

As for the load vector we have

$$\int_\Omega f\varphi_1 = \int_{x_1<0} \varphi_1 + 2\int_{x_1>0} \varphi_1 = 4\cdot\frac{1}{3}\cdot\frac{1}{2} + 2\cdot 4\cdot\frac{1}{3}\cdot\frac{1}{2} = 2/3 + 4/3 = 2.$$

$$\int_\Omega f\varphi_2 = 2\int_{x_1>0} \varphi_2 = 2\cdot 6\cdot\frac{1}{3}\cdot\frac{1}{2} = 2$$

Thus the equation system is given by

$$\begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

b) With the Neumann boundary data we obtain an addition node at $N_3 = (2,1)$ with the obvious corresponding basis function φ_3 which gives rise to an additional row and an additional column viz,

$$\int_\Omega \nabla\varphi_3\cdot\nabla\varphi_3 = 2, \quad \int_\Omega \nabla\varphi_2\cdot\nabla\varphi_3 = \int_\Omega \nabla\varphi_3\cdot\nabla\varphi_2 = -1 \quad \int_\Omega f\varphi_3 = 2\cdot\frac{1}{2}.$$

Consequently the corresponding system reads as

$$\begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2/3 \end{bmatrix}.$$

5. a) Multiply the equation by v, integrate over Ω, partial integrate, and use the boundary data to obtain

$$\int_\Omega fv = -\int_\Omega (\Delta u)v = -\int_\Gamma (\mathbf{n}\cdot\nabla u)v + \int_\Omega \nabla u\cdot\nabla v = \int_\Gamma buv - \int_\Gamma bgv + \int_\Omega \nabla u\nabla v,$$

where $\Gamma := \partial\Omega$. This can be rewritten as

$$\underbrace{\int_\Omega \nabla u\cdot\nabla v + \int_\Gamma buv}_{:=a(u,v)} = \underbrace{\int_\Omega fv + \int_\Gamma bgv}_{:=l(v)}.$$

Let now

$$F(w) = \frac{1}{2} = a(w,w) - l(w) = \frac{1}{2}\int_\Omega \nabla w\cdot\nabla w + \int_\Gamma bww - \int_\Omega fv + \int_\Gamma bgv,$$

and choose $w = u + v$, then

$$F(w) = F(u+v) = F(u) +$$
$$+ \underbrace{\int_\Omega \nabla u\cdot\nabla v + \int_\Gamma buv - \int_\Omega fv + \int_\Gamma bgv}_{=0} + \underbrace{\frac{1}{2}\int_\Omega \nabla v\cdot\nabla v + \frac{1}{2}\int_\Gamma bvv}_{\geq 0} \geq F(u).$$

This gives $F(u) \leq F(w)$ for arbitrary w.

b) Make the discrete ansatz $U = \sum_{j=1}^{M} U_j\varphi_j$, and set $v = \varphi_i$, $i = 1, 2, \ldots, M$ in the variational formulation. Then we get the equation system $AU = B$, where U is the column vector with entries U_j, B is the load vector with elements

$$B_j = \int_\Omega f\varphi_i + \int_\Gamma bg\varphi_i,$$

and A is the matrix with elements

$$A_{ij} = \int_\Omega \nabla\varphi_i\cdot\nabla\varphi_j + \int_\Gamma b\varphi_i\varphi_j.$$

Here $\varphi_j = \varphi_j(x)$ is the basis function (*hat-functions*) for the set of all piecewise linear polynomials functions on a triangulation of the domain Ω.

Finally for the energy-norm $\|v\| = a(v,v)^{1/2}$, using the definition for $U = U(x)$, and the Galerkin orthogonality, we estimate the error $e = u - U$ as

$$\|e\|^2 = a(e,e) = a(e,u-U) = a(e,u) - a(e,U) = a(e,u)$$
$$= a(e,u) - a(e,v) = a(e,u-v) \leq \|e\|\|u-v\|.$$

This gives $\|u - U\| = \|e\| \leq \|u - v\|$, for arbitrary piecewise linear function v, due to the fact that for such U and v Galerkin orthogonality gives $a(e,U) = 0$ and $a(e,v) = 0$: Just notice that both U and v are the linear combination of the basis functions φ_j for which according to the definition of U we have that

$$a(e,\varphi_j) = a(u,\varphi_j) - a(U,\varphi_j) = l(\varphi_j) - l(\varphi_j) = 0.$$

In particular, we may chose the piecewise linear function v to be the interpolant u and hence get

$$\|u - U\| \leq \|u - v\| \leq C\|hD^2u\|,$$

where h is the mesh size and C is an interpolation constant independent of h and u.

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