♣ Markov chains indexed by trees ♣

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1 Introduction

In the study of *Markov chains indexed by trees*, one lets time be the vertices of a tree, instead of the the integers. This generalization leads to a number of more or less surprising properties. For example, one can construct a chain which has an infinite *Green* function, but is transient. In this note, we consider that and some other examples in some detail, and other things in less detail. Since the literature on the subject is rather limited, in essential everything here is based on the papers by Benjamini and Peres [1], [2].

2 Definitions

To be able to make the very definition of a Markov chain indexed by a tree, we first must present several tree related definitions and notations.

2.1 Trees

A *tree* $T = (V(T), E(T))$ is a locally finite connected graph without cycles or loops. We distinguish a vertex $0 \in V(T)$ and call it the *root* of the tree. Since we want the vertex set $V(T)$ to be the index of a random process, it is natural to consider only trees without leaves, that is, we require every vertex except the root to have degree at least 2. Also we need some notation of order for the vertices. Write $|v|$ for the number of edges on the unique path from 0 to v and write $u \leq v$ if u lies on this unique path. Let for $u \neq 0$, $\tilde{u}$ be the unique vertex satisfying $\tilde{u} \leq u$ and $|\tilde{u}| = |u| - 1$. The set of infinite non-self-intersecting paths starting from the root is denoted by ∂T and is called the *boundary* of the tree. An element $\xi \in \partial T$ is called a *ray*. Finally let A_n denote the set $\{u \in V(T) : |u| = n\}$.

2.2 Tree-indexed Markov chains

Let G be a countable state space and T be a tree. A T-indexed Markov chain on G with transition probabilities $\{p(x,y)\}_{x,y\in G}$ and initial state x_0 is a random process $\{S(u)\}_{u\in V}$ with the property that

$$\mathbf{P}[S(v)=y|S(\tilde{v})=x \text{ and } S(u) \,\text{for}\, u\neq v,\ |u|\leq|v|]$$

$$=\mathbf{P}[S(v)=y|S(\tilde{v})=x]=p(x,y)$$

and $S(0)=x_0$. We will call this process a (T,p)-walk on G. Note that if we let the tree T be $\mathbb{N}$ with edges between nearest neighbours, we get the usual definition of a Markov chain. If G is a locally finite connected graph and the usual Markov chain with transition probabilities $\{p(x,y)\}_{x,y\in G}$ on G is a ordinary simple random walk, then we say T-walk instead of (T,p)-walk. Also, we will always assume that the usual Markov chain is reversible and irreducible.

3 Recurrence and the Green function

In the study of Markov chains indexed by trees, it turns out that there are several types of recurrence to consider. They are defined as defined as follows.

Definition 3.1 *The (T,p)-walk is* recurrent *if for some $y\in G$,*

$$\mathbf{P}\left[\sum_{v\in V(T)}\mathbf{1}_{\{S(v)=y\}}=\infty\right]>0.$$

It is strongly recurrent *if the above probability is* 1.

Definition 3.2 *The (T,p)-walk is* ray-recurrent *if for some $y\in G$,*

$$\mathbf{P}\left[\exists\xi\in\partial T,\ \sum_{u\in V(\xi)}\mathbf{1}_{\{S(u)=y\}}=\infty\right]>0.$$

It is strongly ray recurrent *if the above probability is 1.*

Definition 3.3 *The (T,p)-walk has a* guaranteed return *if*

$$\mathbf{P}[\exists u\in V(T)\,:\,S(u)=x_0,\ |u|>0]=1$$

Somewhat surprisingly, the probabilities in the two first definitions do not have to equal 0 or 1. Note that obviously ray-recurrence implies recurrence. The converse however is not true. Also note that for usual Markov chains, the definitions coincide. Furthermore, it is possible to show that recurrence for some y implies recurrence for all y.

The *Green* function $g_T(x, y)$ for a (T, p)-walk is defined by

$$g_T(x, y) = \sum_{u \in V(T)} \mathbf{P}[S(u) = y \mid S(0) = x].$$

The Green function can be rewritten as

$$g_T(x, y) = \sum_{n=0}^{\infty} |A_n| \, \mathbf{P}[X(n) = y \mid X(0) = x] = \sum_{n=0}^{\infty} |A_n| p^n(x, y)$$

where $\{X(n)\}_{n \geq 0}$ is the usual Markov chain on G. For usual Markov chains, the state x is recurrent if and only if $g_T(x, x) = \infty$ (x is transient if and only if $g_T(x, x) < \infty$). However, this does *not* (!) hold for general tree-indexed Markov chains. We shall now find a tree T, a state space G, and transition probabilities p such that the (T, p)-walk on G will have infinite Green function but is not recurrent.

3.1 The exploding tree example

First we construct a tree suitable for our purpose. Let $\{m_i\}_{i=0}^{\infty}$ be a increasing sequence of integers with $m_0 = 1$. Let $\mathbb{E}$ be the set of edges between nearest neighbours in $\mathbb{N}$. Begin with a single ray, a copy of $(\mathbb{N}, \mathbb{E})$. We call this ray the *pivotal* ray of the tree. Denote the vertices along the pivotal ray by $\{\sigma_i\}_{i=0}^{\infty}$. For each $i \geq 1$, attach m_i disjoint rays (copies of $(\mathbb{N}, \mathbb{E}) \backslash \{0\}$) to σ_i.

Next, we let the state space G be the integers $\mathbb{Z}$. Furthermore, we let the transition probabilities be $p(x, x+1) = \alpha = 1 - p(x, x-1)$ for all $x \in \mathbb{Z}$, and suppose $1/2 < \alpha < 1$. Given the sequence m_i we will now find out for which (if any) $\alpha' s$ we have a nonrecurrent (T, p)-walk S (with $S(0) = 0$ on G with infinite Green function. First we calculate the Green function. Obviously $|A_0| = 1$ and for $n \geq 1$

$$|A_n| = \sum_{i=0}^{n-1} m_i.$$

For n odd, $p^n(x, x) = 0$ and for n even,

$$p^n(x, x) = \binom{n}{n/2} \alpha^{n/2} (1 - \alpha)^{n/2}.$$

Thus

$$g_T(x,x) = 1 + \sum_{n=2,4,\ldots}^{\infty} \left(\sum_{i=0}^{n-1} m_i\right) \binom{n}{n/2} \alpha^{n/2}(1-\alpha)^{n/2}. \tag{1}$$

Let $a_n \sim b_n$ mean $a_n/b_n \to 1$ as $n \to \infty$. Then, by Stirling's formula,

$$\binom{n}{n/2} \sim 2^n/\sqrt{\pi n/2}.$$

From this, it is easily seen that $g_T(x,x) = \infty$ if and only if

$$\limsup_{n\to\infty} \left(\sum_{i=0}^{n} m_i\right)^{1/n} \geq (4\alpha(1-\alpha))^{-1/2}. \tag{2}$$

Since m_i is increasing, $\limsup_{n\to\infty}(\sum_{i=0}^n m_i)^{1/n} = \limsup_{n\to\infty}(m_n)^{1/n}$. We now move on to see for which α the (T,p)-walk is nonrecurrent. We remark that obviously, since ∂T is countable, the walk is not ray-recurrent. The strong law of large numbers imply

$$\mathbf{P}[S(\sigma_n) \sim n(2\alpha - 1)] = 1. \tag{3}$$

Thus, *a.s.*, there is N such that $S(\sigma_n) > 0$ for $n \geq N$. Now let $k \geq N$ and $\xi \in \{\eta \in \partial T \,:\, V(\eta) \cap V(\sigma) = \{\sigma_0, ..., \sigma_k\}\}$. Note that the cardinality of this set is m_k. By for example exercise 7.1.c, chapter 4 in [3]

$$\mathbf{P}[\exists u \in \xi \,:\, u \geq \sigma_k,\, S(u) = 0 | \{S(\sigma_n)\}_{n\geq 1}] = ((1-\alpha)/\alpha)^{S(\sigma_k)} \tag{4}$$

If it is the case that

$$\mathbf{P}\left[\sum_{k=1}^{\infty} m_k (\frac{1-\alpha}{\alpha})^{S(\sigma_k)} < \infty\right] = 1, \tag{5}$$

then, using the first Borel-Cantelli lemma with conditioning, the (T,p)-walk is not recurrent. However, by (3), (5) holds if

$$\limsup_{n\to\infty} (m_n)^{1/n} < \left(\frac{\alpha}{1-\alpha}\right)^{2\alpha-1}. \tag{6}$$

It is not difficult to extend this to an if and only if result. Our conclusion is that any (T,p)-walk on G satisfying (2) and (6) has an infinite Green function but is not recurrent. For example, if we let $m_i = 3^i$ then we have non-recurrence but infinite Green function if and only if

$$(4\alpha(1-\alpha))^{-1/2} \leq 3 < \left(\frac{\alpha}{1-\alpha}\right)^{2\alpha-1},$$

i.e. if and only if $0.8365 \leq \alpha < 0.9714$. On the other hand, $m_i = i^2$ grows too slow and $m_i = i^i$ grows too fast to exhibit any interesting α's.

In the next section we consider without proof a useful theorem about strong recurrence.

3.2 Strong recurrence

Sometimes, to check wether a tree indexed chain is strongly recurrent or not can be done by more or less direct calculations. However, if the index tree and the state space are complicated, it could be quite difficult. The following proposition from provides a sufficient condition for strong recurrence which is easy to check. First a definition.

DEFINITION 3.4 *A graph G is* spherically symmetric *around the vertex x if for any two vertices y and z there is a graph-automorphism g on G such that $gz = y$, $gy = z$ and $gx = x$.*

In particular, a tree is spherically symmetric around x if and only if any two vertices on the same distance from x have the same degree.

PROPOSITION 3.5 *(*BENJAMINI AND PERES [1]*) Let T be a tree and G a spherically symmetric graph around x of bounded degree. Let $C_{x,n} = |\{y \in G : d_G(x,y) = n\}|$ where d_G denotes graphical distance and let $A_n = \{u \in T : |u| = n\}$. If*

$$\limsup_{n\to\infty} \frac{|A_n|}{C_{x,n}} > 0$$

then the T-walk on G started at x is strongly recurrent.

A simple application is as follows. Let G be a k-regular tree, that is a tree in which every vertex has degree k. Then $C_{x,n} = k(k-1)^{n-1}$ so letting T be an exploding tree with, say, $m_i > k^i$ for $i \geq 1$. Then by the proposition the T-walk on G is strongly recurrent.

4 Occupation behavior

Suppose S is a T-walk on the integers with $S(0) = 0$. One can think of each ray $\in \partial T$ as the index set of a simple random walk. With this interpretation, two simple random walks move together until their rays separate, and thereafter move independently (by the Markov property). The set

$\{S(u) : |u| = n\}$ should be thought of as the coordinates occupied by simple random walks at time n. It turns out that the occupation behaviour of (say) the origin can be described quite well by the growth of $|A_n|$. To prove a theorem in that direction, we first need an inequality, the proof of which we cannot resist including.

THEOREM 4.1 *(*BENJAMINI AND PERES [2]*) Let S be a (T,p)-walk on G. Let $Q \subset G$. Let $p^n(x,Q) = \sum_{y\in Q} p^n(x,y)$. Then*

$$\mathbf{P}[\forall u \in A_n,\ S(u) \in Q |\, S(0) = x_0] \geq [p^n(x_0,Q)]^{|A_n|} \tag{7}$$

Proof. The proof is done by induction. Trivially, the left and right hand sides of (7) coincide for $n = 1$. Suppose the theorem is true for some $n > 1$. For each $v \in A_1$, let T^v be the subtree of T rooted at v consisting of all vertices $\geq v$ and corresponding edges. If $A_n^v = \{u \in V(T^v) : |u| = n\}$, it is easily seen that

$$|A_{n+1}| = \sum_{v\in T_1} |A_n^v|. \tag{8}$$

Let $\{S^v(u) : u \in T^v\}$ be the (T^v,p)-walk on G. By the Markov property, S^η and S^ν are independent for $\eta \neq \nu$. Now we get using independence, the induction assumption, Jensen's inequality and (8) that

$$\mathbf{P}[\forall u \in A_{n+1},\ S(u) \in Q |\, S(0) = x_0] = \prod_{v\in T_1} \mathbf{P}[\forall u \in A_n^v,\ S(u) \in Q |\, S(0) = x_0]$$

$$= \prod_{v\in T_1} \sum_{z\in G} p(x_0,z)\mathbf{P}[\forall u \in A_n^v,\ S(u) \in Q |\, S(0) = x_0,\ S(v) = z]$$

$$= \prod_{v\in T_1} \sum_{z\in G} p(x_0,z)\mathbf{P}[\forall u \in A_n^v,\ S^v(u) \in Q |\, S^v(v) = z]$$

$$\geq \prod_{v\in T_1} \sum_{z\in G} p(x_0,z)[p^n(z,Q)]^{|A_n^v|} \geq \prod_{v\in T_1} \left(\sum_{z\in G} p(x_0,z)p^n(z,Q)\right)^{|A_n^v|}$$

$$= \left(\sum_{z\in G} p(x_0,z)p^n(z,Q)\right)^{\sum_{v\in T_1}|A_n^v|} = [p^{n+1}(x_0,Q)]^{|A_{n+1}|}$$

This calculation proves the theorem. $\square$

The next theorem gives a condition on the growth of $|A_n|$ guaranteeing that almost surely, the origin will be unoccupied infinitely often.

THEOREM 4.2 (BENJAMINI AND PERES [2]) *Suppose S is a T-walk on the integers and let*

$$V_{2n} = \sum_{u \in A_{2n}} \mathbf{1}_{\{S(u)=0\}}.$$

If

$$\liminf_{n\to\infty} |A_n| n^{-1/2} < \infty \tag{9}$$

then

$$\mathbf{P}[V_{2n} = 0 \; i.o.] = 1.$$

Proof. We show only the weaker result that $\mathbf{P}[V_{2n} = 0\,\text{i.o.}] > 0$. Note that for all even x,

$$p^{2n}(x,0) = \binom{2n}{n + x/2} 2^{-2n} \le \binom{2n}{n} 2^{-2n} \le n^{-1/2}.$$

Theorem (7) with $Q = \mathbb{Z}\backslash\{0\}$ now gives

$$\begin{aligned} \mathbf{P}[V_{2n} = 0] = \mathbf{P}[\forall u \in A_{2n},\, S(u) \in Q] &\ge [p^{2n}(x, Q)]^{|A_{2n}|} \\ = [1 - p^{2n}(x,0)]^{|A_{2n}|} &\ge [1 - n^{-1/2}]^{|A_{2n}|} \end{aligned}$$

Since T has no leaves, $|A_n|$ is an inceasing sequence. Hence, by (9), there is a $c \in (0,\infty)$ and a sequence n_j such that for all $j \ge 1$, $|A_{2n_j}| \le c n_j^{1/2}$. Thus, the previous calculation implies $\limsup_{n\to\infty} \mathbf{P}[V_{2n} = 0] > 0$. Invoking a variant of Fatou's lemma now finishes this part of the proof. □

5 Comment

Here we have studied examples accessible with only elementary probability theory. However, the most surprising results require a bit more. One such result is that it is possible to construct a (T,p)-walk S on a state space G such that S has guaranteed return but is *not* strongly recurrent.

References

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