Pattern and position based permutation statistics

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Abstract

Falling permutation patterns of length two where we allow to mark adjacency both in position and value give rise to three statistics, namely the adjacency ($\pi_i - \pi_{i+1} = 1$), the descent and the inversion. The two former are equidistributed with the position based statistics fixed points and excedances, and the latter is Mahonian and thus equidistributed with many statistics. We prove that the bistatistic of adjacencies and descents is equidistributed with the bistatistic of fixed points and excedances and discuss different possibilities to extend this equidistribution to include some Mahonian statistics as well. In our proof, we use a new bijection between permutations and permutation tableaux that maps descents to rows and adjacencies to empty rows.

1 Introduction

Permutation statistics based on permutation patterns have been studied for quite some time. Indicative of this is naming statistics equidistributed with descents (the pattern 21 in generalised pattern notation, that is $\pi_i > \pi_{i+1}$) **Eulerian** and referring to statistics equinumerous with inversions (the pattern $2-1$, that is $\pi_i > \pi_j$ for $i < j$) as **Mahonian**. These are the only two statistics which can be obtained using generalised patterns of length two.

One may of course define statistics based on other properties of permutations. Comparing each permutation element with its position gives position based statistics, such as the number of fixed points ($\pi_i = i$) and the number of excedances (that is $\pi_i > i$). Interestingly, excedances is an Eulerian statistic and one popular branch of permutation pattern research is finding equidistributed pairs of statistics including one Eulerian statistic and one Mahonian statistic. While the Eulerian statistics used are descents and excedances, there are many Mahonian statistics (see for instance [1, 2] and references therein) and some of these may be defined using position information (for instance the Denert statistic and actually the inversion statistic [2]). There is, however, no obvious position based counterpart of inversions regarded as a pattern based statistic.

A pattern based statistic which has not been studied much as a permutation statistic is **adjacencies**, which are descents descending only one step (that is $\pi_i - \pi_{i+1} = 1$). They are common in bioinformatical studies of gene order, where the number of breakpoints (positions without adjacencies) is an often used measure on the evolutionary difference between two species' gene orders (see for instance several chapters in [5]). In this paper we make the natural generalisation of generalised patterns to be able to include the adjacency among the patterns of length two.

Furthermore, adjacencies are equidistributed with fixed points! This is easy to see by noting that we can add and remove fixed points in the same way as we can remove and add adjacencies, and adjacency free permutations can be shown to obey the same recursion as derangements. This gives a third family of equidistributed statistics to be studied in connection with Eulerian and Mahonian statistics. We initialise these studies by showing that the pattern based pair of adjacencies and descent are equidistributed with the position based pair of fixed points and excedances. This result is obtained by constructing a bijection to permutation tableaux which maps descents to rows and adjacencies to empty rows. With the already known bijection to

Table 1: Doubly generalised patterns, allowing to indicate both for adjacencies in value and in position.

Notation	Description	Common name
21	$\pi_i - \pi_{i+1} = 1$	Adjacency
2–1	$\pi_i - \pi_j = 1, j > i$	
$2\vert 1$	$\pi_i - \pi_{i+1} \geq 1$	Descent
2+1	$\pi_i > \pi_j, i < j$	Inversion

permutation tableaux which maps weak excedances to rows and fixed points to empty rows, the result follows directly.

2 Doubly generalised permutation statistics

We use $\mathfrak{S}_n$ to denote the symmetric group on n elements. For $\pi \in \mathfrak{S}_n$, we use the common notation $\pi_i = \pi(i)$ and brackets for one-line notation: $\pi = [\pi_1 \cdots \pi_n]$. Let $\mathfrak{S}_n^0 = \{\mu = [\pi\ 0] : \pi \in \mathfrak{S}_n\}$ be the set of permutations on n elements with $\mu_{n+1} = 0$. Assuming $\pi \in \mathfrak{S}_n^0$ or $\pi \in \mathfrak{S}_n$, let the set of **adjacencies** in π be $\mathrm{Adj}(\pi) = \{i : \pi_i - \pi_{i+1} = 1\}$. The set of **descents** in π is denoted $\mathrm{Des}(\pi) = \{i : \pi_i > \pi_{i+1}\}$ and the set of **ascents** is $\mathrm{Asc}(\pi) = \{i : \pi_i < \pi_{i+1}\}$. The set of **fixed points** is $\mathrm{Fix}(\pi) = \{i : \pi_i = i\}$ and the set of **weak excedances** is $\mathrm{Exc}(\pi) = \{i : \pi_i \geq i\}$. Finally, the set of **inversions** is $\mathrm{Inv}(\pi) = \{(i,j) : i < j, \pi_i > \pi_j\}$. We also define $\mathrm{adj}(\pi) = |\mathrm{Adj}(\pi)|$ and $\mathsf{Adj}(n,k) = \{\pi \in \mathfrak{S}_n^0 : \mathrm{adj}(\pi) = k\}$. The numbers $\mathrm{des}(\pi)$, $\mathrm{asc}(\pi)$ et cetera and the sets $\mathsf{Des}(n,k)$, $\mathsf{Asc}(n,k)$ et cetera are defined analogously.

Babson and Steingrímsson [1] introduced generalised patterns, which differentiate between patterns where the elements in the pattern must be adjacent in position (the pattern 21 indicates a descent) and the patterns where the elements may have any gap between them (the pattern 2–1 indicates an inversion). However, there is no way to indicate adjacency in value using generalised patterns. Of course, a positional adjacency in π will be an adjacency in value in π^{-1}. But the adjacencies defined above, which are adjacent in both position and value are harder to describe. We thus introduce the notation of doubly generalised patterns given in Table 1. A horizontal line indicates that the difference in position between the two elements in the pattern may be larger than 1, and a vertical line indicates that the difference in value between the two elements may be larger than 1.

We compare pattern based statistics on $\mathfrak{S}_n^0$ with position based statistics on $\mathfrak{S}_n$ and $\mathfrak{S}_n^0$. In particular, we show that the bistatistics $\{\mathsf{Adj}, \mathsf{Des}\}$ and $\{\mathsf{Fix}, \mathsf{Exc}\}$ on $\mathfrak{S}_n^0$ are equidistributed. We also discuss possible extensions to include Mahonian statistics as well.

3 Neighbour crossing partitions and pattern based permutation tableaux

Permutation tableaux are great for highlighting permutation statistics such as fixed points and excedances, but they work less well for pattern statistics such as adjacencies and descents. This was partly remedied by the tableaux bijection of Corteel [3] which had the number of rows in the tableaux to equal the number of descents in the corresponding permutation. We wish to take this one step further, such that the number of empty rows also equals the number of adjacencies. To reach that goal, we must however take a detour through yet another representation of a permutation, in the shape of the neighbour crossing partition.

Definition 3.1. *Consider* $u : \{0, \ldots, n+1\} \to [k]$ *as the vector* $(u(0), u(1), \ldots, u(n))$ *and let* $u^{-1}(m) = \{j : u(j) = m\}$. *A* **neighbour crossing partition** *(ncp) on* n *elements in* k *parts is a surjective function* $u : \{0, \ldots, n\} \to [k]$ *such that* $u(0) = k$ *and for* $1 \leq m \leq k-1$, $\min u^{-1}(m) <$

$\max u^{-1}(m+1)$. *The set of such neighbour crossing partitions is denoted* $\mathrm{NeighCP}(n,k)$ *and the number of parts in the partition is denoted* $k(u)$.

Lemma 3.1. *We have* $|\mathrm{NeighCP}(n,k)| = A(n,k)$, *where* $A(n,k)$ *are the Eulerian numbers, that is the number of permutations* $\pi \in \mathfrak{S}_n$ *with* $k-1$ *descents.*

Proof. We note that $A(n,k)$ can also be characterised as the number of permutations $\pi \in \mathfrak{S}_n$ with $k-1$ ascents, which is equinumerous to the number of permutations $\mu \in \mathfrak{S}_n^0$ with $k-1$ ascents, since for $\mu = [\pi\ 0]$, $\mathrm{asc}(\mu) = \mathrm{asc}(\pi)$. We will now display a bijection between $\mathrm{NeighCP}(n,k)$ and $\mathsf{Asc}(n,k-1)$.

Assume $\mu \in \mathsf{Asc}(n,k-1) \subseteq \mathfrak{S}_n^0$. Define $\Gamma : \mathsf{Asc}(n,k-1) \to \mathrm{NeighCP}(n,k)$ by $(\Gamma(\mu))(\ell) = |\{a \in \mathrm{Asc}(\mu) : a < \mu^{-1}(\ell)\}| + 1$. We need to show $\Gamma(\mu) \in \mathrm{NeighCP}(n,k)$. Let $u = \Gamma(\mu)$. It is clear that u is surjective, since $u(\mu(1)) = 1$, $u(\mu(n)) = k$ and $u(\mu(j+1)) - u(\mu(j)) \in \{0,1\}$ for $1 \le j \le n-1$. It is also clear that $u(0) = k$. Further, let j be the mth smallest element in $\mathrm{Asc}(\pi)$. Then, $\min u^{-1}(m) = \mu(j) < \mu(j+1) = \max u^{-1}(m+1)$. Hence we have obtained a neighbour crossing partition. It is not hard to see that the algorithm can be easily reversed. $\square$

Example 3.1. *Consider the permutation* $\mu = [7\ 5\ 2\ 1\ 6\ 3\ 4\ 0] \in \mathsf{Asc}(7,3)$. *In* $u = \Gamma(\mu)$, *the positions* $\{7,5,2,1\}$ *are given the value* 1 *since they are in the first descending sequence,* 6 *and* 3 *get the value* 2 *since the are in the second descending sequence and* 4 *and* 0 *gets the value* 3. *Hence we get* $u = (3,1,1,2,3,1,2,1)$. *From* u, *we see that the first descending sequence contains* 1, 2, 5 *and* 7 *and sorting these in descending order gives the beginning of* μ. *Continuing in a similar fashion, it is not hard to restore* μ.

Permutation tableaux have been extensively studied during the last couple of years. They consist of a Young tableau Y_λ, allowing zero parts in λ, where each cell is either filled or empty. We put the largest part on top and let $\ell(\lambda)$ denote the number of parts in λ, including the zero parts. Also, the **shape** of the tableaux $\mathcal{T}$ is written $\mathrm{sh}(\mathcal{T}) = \lambda$. The tableau must also adhere to the following two rules: Each column of the tableau must have at least one filled cell and if a cell has at least one filled cell above and at least one filled cell to its right, then it must be filled.

Let $\mathfrak{T}_n^1$ denote the set of permutation tableaux such that $\lambda_1 + \ell(\lambda) = n$ for $\mathrm{sh}(\mathcal{T}) = \lambda$. There is a natural bijection $\Phi_1 : \mathfrak{T}_n^1 \to \mathfrak{S}_n$ [6]. Starting in the north-east corner, label the south-east boundary of Y_λ with the numbers 1 to n, where $n = \lambda_1 + \ell(\lambda)$. Extend these labels to the rows and columns they appear in. Now, consider the zig-zag path starting at the row or column labelled i and turning south or east every time it hits a filled cell. Letting $\pi(i) = j$ if the row or column where the path ends is labelled j gives a permutation with the properties that each empty row corresponds to a fixed point in the permutation, which is obvious from the description of the bijections, and each row corresponds to a weak excedance in the partition, which is less obvious, but not hard to see. Since these properties are related to each element's position in the permutation, we call the tableaux related to π via this bijection **position based permutation tableaux**.

We will now define a bijection $\Lambda : \mathrm{NeighCP}(n,k) \to \mathfrak{T}_n^1$ such that the number of rows in $\Lambda(u)$ is given by the number of descents in $\Gamma^{-1}(u)$ and the number of empty rows in $\Lambda(u)$ is given by the number of adjacencies in $\Gamma^{-1}(u)$. Since these properties are related to doubly generalised patterns in the permutation, we will call the related tableaux **pattern based permutation tableaux**.

We start by constructing the integer partition $\lambda \vdash n$ which gives the shape of the tableau. Traverse through u from the left. To obtain the south-east border of Y_λ, we start in the south-west corner. Position $i > 0$ in u contributes a horizontal step if i is the leftmost position in u with value $u(i) < k$ and a vertical step otherwise.

Example 3.2. *We will use the permutation* $\mu = [7\ 9\ 3\ 8\ 2\ 4\ 6\ 5\ 1\ 0] \in \mathfrak{S}_9^0$ *to exemplify how the bijection* Λ *works. We have*

$$\Gamma([7\ 9\ 3\ 8\ 2\ 4\ 6\ 5\ 1\ 0]) = (5,5,3,2,4,5,5,1,3,2).$$

Disregarding $u(0)$, *we traverse* u *from the left and first find an element with the top value* $k=5$, *which gives a vertical step. Then we find three positions all containing elements less than* k

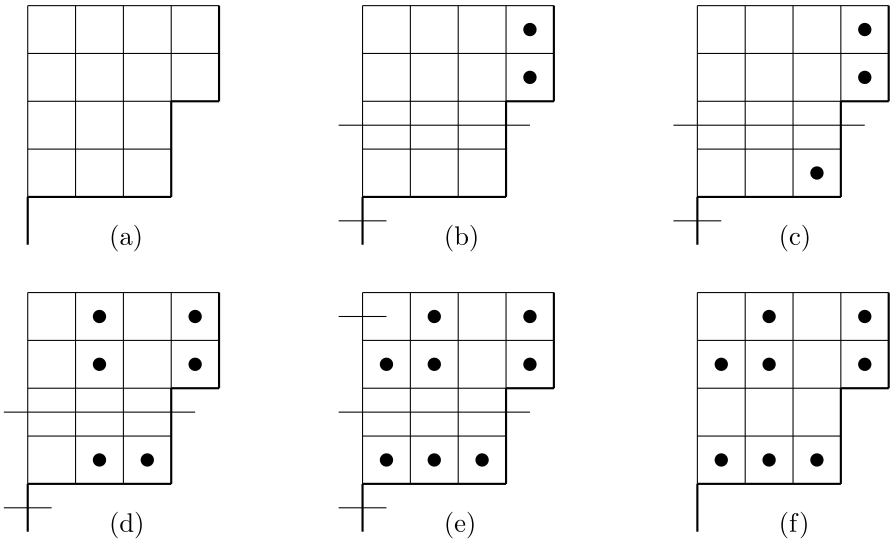
(a) (b) (c) (d) (e) (f)

Figure 1: The pattern based permutation tableaux of $u = (5, 5, 3, 2, 4, 5, 5, 1, 3, 2)$, where $\Gamma^{-1}(u) = [7\ 9\ 3\ 8\ 2\ 4\ 6\ 5\ 1\ 0]$. We iteratively fill cells in the columns from right to left. The line over a row indicates that that row is inactive and should not have any more filled cells.

which are the leftmost such elements. Thus, we continue with three horizontal steps. In the forth and fifth positions, the elements are k *and thus render vertical steps. The sixth position is the leftmost occurrence of* 1*, giving a horizontal step, and the last two positions have elements which can also be found to their left, ending the path with two vertical steps. The final shape can be found in Figure 1 (a).*

Given the shape of the tableau $\mathcal{T}$, we work out which cells to fill one column at the time, moving from right to left. We will also make rows inactive when we find that there should be no more filled cells in them. To this end, consider u. If there are any repeated elements in adjacent positions, remove the rightmost of the duplicates. Let $\ell(u)$ be the number of positions in u. If the removed element was in position $\ell(u)-j+1$, then the row at the jth topmost active segment of the south-east border is to be made inactive.

Having reduced u to a vector without duplicates, find the position

$$p = \max_{1\leq j\leq k(u)-1} \min u^{-1}(j).$$

Assume that $u(p) = a$ and $u(p+1) = b$. Find the longest consecutive alternating pattern $abab\ldots$ of a and b and change each a into b and vice versa. This pattern may end in either a or b. Then, the leftmost occurrence of a is at position $p + 1$ and we repeat the procedure at that position until the vector no longer is neighbour crossing. This occurs when all occurrences of a are to the right of all occurrences of $a + 1$.

For each step above, we fill the cell in the bottommost unfilled active row in column $k(u)-1$. Then, remove the leftmost occurrence of a and reduce all elements larger than a by one. Also, make column $k(u) - 1$ inactive. Now iterate this procedure, removing repeated elements in adjacent positions and filling the cells of the next column until the leftmost column has been filled.

Example 3.3. *We continue with the tableau whose shape we computed in the previous example. The neighbour crossing partition we considered was* $u = (5, 5, 3, 2, 4, 5, 5, 1, 3, 2)$*. It contains two repetitions, namely the element* 5 *in positions* 0 *and* 1 *and in* 5 *and* 6*. We first remove the* 5 *in position* $6 = 9 - 4 + 1$ *and make the fourth topmost segment along the south-east border, that is the third row, inactive. Then, removing the* 5 *in position* $1 = 8 - 8 + 1$ *tells us to inactivate the* 8*th active segment from the top, which is the fifth row. We now have* $u = (5, 3, 2, 4, 5, 1, 3, 2)$*.*

The position p we seek is 5, which contains the leftmost occurrence of the element $a = 1$. When moving this element to the right, the longest consecutive segments are only two positions long, so we just exchange positions of 1 and 3, and of 1 and 2, respectively, until all occurrences of $2 = 1+1$ are to the left of the single occurrence of 1: $u = (5, 3, 2, 4, 5, 3, 2, 1)$. We then remove the 1 and reduce all larger elements by one to obtain $u = (4, 2, 1, 3, 4, 2, 1)$. Since 1 was moved twice, we fill the two bottommost active rows in column 4 (see Figure 1 (b)).

This time u contains no repetitions, and we proceed directly with the element 3 in position 3. We exchange the positions of elements 3 and 4, remove the 3 and reduce the higher elements by one to obtain $u = (3, 2, 1, 3, 2, 1)$. One cell should be filled in that column (Figure 1 (c)).

For column 2 we have $p = 2$ and $a = 1$. Exchanging 1 and 3 gives $u = (3, 2, 3, 1, 2, 1)$. Then, the longest consecutive segment of 1 and 2 is three positions long, and exchanging elements gives $u = (3, 2, 3, 2, 1, 2)$. We need one more exchange before all occurrences of 2 are to the left of the rightmost 1, giving $u = (3, 2, 3, 2, 2, 1)$. We then remove 1, reduce the larger elements by one to obtain $u = (2, 1, 2, 1, 1)$ and fill three cells in column 2 (Figure 1 (d)).

For the last column, we start by removing the last 1 in u and make the topmost row inactive. We then transform $u = (2, 1, 2, 1)$ into $u = (2, 2, 1, 2)$ and $u = (2, 2, 2, 1)$, adding to filled cells in column 1 (Figure 1 (e) and (f)). We are done!

Theorem 3.2. *The map Λ described above is a bijection.*

Proof. The resulting tableau is clearly a permutation tableau. It is also quite clear that the algorithm can be applied to all neighbour crossing partitions, since as soon as some neighbours get uncrossed, the lower one of them is removed. But can the map be inverted?

Given the ncp u obtained after making the first i columns active, we will show that we can obtain u after restoring the next column. We must insert the element removed by Λ. But the number j of empty active rows in column $i + 1$ gives the position from the right where the element should be inserted, and the value $a = u(\ell(u) - j + 1)$ should be the same as is already present there. We thus increase all values a or larger and then insert a at position $\ell(u) - j$.

The procedure where the segment of a and b exchanges values is obviously invertible, since the number of filled cells in column $i + 1$ gives the number of times we iterate it. Finally, we need to insert duplicates if there are any rows becoming active after column $i + 1$. But again, the position of the duplicates is given by the number of active rows above, and the value is the same as the one already present there. Hence we have the ncp u after making $i + 1$ columns active and conclude that Λ is invertible. □

Theorem 3.3. *For $\mu \in \mathfrak{S}_n^0$, the number of rows in $(\Lambda \circ \Gamma)(\mu)$ is given by $\operatorname{des}(\mu)$ and the number of empty rows in is given by $\operatorname{adj}(\mu)$.*

The proof is straightforward.

Corollary 3.4. *The bistatistics $\{\mathsf{Adj}, \mathsf{Des}\}$ and $\{\mathsf{Fix}, \mathsf{Exc}\}$ on $\mathfrak{S}_n^0$ are equidistributed.*

Proof. The statistics Fix and Exc have the same distribution on $\mathfrak{S}_n^0$ as on $\mathfrak{S}_n$. Let $\Theta = (\Phi_1 \circ \Lambda \circ \Gamma)$. Θ is a bijection from $\mathfrak{S}_n^0$ to $\mathfrak{S}_n$ such that $\operatorname{fix}(\Theta(\mu)) = \operatorname{adj}(\mu)$ and $\operatorname{exc}(\Theta(\mu)) = \operatorname{des}(\mu)$. □

4 Inversions in position based tableaux

A natural question to ask is what properties of a permutation can be easily found in its position based or pattern based permutation tableau. We know that the number of rows and empty rows in the position based tableau give the number of fixed points and weak excedances, and in the pattern based tableau these statistics equal the number of adjacencies and descents. Unfortunately, the number of inversions seems hard to compute from the patterns based tableau, although for instance the single column case is completely understood, but we can actually compute the number of inversions in a permutation from its position based tableau.

Consider the position based tableau $\mathcal{T} = \Phi_1^{-1}(\pi)$ of the permutation $\pi \in \mathfrak{S}_n$. For each unfilled cell, label it EE if both paths crossing there originated from rows, NN if both paths originated from columns and EN if one of the paths originated from a row and the other from a column. Given $\lambda = \mathrm{sh}(\mathcal{T})$, all cells in the rectangle $(\lambda_1^{\ell(\lambda)})$ but not in λ are labelled NE. Finally, let $\mathrm{NE}(\mathcal{T})$ be the number of cells labelled NE in $\mathcal{T}$, and similarly for the other labels. These statistics on $\mathcal{T}$ are closely related to statistics on π defined by Sylvie Corteel [4].

It is not hard to show that, given $k = \mathrm{exc}(\pi)$, we have

$$\mathrm{inv}(\pi) = k(n-k) - \mathrm{EN}(\mathcal{T}) - \mathrm{NE}(\mathcal{T}) + \mathrm{EE}(\mathcal{T}) + \mathrm{NN}(\mathcal{T}).$$

We define

$$\mathrm{dnv}(\pi) = \mathrm{des}(\pi)(n - \mathrm{des}(\pi)) - \mathrm{EN}((\Lambda\circ\Gamma)(\pi)) - \mathrm{NE}((\Lambda\circ\Gamma)(\pi)) + \mathrm{EE}((\Lambda\circ\Gamma)(\pi)) + \mathrm{NN}((\Lambda\circ\Gamma)(\pi)).$$

Then, the following theorem follows directly from the preceding discussion.

Theorem 4.1. *The triple statistics* $\{\mathsf{Fix}, \mathsf{Exc}, \mathsf{Inv}\}$ *and* $\{\mathsf{Adj}, \mathsf{Des}, \mathsf{Dnv}\}$ *on* $\mathfrak{S}_n^0$ *are equidistributed.*

5 Conjectures and open problems

- We conjecture that $(\mathsf{Adj}, \mathsf{Des}, \mathsf{Stat})$ (the statistic Stat is defined in [1]) is equidistributed with $(\mathsf{Adj}, \mathsf{Des}, \mathsf{Maj})$.
- Is there a simpler description for the Dnv statistic?
- What known Mahonian statistics can be interpreted for position based tableaux (except for inversions) and pattern based tableaux? In particular, we seek an interpretation for inversions in pattern based tableaux.
- The alternative tableaux of Viennot [7] seem very interesting in this context. For instance, we conjecture that given the statistic Astat given by

  $$\mathrm{Astat}(\mathcal{T}) = \binom{\mathrm{rows}(\mathcal{T})+1}{2} + \text{red or blue cells} + \text{cells covered by red or blue cells}$$

  is Mahonian. We further conjecture that $(\mathrm{rows}, \mathsf{Astat})$ is equidistributed with $(\mathsf{Des}, \mathsf{Maj})$ and that $(\mathrm{emptyrows}, \mathrm{rows})$ is equidistributed with $(\mathsf{Adj}, \mathsf{Des})$ on $\mathfrak{S}_n$.

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