Combinatorial proofs for some forest hook length identities

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Abstract

Chen, Gao and Guo gave in a recent paper many interesting identities involving hook lengths of trees and forests using an extension of Han's expansion technique. We give combinatorial proofs of some of these identities.

1 Introduction

In a rooted tree, the **hook length** h_u of a vertex u is the number of descendants of u, counting u. Let $\mathcal{H}(F)$ be the multiset of hook lengths of vertices in the forest F. There are many identities on forests involving hook lengths. In a recent paper, Chen, Gao and Guo [1] derive a multitude of such formulae using extensions Han's expansion technique [2]. Most of these identities are fairly complicated, but some are simpler, such as

$$\sum_{F \in \mathcal{F}([n])} \prod_{h \in \mathcal{H}(T)} \frac{1}{h^2} = \frac{(n+1)!}{2^n}, \tag{1}$$

where the sum runs over all labelled forests with labels in $[n]$, and

$$n! \sum_{T \in \mathcal{PF}(n)} \prod_{h \in \mathcal{H}(T)} \left(1 - \frac{1}{h}\right)^{h-1} = (n+1)^{n-1}, \tag{2}$$

where the sum runs over all plane forests with n vertices. The authors asked whether there was a combinatorial interpretation of these simpler formulae. The purpose of this note is to contribute combinatorial proofs of these, and a few related, identities.

We start with labelled forests by giving a bijection from forests with increasing labels in each tree to $\mathfrak{S}_n$. Since the probability that a forest has increasing labels is $\prod h^{-1}$, this explains one of the hook lengths in the denominator. We can then treat every cycle in the permutation independently and use induction to prove the identity (1). Full proofs and necessary definitions are given in Section 2.

Proceeding with plane forests, we note that these are counted by the Catalan numbers C_n and that the number of plane forests with n vertices and $k+1$ vertices in its leftmost tree is $C_k C_{n-k-1}$. By induction the left hand side of equation (2) transform into a sum where each term equals the number of forests with n labels and $n-k$ vertices in the tree with lowest root label. Since the total number of labelled forests with n vertices is $(n+1)^{n-1}$, we are done. All details are given in Section 2.

2 Identities involving labelled forests

A **labelled forest** is a collection of rooted trees with distinct labels on each vertex. Let $\mathcal{T}(S)$ be the set of labelled trees with labels taken from the set S and $\mathcal{T}^{\uparrow}(S)$ the set of labelled trees whose labels increase downwards the tree. Thus, the root is $\mathcal{T}^{\uparrow}([n])$ is always labelled 1. Similarly,

$\mathcal{F}(S)$ is the set of forests with labels from S and $\mathcal{F}^{\uparrow}(S)$ the set of increasing forests with labels from S. Given $F \in \mathcal{F}(S)$ and a set U such that no label $u \in U$ is above a label $v \in (S \setminus U)$ in F, we use $F(U)$ to denote the restriction of F to U.

To prove the identities on labelled forests, we will give a bijection $\alpha : \mathcal{F}^{\uparrow}([n]) \to \mathfrak{S}_n$. This bijection is closely related to the one given by Hugh Thomas and presented in [1], but somewhat simplified. We recursively define $\pi = \alpha(F)$ by letting π_1 equal the greatest root label a in F and the rest of π be given by the restriction all elements but a in F, that is π_i is the greatest root label in $F([n] \setminus \{\pi_1, \ldots, \pi_{i-1}\}$. This is clearly a bijection, since we can rebuild F by traversing $\pi = \alpha(F)$ from the right, making π_i the root of all trees with higher labels.

We also define $\beta : \mathfrak{S}_n \to \mathfrak{S}_n$ to be the classical bijection on $\mathfrak{S}_n$ mapping left-to-right minima to cycles: given $\pi \in \mathfrak{S}_n$, insert a left parenthesis before every left-to-right minimum in π and right parenthesis where appropriate (see for instance [4]. This gives a permutation $\beta(\pi)$ in standard cycle representation, that is with every cycle starting with its smallest element first, and cycles ordered by decreasing first elements.

Lemma 2.1. *The bijection $\gamma = \beta \circ \alpha$ maps forests in $\mathcal{F}^{\uparrow}([n])$ consisting of k trees to permutations in $\mathfrak{S}_n$ with k cycles.*

Proof. It is not hard to see that we find a new left-to-right minimum in $\alpha(F)$ for every root label in F. Indeed, roots are chosen with decreasing root labels and non-roots have higher labels than there roots and can thus not contribute a minimum. By β, minima are converted into cycles. ☐

Either one of the bijections α and γ immediately give two theorems in [1]. These were given combinatorial interpretations by Thomas' bijection, but are restated here for completeness.

Theorem 2.2 (Theorems 4.3 and 4.9 in [1]). *For $n \geq 1$ we have*

$$\sum_{T \in \mathcal{T}([n])} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = (n-1)!$$

and

$$\sum_{F \in \mathcal{F}([n])} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = n!.$$

Proof. For each $F \in \mathcal{F}^{\uparrow}([n])$, we obtain $\prod_{h \in \mathcal{H}(F)} h$ forests in $\mathcal{F}([n])$ by simply rearranging the labels of each tree. Since every forest in $\mathcal{F}([n])$ is obtained exactly once by this procedure, we can conclude that

$$\sum_{F \in \mathcal{F}([n])} \prod_{h \in \mathcal{H}(T)} \frac{1}{h} = \sum_{F \in \mathcal{F}^{\uparrow}([n])} 1 = n!.$$

Since $\alpha(F)_1 = 1$ if and only if F is a tree, the first identity holds as well. ☐

Let $\lambda \vdash n$ be an **integer partition** of n, denoted $\lambda = (\lambda_1, \ldots, \lambda_{\ell(\lambda)})$ where the **length** $\ell(\lambda)$ is the number of parts in λ. We let $m_i(\lambda)$ be the number of parts in λ which equal i, and write m_i if λ can easily be identified from the context. The **type** of $\pi \in \mathfrak{S}_n$ is the integer partition $\lambda \vdash n$ defined by $m_i(\lambda)$ being the number of cycles with i elements in π. The number of permutations in $\mathfrak{S}_n$ with type λ is given by (see for instance [3])

$$c_\lambda = \frac{n!}{1^{m_1} m_1! 2^{m_2} m_2! \cdots n^{m_n} m_n!}.$$

The number of $\pi \in \mathfrak{S}_n$ with k cycles is often called the **signless Stirling numbers of the first kind** and are given by

$$c(n,k) = \sum_{\lambda \vdash n, \ell(\lambda) = k} c_\lambda.$$

It is well-known ([4]) that

$$\sum_{k=0}^{n} c(n,k)x^k = x(x+1)(x+2)\cdots(x+n-1).$$

We can now proceed with identity (1). To this end, we define hook lengths on permutations. The **one-line hook length** of $i \in [n]$ in $\pi \in \mathfrak{S}_n$ is

$$h_i = \min\{k \geq 1 : \pi_{i+k} < \pi_i\},$$

where we tacitly assume $\pi_{n+1} = 0$. The **cycle hook length** of i in π is given by the one-line hook length of i in $\beta^{-1}(\pi)$. The sets of hook lengths are denoted $\mathcal{H}_{\text{line}}(\pi)$ and $\mathcal{H}_{\text{cyc}}(\pi)$, respectively. It is easy to see that $\prod_{h\in\mathcal{H}(T)} h = \prod_{h\in\mathcal{H}_{\text{line}}(\alpha(T))} h = \prod_{h\in\mathcal{H}_{\text{cyc}}(\alpha(T))} h$.

Theorem 2.3 (Theorem 4.10 in [1]). *For $n \geq 0$ we have*

$$\sum_{F\in\mathcal{F}([n])} \prod_{h\in\mathcal{H}(T)} \frac{1}{h^2} = \frac{(n+1)!}{2^n}.$$

Proof. One easily gather that both sides give 1 for $n = 0$. We can thus proceed inductively with $n \geq 1$.

From the proof of Theorem 2.2, we know that

$$\sum_{F\in\mathcal{F}([n])} \prod_{h\in\mathcal{H}(T)} \frac{1}{h^2} = \sum_{F\in\mathcal{F}^{\uparrow}([n])} \prod_{h\in\mathcal{H}(T)} \frac{1}{h} = \sum_{\pi\in\mathfrak{S}_n} \prod_{h\in\mathcal{H}_{\text{cyc}}(\pi)} \frac{1}{h}.$$

Consider a permutation $\pi \in \mathfrak{S}_n$ of type λ in standard cycle representation and any cycle τ of some length k in π. Let S be the elements in τ and let A be the set of permutations obtained by rearranging the elements in τ, except for the first one. Then, summing over the permutations in A, all hook lengths are fixed except for those in τ, for which we could just as well sum over all permutations of $[k]$. Thus, by induction we have

$$\sum_{\pi\in A} \prod_{h\in\mathcal{H}_{\text{cyc}}(\pi)} \frac{1}{h} = \frac{k!}{2^{k-1}}\frac{1}{k} \prod_{h\in\mathcal{H}_{\text{cyc}}(\pi\setminus S)} \frac{1}{h}.$$

We now expose every cycle to this inductive simplification. To assign elements to each cycle in π of type λ, we first choose which elements goes into which cycle and then divide by the number of ways to permute cycles of equal length. We thus get

$$\begin{aligned}
\sum_{\pi\in\mathfrak{S}_n} \prod_{h\in\mathcal{H}_{\text{cyc}}(\pi)} \frac{1}{h} &= \sum_{\lambda\vdash n} \frac{\binom{n!}{\lambda_1,\lambda_2,\ldots,\lambda_{\ell(\lambda)}}}{m_1!m_2!\cdots m_n!} \frac{\lambda_1!}{2^{\lambda_1-1}\lambda_1} \cdots \frac{\lambda_{\ell(\lambda)}!}{2^{\lambda_{\ell(\lambda)}-1}\lambda_{\ell(\lambda)}} \\
&= \sum_{\lambda\vdash n} \frac{n!}{\lambda_1!\lambda_2!\cdots\lambda_{\ell(\lambda)}!m_1!m_2!\cdots m_n!} \frac{\lambda_1!}{2^{\lambda_1-1}\lambda_1} \cdots \frac{\lambda_{\ell(\lambda)}!}{2^{\lambda_{\ell(\lambda)}-1}\lambda_{\ell(\lambda)}} \\
&= \frac{1}{2^n} \sum_{\lambda\vdash n} \frac{n!}{\lambda_1\lambda_2\cdots\lambda_{\ell(\lambda)}m_1!m_2!\cdots m_n!} 2^{\ell(\lambda)} \\
&= \frac{1}{2^n} \sum_{\lambda\vdash n} c_\lambda 2^{\ell(\lambda)} \\
&= \frac{1}{2^n} \sum_{k=1}^{n} c(n,k)2^k = \frac{1}{2^n} 2\cdot 3\cdots(n+1) = \frac{(n+1)!}{2^n}.
\end{aligned}$$

□

3 Identities involving plane forests

While general forests can easily be related to permutations, plane forests belong to the large class of objects counted by the **Catalan numbers** $C_n = \frac{1}{n+1}\binom{2n}{n}$. We will use their recursion to again inductively show some hook length identities.

Theorem 3.1 (Theorem 3.3 in [1]). *For $n \geq 1$ we have*

$$n! \sum_{T \in \mathcal{PT}(n)} \prod_{h \in \mathcal{H}(T)} \left(1 - \frac{1}{h}\right)^{h-1} = (n-1)^{n-1}.$$

Theorem 3.2 (Theorem 3.10 in [1]). *For $n \geq 0$ we have*

$$n! \sum_{T \in \mathcal{PF}(n)} \prod_{h \in \mathcal{H}(T)} \left(1 - \frac{1}{h}\right)^{h-1} = (n+1)^{n-1}.$$

Since $(n+1)^{n-1}$ is the number of labelled forests with n vertices, we expect that it may prove useful to study them. The following lemma is what we need.

Lemma 3.3. *The number of labelled forests with n vertices and $n-k$ vertices in the tree with lowest root label is given by*

$$|\mathcal{F}([n], k)| = \binom{n}{k+1}(n-k)^{n-k-2}k^k.$$

Proof. For trees, we have $k = 0$. There are n^{n-1} labelled trees on n vertices and the formula says $|\mathcal{F}([n], 0)| = \binom{n}{1}n^{n-2}0^0 = n^{n-1}$. It only to remains to show the identity for forests with at least two trees.

To get a forest with $m+1$ trees, one of which has $n-k$ vertices, we choose disjoint sets $S_1, S_2, S_3 \in [n]$ such that $|S_1| = n-k-1$, $|S_2| = k$ and $|S_3| = 1$. Then, we choose a forest from $\mathcal{F}(S_1)$, add a root labelled with the element in S_3 to the chosen forest and then add a forest with m trees taken from $\mathcal{F}(S_2)$. The probability that the tree containing the labels in $S_1 \cup S_3$ has the lowest root is $1/(m+1)$. Using the fact that the number of labelled forests with n vertices and j trees is $\binom{n-1}{j-1}n^{n-j}$ [5], we get

$$\begin{aligned}
|\mathcal{F}([n], k)| &= (n-k)^{n-k-2} \sum_{m \geq 1} \binom{k-1}{m-1} k^{k-m} \binom{n}{n-k-1, k, 1} \frac{1}{m+1} \\
&= \binom{n}{k+1}(n-k)^{n-k-2}k^k \sum_{m \geq 1} \binom{k-1}{m-1} \frac{k+1}{k^m(m+1)} \\
&= \binom{n}{k+1}(n-k)^{n-k-2}k^k \sum_{m \geq 1} \binom{k+1}{m+1} \frac{m}{k^{m+1}}.
\end{aligned}$$

We need to show that the last sum equals 1. Consider $F(x) = \sum_{m \geq 1} \binom{k+1}{m+1} m x^{-(m+1)}$. We get

$$\begin{aligned}
F(x) &= D\left(-\sum_{m \geq 1} \binom{k+1}{m+1} x^{-m}\right) \\
&= D\left(-x\left(\left(1 + \frac{1}{x}\right)^{k+1} - \frac{k+1}{x} - 1\right)\right) \\
&= D\left(x + (k+1) - x\left((1 + \frac{1}{x}\right)^{k+1}\right) \\
&= 1 - \left(\left(1 + \frac{1}{x}\right) - \frac{k+1}{x}\right)\left(1 + \frac{1}{x}\right)^k \\
&= 1 - \left(1 - \frac{k}{x}\right)\left(1 + \frac{1}{x}\right)^k.
\end{aligned}$$

We immediately get $F(k) = 1$, which proves the lemma. ☐

It is now easy to show Theorems 3.1 and 3.2 by induction.

Proof. (Theorems 3.2 and 3.1) We first note that for $n = 0$, both the left hand side and the right hand side of the equality gives 1.

Any plane forest on n vertices can be constructed by adding a root to a plane forest with k vertices and then letting another plane forest with $n-k-1$ vertices follow to its right. This gives the Catalan recursion

$$|\mathcal{PF}(n)| = \sum_{k=0}^{n-1} |\mathcal{PF}(n-k-1)||\mathcal{PF}(k)|.$$

By induction, we thus get

$$\begin{aligned}
&n! \sum_{T\in\mathcal{PF}(n)} \prod_{h\in\mathcal{H}(T)} \left(1-\frac{1}{h}\right)^{h-1} \\
&= n! \sum_{k=0}^{n-1} \left(\sum_{\mathcal{PF}(n-k-1)} \prod \left(1-\frac{1}{h}\right)^{h-1} \right) \left(\sum_{\mathcal{PF}(k)} \prod \left(1-\frac{1}{h}\right)^{h-1} \right) \left(\frac{k}{k+1}\right)^{k} \\
&= \sum_{k=0}^{n-1} \binom{n}{k+1} \frac{(n-k)^{n-k-2}(k+1)^{k-1}k^k(k+1)}{(k+1)^k} \\
&= \sum_{k=0}^{n-1} \binom{n}{k+1} (n-k)^{n-k-2}k^k.
\end{aligned}$$

By Lemma 3.3, we are done, since we sum over all sizes of the tree with the minimal root and thus over all labelled forests on n vertices, which is given by $(n+1)^{n-1}$.

For the plane tree identity, we note that plane trees correspond to $k = n-1$ in the recursion above. Lemma 3.3 gives

$$|\mathcal{F}([n], n-1)| = \binom{n}{n} 1^{-1}(n-1)^{n-1} = (n-1)^{n-1}.$$

☐

References

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