1.6 sid 23 a) Skriv ekvationen $u_{xx}+2u_{xy}+u_{yy}=0$, i nya koordinater:

$$s=x, \qquad t=x-y$$

Lösning: Använd kedjeregeln:

i) $\underline{\underline{u_x}} = \frac{\partial u}{\partial s}\cdot\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\cdot\frac{\partial t}{\partial x} = \left\{\frac{\partial s}{\partial x}=1;\ \frac{\partial t}{\partial x}=1\right\} = \underline{\underline{u_s+u_t}}.$

Alltså $\underline{\underline{\partial_x(\circ) = (\partial_s+\partial_t)(\circ)}}$ $(*)$

$(*) \Rightarrow \underline{\underline{u_{xx}}} = \partial_x(u_x) = (\partial_s+\partial_t)(u_x) \overset{(*)}{=} \partial_s(u_s+u_t)+\partial_t(u_s+u_t)$

$= \underline{\underline{u_{ss}+2u_{st}+u_{tt}}}$

ii) $\underline{\underline{u_y}} = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\overset{0}{\cancel{\frac{\partial s}{\partial y}}} + \frac{\partial u}{\partial t}\overbrace{\frac{\partial t}{\partial y}}^{=-1} = -\frac{\partial u}{\partial t} \Rightarrow \underline{\underline{u_y=-u_t.}}$

Alltså $\partial_y(\circ) = -\partial_t(\circ)$ $(**)$

$\Rightarrow \underline{\underline{u_{xy}}} = \partial_x(\partial_y u) \overset{(**)}{=} \partial_x(-u_t) \overset{(*)}{=} \partial_s(-u_t)+\partial_t(-u_t) = \underline{\underline{-u_{st}-u_{tt}}}.$

$\underline{\underline{u_{yy}}} = \frac{\partial}{\partial y}(\partial_y u) = \frac{\partial}{\partial y}(-u_t) \overset{(**)}{=} -\frac{\partial}{\partial t}(u_t) = \underline{\underline{+u_{tt}}}$

$\therefore u_{xx}+2u_{xy}+u_{yy} = u_{ss}+\cancel{2u_{st}}+\cancel{u_{tt}}-\cancel{2u_{st}}-\cancel{2u_{tt}}+\cancel{u_{tt}} = u_{ss}=0$

b) Finn allmänna lösningen:

$u_{ss}=0 \Rightarrow u_s(s,t)=A(t) \Rightarrow u(s,t)=A(t)s+B(t)$

$\therefore \underline{\underline{u(x,y)=A(x-y)s+B(x-y)}}$

c) Skriv om ekvationen $u_{xx}-2u_{xy}+5u_{yy}=0$ i koordinater:

$s=x+y \wedge t=2x$

p.s.s. i a) ... $u_x = \frac{\partial u}{\partial s}\cdot\overbrace{\frac{\partial s}{\partial x}}^{1} + \frac{\partial u}{\partial t}\cdot\overbrace{\frac{\partial t}{\partial x}}^{2} = u_s+2u_t \wedge u_y = \frac{\partial u}{\partial s}\overbrace{\frac{\partial s}{\partial y}}^{=1}+\frac{\partial u}{\partial t}\overset{0}{\cancel{\frac{\partial t}{\partial y}}} = u_s$

Alltså $\partial_x(\circ) \underset{(C*)}{=} (\partial_s+2\partial_t)(\circ) \wedge \partial_y(\circ) \underset{(C**)}{=} \partial_s(\circ)$

$$\underline{\underline{u_{xx}}} = \partial_x(u_x) = (\partial_s + 2\partial_t)(u_s + 2u_t)$$

$$= u_{ss} + 2u_{st} + 2u_{ts} + 4u_{tt} = \underline{\underline{u_{ss} + 4u_{st} + 4u_{tt}}} \quad (C1)$$

$$\underline{\underline{u_{xy}}} = \partial_x(u_s) \underset{(c*)}{=} (\partial_s + 2\partial_t)(u_s) = \underline{\underline{u_{ss} + 2u_{st}}}. \quad (C2)$$

$$\underline{\underline{u_{yy}}} = \partial_y(\partial_y u) \underset{(c**)}{=} \frac{\partial}{\partial y}(u_s) \underset{(c**)}{=} \frac{\partial}{\partial s}(u_s) = \underline{\underline{u_{ss}}}. \quad (C3)$$

(C1), (C2) & (C3) $\Rightarrow$

$$0 = u_{xx} - 2u_{xy} + 5u_{yy} \Rightarrow u_{ss} + \cancel{4u_{st}} + 4u_{tt} - 2u_{ss} - \cancel{4u_{st}} + 5u_{ss} = 0$$

$$\Leftrightarrow 4u_{ss} + 4u_{tt} = 0 \Leftrightarrow \underline{\underline{u_{ss} + u_{tt} = 0}}.$$

Mao: om $u(x,y) := v(s,t)$, så $v_{ss} + v_{tt} = 0$. $\boxtimes$