A pairwise averaging procedure with application to consensus formation in the Deffuant model

Olle Häggström*

December 6, 2011

Abstract

Place a water glass at each integer point, the one at the origin being full and all others empty, and consider averaging procedures where we repeatedly pick a pair of adjacent glasses and pool their contents, leaving the two glasses with equal amounts but with the total amount unchanged. Some simple results are derived for what kinds of configurations of water levels are obtainable via such procedures. These are applied in the analysis of the so-called Deffuant model for social interaction, where individuals have opinions represented by numbers between 0 and 1, and whenever two individuals interact they take a step towards equalizing their opinion, unless their opinions differ beyond a fixed amount θ in which case they make no adjustment. In particular, we reprove and sharpen the recent result of Lanchier which identifies the critical value θ_c for consensus formation in the Deffuant model on $\mathbb{Z}$ to be $\frac{1}{2}$.

1 Introduction

Let $G = (V, E)$ be a graph which may be either finite or infinite with bounded degree. **The Deffuant model** [2] on G with parameters $\mu \in (0, \frac{1}{2}]$ and $\theta \in (0, 1)$ is defined as follows. At time $t = 0$, the vertices are assigned i.i.d. values, uniformly distributed on $[0, 1]$. Independently of this, each edge $e \in E$ is independently assigned a unit rate Poisson process. The value at $v \in V$ at time t is denoted $\eta_t(v)$, and remains unchanged as long as no Poisson event happens for any of the edges incident to v. When at some time t the Poisson clock rings at edge $e = \langle u, v \rangle$ such that $\eta_{t-}(u) = a$ (meaning $\lim_{s \uparrow t} \eta_s(u) = a$) and $\eta_{t-}(v) = b$, we set

$$\eta_t(u) = \begin{cases} a + \mu(b - a) & \text{if } |a - b| \leq \theta \\ a & \text{otherwise} \end{cases} \tag{1}$$

and

$$\eta_t(v) = \begin{cases} b + \mu(a - b) & \text{if } |a - b| \leq \theta \\ b & \text{otherwise.} \end{cases} \tag{2}$$

In the case $|a - b| > \theta$, we will speak of the Poisson clock being **censored**. For finite G, the well-definedness of this process is trivial from the facts that a.s. there will be only finitely many Poisson events in any finite time interval, and none of them will

*Work supported by the Swedish Research Council.

be simultaneous. The extension to bounded degree graphs is standard; see, e.g., [6, Chapter I, Thm. 3.9].

This can be thought of as a model for consensus formation in a social network. Each vertex $v \in V$ represents an individual, and $\eta_t(v)$ represents her belief or opinion on some matter. The dynamics defined in (1) and (2) is a crude model for the following phenomenon. Suppose I believe that the universe is 13.5×10^9 years old, and that I encounter Lisa who believes the universe is 13.0×10^9 years old. Her estimate sounds not unreasonable, and she thinks the same about mine. This causes me to adjust my estimate down to 13.4×10^9 years, while she adjusts her estimate up to 13.1×10^9 years. If instead I encounter Sarah who thinks the universe is merely 6000 years old, then I conclude that she is nuts, a view she reciprocates, so neither of us finds any reason to adjust our estimates. More generally, we are prone to take other people's opinions seriously only if they fall within some given range θ of our own.

The Deffuant model is just one among many mathematical models for social dynamics; see [1] for a survey. In a recent paper, Lanchier [5] considers the Deffuant model on $\mathbb{Z}$, or more precisely on the graph $G = (V, E)$ where $V = \mathbb{Z}$ and $E = \{\langle x, x+1\rangle : x \in \mathbb{Z}\}$. We say that the process $\{\eta_t(x)\}_{x\in\mathbb{Z}, t\geq 0}$ **approaches compatibility** if for some (hence, by translation invariance, any) $x \in \mathbb{Z}$ we have

$$\lim_{t\to\infty} \mathbb{P}(|\eta_t(x) - \eta_t(x+1)| \leq \theta) \;=\; 1\,, \tag{3}$$

i.e., if any two neighboring vertices are "on speaking terms" (compatible) at time t with a probability that tends to 1 as $t \to \infty$. The main result in [5] supports the intuition that the larger θ is, the easier is it for the individuals to converge towards agreement. It states that regardless of the value of μ, the model exhibits a critical phenomenon at $\theta = \frac{1}{2}$, in the following sense.

Theorem 1.1 (Lanchier) *Consider the Deffuant model with fixed $\mu \in (0, \frac{1}{2}]$. If $\theta > \frac{1}{2}$, then the model approaches compatibility. If on the other hand $\theta < \frac{1}{2}$, then the model does not approach compatibility.*

In the present paper, I will present an analysis of the Deffuant model based on invoking a related but nonrandom pairwise averaging procedure on $\mathbb{Z}$ which I propose colloquially to call **Sharing a drink (SAD)**. In Section 2 I will define the SAD procedure and prove a couple of basic results on what final states can be achieved by it. Then, after having given the basic lemma linking the SAD procedure to the Deffuant model in Section 3 (plus some further preliminaries in Section 4), I will in Sections 5 and 6 exploit those results in order to reprove Lanchier's result. I do not wish to overstate the novelty of my approach, as several steps in my proofs have analogues in Lanchier's proof; in particular, the set of two-sidedly ε-flat points (see Definition 4.1) will be a key quantity that is distinct from but will play a similar role here as the set denoted by Lanchier as Ω_j does in [5]. On the other hand, I would like to think that some readers might find my approach a bit easier to digest.

Some of my conclusions go slightly further than Lanchier's. In the $\theta > \frac{1}{2}$ case, it will turn out that the convergence-in-probability of (3) can be strengthened to almost sure convergence, and that the discrepancy $|\eta_t(x) - \eta_t(x+1)|$ in (3) will a.s. drop not only below θ, but eventually below any fixed $\varepsilon > 0$. Thus, I will show (still in the $\theta > \frac{1}{2}$ regime) that for any x,

$$\mathbb{P}\left(\lim_{t\to\infty} |\eta_t(x) - \eta_t(x+1)| = 0\right) \;=\; 1\,, \tag{4}$$

something we may call **asymptotic consensus**. Since, by the Strong Law of Large Numbers, the spatial average of the initial configuration $\{\eta_0(x)\}_{x\in\mathbb{Z}}$ is a.s. $\frac{1}{2}$, it is tempting to conclude from (4) that a.s. each $\eta_t(x)$ will tend to $\frac{1}{2}$ as $t \to \infty$. This, however, does not follow immediately from (4), because we might imagine a scenario where $\{\eta_t(x)\}_{x\in\mathbb{Z}}$ exhibits wave-like patterns on longer and longer spatial scales but nonvanishing amplitude as $t \to \infty$. Nevertheless, as I shall demonstrate in Theorem 6.5, the hoped-for convergence $\lim_{t\to\infty} \eta_t(x) = \frac{1}{2}$ does hold.

2 Sharing a drink on $\mathbb{Z}$

Define $\{\xi_0(x)\}_{x\in\mathbb{Z}}$ by setting

$$\xi_0(x) = \begin{cases} 1 & \text{for } x = 0 \\ 0 & \text{for } x \neq 0\,. \end{cases} \tag{5}$$

We can think of having a glass at each $x \in \mathbb{Z}$, and of the initial profile $\{\xi_0(x)\}_{x\in\mathbb{Z}}$ as telling us that the glass at 0 is full while all others are empty. The profiles $\{\xi_1(x)\}_{x\in\mathbb{Z}}$, $\{\xi_2(x)\}_{x\in\mathbb{Z}}, \ldots$ at all later times are obtained iteratively via a procedure which, given the parameters $x_1, x_2 \ldots \in \mathbb{Z}$ and $\mu_1, \mu_2, \ldots \in (0, \frac{1}{2}]$, is deterministic. The profile $\{\xi_1(x)\}_{x\in\mathbb{Z}}$ is obtained from $\{\xi_0(x)\}_{x\in\mathbb{Z}}$ by picking two adjacent glasses and pouring some water from the glass with the higher level to the glass with the lower level, but never more than what it takes to equalize the levels in the two glasses. This is then iterated. More precisely, given the profile $\{\xi_{i-1}(x)\}_{x\in\mathbb{Z}}$ and the parameters $x_i \in \mathbb{Z}$ and $\mu_i \in (0, \frac{1}{2}]$, we obtain the next profile $\{\xi_i(x)\}_{x\in\mathbb{Z}}$ by setting

$$\xi_i(x) = \begin{cases} \xi_{i-1}(x) + \mu_i(\xi_{i-1}(x+1) - \xi_{i-1}(x)) & \text{for } x = x_i \\ \xi_{i-1}(x) + \mu_i(\xi_{i-1}(x-1) - \xi_{i-1}(x)) & \text{for } x = x_i + 1 \\ \xi_{i-1}(x) & \text{for } x \notin \{x_i, x_i + 1\}\,. \end{cases}$$

This is the SAD procedure. The pairwise averaging that we perform in going from $\{\xi_{i-1}(x)\}_{x\in\mathbb{Z}}$ to $\{\xi_i(x)\}_{x\in\mathbb{Z}}$ is of course highly reminiscent of what goes on in the Deffuant model, but there are several differences. First, unlike the Deffuant model, the SAD procedure postulates no stochastic model for which pairs are to be averaged, when, and by how much. Second, the initial profile is very different. Third, the SAD procedure has no threshold $\theta \in (0, 1)$ preventing averaging between sites whose values differ by more than θ. (A fourth, but unimportant, difference is that the SAD procedure runs in discrete time.)

Next follow some results on what kinds of profiles are achievable through SAD procedures. The main motivation here is that they will be needed in the analysis of the Deffuant model in Sections 5 and 6, but they strike me as somewhat intriguing in their own right. If the restriction that $\mu_i \in (0, \frac{1}{2}]$ is relaxed to $\mu_i \in (0, 1]$, it is trivial to see that we would be able to achieve any nonnegative profile that sums to 1 and that is nonzero at only finitely many sites. With $\mu_i \in (0, \frac{1}{2}]$, the class of achievable profiles is much more restricted. The first result concerns the case where water is never sent to the left of 0.

Lemma 2.1 *If $\{\xi_i(x)\}_{x\in\mathbb{Z}}$ is obtained via a SAD procedure such that $x_j \neq -1$ for $j = 1, \ldots, i$, then*

$$\xi_i(0) \geq \xi_i(1) \geq \xi_i(2) \geq \cdots\,. \tag{6}$$

Proof. It suffices to show that if sites -1 and 0 never exchange liquids, then property (6) is preserved by the steps of the SAD procedure. Assume that (6) holds for $i = k-1$, and fix x_k and μ_k. Clearly, if (6) is to fail at all for $i = k$, failure has to happen in at least one of the three inequalities

$$\xi_k(x_k - 1) \geq \xi_k(x_k) \geq \xi_k(x_k + 1) \geq \xi_k(x_k + 2) . \tag{7}$$

The induction hypothesis in conjunction with $\mu_k > 0$ yields

$$\begin{aligned} \xi_k(x_k - 1) &= \xi_{k-1}(x_k - 1) \\ &\geq \xi_{k-1}(x_k) + \mu_k(\xi_{k-1}(x_k + 1) - \xi_{k-1}(x_k)) = \xi_k(x_k) , \end{aligned}$$

so the first inequality in (7) holds. The third inequality in (7) follows in the same way. Finally, the middle inequality in (7) follows from the induction hypothesis combined with $\mu_k \leq \frac{1}{2}$ by noting that

$$\begin{aligned} \xi_k(x_k) &= \xi_{k-1}(x_k) + \mu_k(\xi_{k-1}(x_k + 1) - \xi_{k-1}(x_k)) \\ &\geq \xi_{k-1}(x_k) + (1 - \mu_k)(\xi_{k-1}(x_k + 1) - \xi_{k-1}(x_k)) = \xi_k(x_k + 1) . \end{aligned}$$

□

Given the assumption that $\xi_i(x) = 0$ for all $x < 0$, to say that $\{\xi_i(x)\}_{x\in\mathbb{Z}}$ is decreasing on $\{0, 1, 2, \ldots\}$ is the same as saying that it is unimodal with the mode at 0. The first part of this survives when we drop the requirement to avoid the negative integers:

Lemma 2.2 *Any $\{\xi_i(x)\}_{x\in\mathbb{Z}}$ obtained via the SAD procedure is unimodal, meaning that there exists a $y \in \mathbb{Z}$ such that*

$$\cdots \leq \xi_i(y - 2) \leq \xi_i(y - 1) \leq \xi_i(y) \tag{8}$$

and

$$\xi_i(y) \geq \xi_i(y + 1) \geq \xi_i(y + 2) \geq \cdots . \tag{9}$$

Proof. Again induction is the way. Assume that $\{\xi_{k-1}(x)\}_{x\in\mathbb{Z}}$ is unimodal, and fix $y \in \mathbb{Z}$ such that $\xi_{k-1}(x)$ is maximized for $x = y$. Now, for each of the cases $y < x_k$, $y = x_k$, $y = x_k + 1$ and $y > x_k + 1$, a calculation similar to the one in the proof of Lemma 2.1 shows that $\{\xi_k(x)\}_{x\in\mathbb{Z}}$ inherits the unimodality from $\{\xi_{k-1}(x)\}_{x\in\mathbb{Z}}$. □

Will the mode stay at 0 also in the more general situation of Lemma 2.2? No. Consider the example where first 0 and 1 exchange liquids with $\mu_1 = \frac{1}{2}$, followed by 0 and -1 exchanging liquids with $\mu_2 = \frac{1}{2}$; this results in a profile $\{\xi_2(x)\}_{x\in\mathbb{Z}}$ with $\xi_2(-1) = \xi_2(0) = \frac{1}{4}$ and $\xi_2(1) = \frac{1}{2}$. In fact, it is not hard to construct examples to show that the mode can move arbitrarily far away from 0.

The fact that the mode can walk away from 0 will be a bit of a nuisance in Section 6, and because of that we will need to control the height of the mode. What is the largest possible value that can be obtained at a given location x? For $x \in \mathbb{Z}$, define M_x as the supremum of $\xi_i(x)$ over all i and all possible SAD procedures.

Theorem 2.3 *For any $x \in \mathbb{Z}$, we have $M_x = \frac{1}{|x|+1}$.*

All we will need in the application in Section 5 is the upper bound $M_x \leq \frac{1}{|x|+1}$, but since the exact value is within easy reach, why not do it?

For the proof, the notion of domination between two profiles will be convenient. For two sequences $\xi = \{\xi(x)\}_{x\in\mathbb{Z}}$ and $\xi' = \{\xi'(x)\}_{x\in\mathbb{Z}}$ of nonnegative numbers summing to 1, we say that ξ' **dominates** ξ, writing $\xi \preceq \xi'$, if $\sum_{x=k}^{\infty} \xi(x) \leq \sum_{x=k}^{\infty} \xi'(x)$ for all $k \in \mathbb{Z}$. If we think of ξ and ξ' as probability distributions on $\mathbb{Z}$, then $\xi \preceq \xi'$ is the same as the familiar notion of stochastic domination (see, e.g., [6]) between two random variables with respective distributions ξ and ξ'.

Lemma 2.4 *Suppose that $\xi_i = \{\xi_i(x)\}_{x\in\mathbb{Z}}$ and $\xi_i' = \{\xi_i'(x)\}_{x\in\mathbb{Z}}$ are exposed to the same SAD move, i.e. ξ_{i+1} and ξ_{i+1}' are obtained by picking the same pair of vertices $(x, x+1)$ to exchange liquids, with the same $\mu \in (0, \frac{1}{2}]$. If $\xi_i \preceq \xi_i'$ then $\xi_{i+1} \preceq \xi_{i+1}'$.*

Proof. Define, for each $y \in \mathbb{Z}$, $S_i(y) = \sum_{k=y}^{\infty} \xi_i(k)$, and define S_i', S_{i+1} and S_{i+1}' similarly. We have $S_i(y) \leq S_i'(y)$ for each y, and need to show that $S_{i+1}(y) \leq S_{i+1}'(y)$ for each y. For $y \neq x+1$ this is trivial (because $S_{i+1}(y) = S_i(y)$ and $S_{i+1}'(y) = S_i'(y)$). For $y = x+1$ we get

$$
\begin{aligned}
S_{i+1}(x+1) &= S_i(x+1) + \mu(\xi_i(x) - \xi_i(x+1)) \\
&= S_i(x+1) + \mu(S_i(x) - S_i(x+1) - S_i(x+1) + S_i(x+2)) \\
&= (1-2\mu)S_i(x+1) + \mu(S_i(x) + S_i(x+2))
\end{aligned}
$$

and similarly for $S_{i+1}'(x+1)$. Hence

$$
\begin{aligned}
S_{i+1}'(x+1) - S_{i+1}(x+1) &= (1-2\mu)(S_i'(x+1) - S_i(x+1)) \\
&\quad + \mu(S_i'(x) - S_i(x)) + \mu(S_i'(x+2) - S_i(x+2)) \\
&\geq 0
\end{aligned}
$$

as needed. □

Proof of Theorem 2.3. The case $x = 0$ is trivial, and obviously $M_x = M_{-x}$, so it suffices to consider the case $x > 0$. The first task is to show that $M_x \leq \frac{1}{|x|+1}$.

Fix $x > 0$, and suppose that m is a value that is achievable at x in a finite number of iterations. Then there is a smallest i such that $\xi_i(x) \geq m$ is possible. Fix i in such a way, and let $\{\xi_i(y)\}_{y\in\mathbb{Z}}$ be a configuration such that $\xi_i(x) \geq m$. We need to show that $m \leq \frac{1}{x+1}$.

As a preliminary step, we first show that in such a SAD scheme,

$$
\begin{array}{l}\text{there will up to time } i \text{ never be an} \\ \text{exchange of liquids between } x \text{ and } x+1.\end{array} \tag{10}
$$

Suppose for contradiction that such an exchange happens. Then the first time j this happens x sends liquid to $x+1$. Then we either have (a) that at some later time (but no later than i) x receives liquid from $x+1$, or (b) not.

Scenario (a) requires that $\xi_k(x+1) > \xi_k(x)$ at the first time $k \in \{j, j+1, \ldots, i-1\}$ at which $x+1$ is about to send liquid to x. By unimodality (Lemma 2.2), this implies that $\max_{y\in\mathbb{Z}} \xi_k(y)$ is attained to the right of x. But since the height of the mode is decreasing in time it must be at least m at time k, but in order to attain such a value to the right of k we must have had $\xi_l(x) \geq m$ at some time $l < k$. This contradicts the choice of i, so scenario (a) is impossible.

Suppose on the other hand that scenario (b) happens, and consider modifying the SAD procedure by replacing the move of liquid from x to $x+1$ at time j by a "null" move (i.e., say, an exchange between two empty glasses), and otherwise making the same moves as in the original SAD procedure. Write

$$\{\xi'_j(y)\}_{y\in\mathbb{Z}}, \{\xi'_{j+1}(y)\}_{y\in\mathbb{Z}}, \ldots, \{\xi'_i(y)\}_{y\in\mathbb{Z}}$$

for the resulting levels in the modified procedure. Since we are in scenario (b), the liquid sent to $x+1$ at time j in the original procedure is lost to the set of sites $\{\ldots, x-1, x\}$ forever, and it follows by induction over k that $\xi'_k(y) \geq \xi_k(y)$ for all $y \leq x$ and all $k \in \{j, j+1, \ldots i\}$. In particular $\xi'_i(x) \geq \xi_i(x) \geq m$. But the modified procedure can be modified one step further by skipping the null move at time j and doing all subsequent moves one time unit earlier. This yields a SAD procedure that gives site x level at least m at time $i-1$, contradicting the choice of i. So scenario (b) is impossible as well, and (10) is established.

We move on to showing that no move is made between sites 0 and -1 in the SAD procedure leading to $\xi_i(x) \geq m$. Suppose for contradiction that such a move happens. The first time j this happens liquid is sent from 0 to -1. Consider (similarly as in scenario (b) above) the modified procedure where this move is replaced by a null move and keeping all other moves intact, and write $\{\xi''_j(y)\}_{y\in\mathbb{Z}}, \{\xi''_{j+1}(y)\}_{y\in\mathbb{Z}}, \ldots, \{\xi''_i(y)\}_{y\in\mathbb{Z}}$ for the profiles resulting from this modification. We have $\{\xi_j(y)\}_{y\in\mathbb{Z}} \preceq \{\xi''_j(y)\}_{y\in\mathbb{Z}}$, and by iterated use of Lemma 2.4 we get $\{\xi_k(y)\}_{y\in\mathbb{Z}} \preceq \{\xi''_k(y)\}_{y\in\mathbb{Z}}$ for $k = j+1, \ldots, i$. In particular,

$$\sum_{y=x}^{\infty} \xi_i(y) \;\leq\; \sum_{y=x}^{\infty} \xi''_i(y)\,. \tag{11}$$

But we already know from (10) that $\xi_i(y) = \xi''_i(y) = 0$ for all $y \geq x+1$. This, in combination with (11), implies that $\xi''_i(x) \geq \xi_i(x) \geq m$. Now (again as in scenario (b)) we can modify the modified SAD procedure one step further by skipping the null move at time j and doing all subsequent moves one time unit earlier, thus obtaining a procedure that gives site x level at least m at time $i-1$, contradicting the choice of i. We can thus conclude that the negative axis $\{\ldots, -2, -1, 0\}$ is never touched in the scheme yielding $\xi_i(x) = m$. But that means that Lemma 2.1 is in force, so that

$$\xi_i(0) \;\geq\; \xi_i(1) \;\geq \cdots \geq \xi_i(x)$$

which, since $\sum_{y=0}^{x} \xi_i(y) = 1$, implies $m \leq \frac{1}{|x|+1}$, and the desired inequality $M_x \leq \frac{1}{|x|+1}$ follows.

The next task is to show the complementary inequality: $M_x \geq \frac{1}{|x|+1}$. The case $x=0$ is trivial from the original configuration, and the cases $x = -1, 1$ immediate from moving the maximum amount of mass from 0 at time $i=1$. So we only need to consider $|x| > 1$, and by symmetry $x > 1$ is enough.

Fix a site $x > 1$, and consider the following infinite sequence of SAD moves. Never pass any liquid outside of the interval $\{0, 1, \ldots, x\}$. Instead, at each time i, pick the $y \in \{0, 1, \ldots, x-1\}$ that maximizes $|\xi_{i-1}(y) - \xi_{i-1}(y+1)|$ (with an arbitrary tie-breaking convention), and average the levels at sites y and $y+1$ with $\mu_i = \frac{1}{2}$, so that $\xi_i(y) = \xi_i(y+1) = \frac{1}{2}(\xi_{i-1}(y) + \xi_{i-1}(y+1))$.

To analyze this SAD procedure, we need the concept of **energy** W of a profile $\{\xi_i(y)\}_{y\in\mathbb{Z}}$, defined by

$$W(i) \;=\; W(\{\xi_i(y)\}_{y\in\mathbb{Z}}) \;=\; \sum_{y\in\mathbb{Z}} (\xi_i(y))^2\,.$$

Note that $W(i)$ is nonnegative. A direct calculation shows that if $|\xi_{i-1}(y) - \xi_{i-1}(y+1)| = a$ and the SAD procedure selects y and $y+1$ to be averaged, then $W(i) - W(i-1) = -\frac{a^2}{2}$. It follows that $W(i)$ is decreasing in i, and if $\max_{y \in \{0,1,\ldots,x-1\}} |\xi_i(y) - \xi_i(y+1)|$ exceeds some given $a > 0$ infinitely often, then $\lim_{i\to\infty} W(i) = -\infty$, which is impossible. Hence

$$\lim_{i\to\infty} \max_{y\in\{0,1,\ldots,x-1\}} |\xi_i(y) - \xi_i(y+1)| \, = \, 0 \, . \tag{12}$$

Since $\sum_{y\in\{0,1,\ldots,x\}} \xi_i(y) = 1$ for each i, (12) implies that $\lim_{i\to\infty} \xi_i(y) = \frac{1}{x+1}$ for each $y \in \{0,1,\ldots,x\}$, so that in particular $\lim_{i\to\infty} \xi_i(x) = \frac{1}{|x|+1}$, and $M_x \geq \frac{1}{|x|+1}$ as desired. □

The obvious follow-up question, given the identification of M_x in Theorem 2.3, is whether, for some i, $\xi_i(x)$ can attain the critical value M_x. For $x = 0$ and for $x = 1$, the answer is obviously yes with attainment for $i = 0$ and $i = 1$, repectively. For $x \geq 2$, the answer is no, for the following reason.

Fix $x \geq 2$, suppose for contradiction that $\xi_i(x) = \frac{1}{x+1}$ is attainable, and let i be the first time at which this happens. Then, from the arguments leading up to the $M_x \leq \frac{1}{|x|+1}$ half of Theorem 2.3, we have $\xi_i(y) = \frac{1}{x+1}$ for $y = 0,\ldots,x$. But on the last move, x must have received liquid from $x-1$, so that

$$\xi_{i-1}(x-1) \, > \, \frac{1}{|x|+1} \, ,$$

contradicting Lemma 2.1.

3 The link between SAD and Deffuant

The usefulness of the SAD procedure for analysing Deffuant's model arises from the fact that in the latter, the state $\eta_t(0)$ can be written as a weighted average of $\{\eta_0(y)\}_{y\in\mathbb{Z}}$, with weights given by the profile $\{\xi_i(y)\}_{y\in\mathbb{Z}}$ of a carefully chosen SAD procedure. The exact relation is as follows.

Consider the Deffuant model with parameters $\mu \in (0, \frac{1}{2}]$ and $\theta \in (0,1)$, and fix $t > 0$. For each edge $e = \langle x, x+1 \rangle$ there is a probability $\exp(-t)$ that its Poisson clock does not ring before time t, independently of all edges. Hence there are a.s. infinitely many edges to the left of 0 whose clocks have not rung by time t, and infinitely many to the right. Define

$$Z_- = \max\{x \leq 0 : \langle x-1, x \rangle \text{ has not rung by time } t\}$$

and

$$Z_+ = \min\{x \geq 0 : \langle x, x+1 \rangle \text{ has not rung by time } t\} \, .$$

The point here is that identifying Z_- and Z_+ reduces what happens between them up to time t to a finite system. Given the initial states $\eta_0(Z_-), \eta_0(Z_- + 1), \ldots, \eta_0(Z_+)$ and the rings of the Poisson clocks between these vertices up to time t, nothing else is needed to find $\eta_s(Z_-), \eta_s(Z_- + 1), \ldots, \eta_s(Z_+)$ for $s \in [0,t]$. Write N for the number of Poisson rings not censored by the θ parameter in the interval $\{Z_-, Z_- + 1, \ldots, Z_+\}$ up to time t, and write $\tau_1, \tau_2, \ldots, \tau_N$ for the times of these non-censored Poisson rings in *reverse* chronological order, so that

$$\tau_N \, \leq \, \tau_{N-1} \, \leq \, \cdots \, \leq \, \tau_1 \, .$$

For $i = 1, \ldots, N$, define x_i to be the left endpoint of the edge $\langle x_i, x_i + 1\rangle$ for which the Poisson clock rings at time τ_i. It will be notationally convenient also to define $\tau_{N+1} = 0$.

Given these Poisson rings and their locations, define the SAD procedure given by, at each stage $i = 1, 2, \ldots, N$, choosing vertices x_i and $x_i + 1$ to exchange liquids, with $\mu_i = \mu$. For $i = 0, 1, \ldots, N$, write $\xi_i = \{\xi_i(y)\}_{y\in\mathbb{Z}}$ for the resulting profile at stage i.

Lemma 3.1 *For $i = 0, 1, \ldots, N$, $\eta_t(0)$ can be decomposed as*

$$\eta_t(0) \;=\; \sum_{y\in\mathbb{Z}} \xi_i(y)\eta_{\tau_{i+1}}(y)\,. \tag{13}$$

In particular, $\eta_t(0) = \sum_{y\in\mathbb{Z}} \xi_N(y)\eta_0(y)$.

Proof. We proceed by induction over i. The base case $i = 0$ follows from (5) and the fact that $\eta_{\tau_1}(0) = \eta_t(0)$. Assuming now that (13) holds for $i = j - 1$, i.e. that

$$\eta_t(0) \;=\; \sum_{y\in\mathbb{Z}} \xi_{j-1}(y)\eta_{\tau_j}(y)\,, \tag{14}$$

we need to show that it holds for $i = j$ as well.

In the SAD procedure, we have

$$\begin{cases} \xi_j(x_j) &=& \xi_{j-1}(x_j) + \mu(\xi_{j-1}(x_j+1) - \xi_{j-1}(x_j)) \\ \xi_j(x_j+1) &=& \xi_{j-1}(x_j+1) + \mu(\xi_{j-1}(x_j) - \xi_{j-1}(x_j+1)) \\ \xi_j(y) &=& \xi_{j-1}(y) \text{ for all } y \notin \{x_j, x_{j+1}\} \end{cases}$$

while in the Deffuant model we have

$$\begin{cases} \eta_{\tau_j}(x_j) &=& \eta_{\tau_{j+1}}(x_j) + \mu(\eta_{\tau_{j+1}}(x_j+1) - \eta_{\tau_{j+1}}(x_j)) \\ \eta_{\tau_j}(x_j+1) &=& \eta_{\tau_{j+1}}(x_j+1) + \mu(\eta_{\tau_{j+1}}(x_j) - \eta_{\tau_{j+1}}(x_j+1)) \\ \eta_{\tau_j}(y) &=& \eta_{\tau_{j+1}}(y) \text{ for all } y \notin \{x_j, x_{j+1}\}\,. \end{cases}$$

Plugging these relations into (14) yields

$$\begin{aligned} \eta_t(0) &= \sum_{y\in\mathbb{Z}} \xi_{j-1}(y)\eta_{\tau_j}(y) \\ &= \xi_{j-1}(x_j)\eta_{\tau_j}(x_j) \;+\; \xi_{j-1}(x_j+1)\eta_{\tau_j}(x_j+1) \;+\; \sum_{y\notin\{x_j,x_{j+1}\}} \xi_{j-1}(y)\eta_{\tau_j}(y) \\ &= \xi_{j-1}(x_j)[\eta_{\tau_{j+1}}(x_j) + \mu(\eta_{\tau_{j+1}}(x_j+1) - \eta_{\tau_{j+1}}(x_j))] \\ &\quad + \xi_{j-1}(x_j+1)[\eta_{\tau_{j+1}}(x_j+1) + \mu(\eta_{\tau_{j+1}}(x_j) - \eta_{\tau_{j+1}}(x_j+1))] \\ &\quad + \sum_{y\notin\{x_j,x_{j+1}\}} \xi_j(y)\eta_{\tau_{j+1}}(y) \\ &= \eta_{\tau_{j+1}}(x_j)[\xi_{j-1}(x_j) + \mu(\xi_{j-1}(x_j+1) - \xi_{j-1}(x_j))] \\ &\quad + \eta_{\tau_{j+1}}(x_j+1)[\xi_{j-1}(x_j+1) + \mu(\xi_{j-1}(x_j) - \xi_{j-1}(x_j+1))] \\ &\quad + \sum_{y\notin\{x_j,x_{j+1}\}} \xi_j(y)\eta_{\tau_{j+1}}(y) \\ &= \eta_{\tau_{j+1}}(x_j)\xi_j(x_j) \;+\; \eta_{\tau_{j+1}}(x_j+1)\xi_j(x_j+1) \;+\; \sum_{y\notin\{x_j,x_{j+1}\}} \xi_j(y)\eta_{\tau_{j+1}}(y) \\ &= \sum_{y\in\mathbb{Z}} \xi_j(y)\eta_{\tau_{j+1}}(y)\,, \end{aligned}$$

so that (13) holds for $i = j$ as desired. □

4 Flat points

Besides Lemma 3.1, another useful tool for the proof of Theorem 1.1 and related results on the Deffuant model is the notion of **ε-flat points** in the initial configuration $\{\eta_0(y)\}_{y\in\mathbb{Z}}$. Recall that, by definition, the $\eta_0(y)$ variables are i.i.d. uniform on $[0,1]$.

Definition 4.1 *Given $\varepsilon > 0$ and the Deffuant model initial configuration $\{\eta_0(y)\}_{y\in\mathbb{Z}}$, we say that x is an* **ε-flat point to the right** *if for all $n \geq 0$ we have*

$$\frac{1}{n+1}\sum_{y=x}^{x+n}\eta_0(y) \in [\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}+\varepsilon]\,.$$

Similarly, x is said to be **ε-flat point to the left** *if for all $n \geq 0$ we have*

$$\frac{1}{n+1}\sum_{y=x-n}^{x}\eta_0(y) \in [\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}+\varepsilon]\,.$$

Finally x is said to be **two-sidedly ε-flat** *if for all $m, n \geq 0$ we have*

$$\frac{1}{m+n+1}\sum_{y=x-m}^{x+n}\eta_0(y) \in [\tfrac{1}{2}-\varepsilon, \tfrac{1}{2}+\varepsilon]\,.$$

Translation invariance of the random configuration $\{\eta_0(y)\}_{y\in\mathbb{Z}}$ guarantees, of course, that the probability of site x being flat (in whichever of the three senses we have in mind) is independent of x. The important point here is that the probability is strictly positive for any $\varepsilon > 0$, as stated in the following two lemmas.

Lemma 4.2 *For any $\varepsilon > 0$ and any $x \in \mathbb{Z}$, we have*

$$\mathbb{P}(x \textit{ is } \varepsilon\textit{-flat to the right}) \;>\; 0\,.$$

Proof. The Strong Law of Large Numbers tells us that

$$\lim_{n\to\infty}\frac{1}{n+1}\sum_{y=x}^{x+n}\eta_0(y) = \frac{1}{2}\ \text{a.s.}$$

Hence, for fixed $\varepsilon > 0$, there exists an $N < \infty$ such that

$$\mathbb{P}\left(\frac{1}{n+1}\sum_{y=x}^{x+n}\eta_0(y) \in [\tfrac{1}{2}-\tfrac{\varepsilon}{3}, \tfrac{1}{2}+\tfrac{\varepsilon}{3}] \text{ for all } n \geq N\right) > 0\,. \tag{15}$$

Fix such an N. To get from here to the statement of the lemma, we employ a technique known in percolation theory as **local modification**. A coupling formulation is the following; cf. [4, Coupling 2.5]. Let $\{\eta_0(y)\}_{y\geq x}$ and $\{\eta_0'(y)\}_{y\geq x}$ be i.i.d. sequences of uniform $[0,1]$ random variables coupled in such a way that for each $y \in \{x, x+1, \ldots\}$ independently, the pair $(\eta_0(y), \eta_0'(y))$ is chosen so that both $\eta_0(y)$ and $\eta_0'(y)$ have the correct marginal (uniform on $[0,1]$), and so that

$$\begin{cases} \eta_0(y) \text{ and } \eta_0'(y) \text{ are independent} & \text{if } y \in \{x, x+1, \ldots, x+N\} \\ \eta_0(y) \;=\; \eta_0'(y) & \text{if } y \in \{x+N+1, x+N+2, \ldots\}\,. \end{cases}$$

By (15) we have that $\mathbb{P}(B) > 0$, where B is the event that $\frac{1}{n+1}\sum_{y=x}^{x+n} \eta_0'(y) \in [\frac{1}{2} - \frac{\varepsilon}{3}, \frac{1}{2} + \frac{\varepsilon}{3}]$ for all $n \geq N$. Also, $\mathbb{P}(C) = (\frac{2\varepsilon}{3})^{N+1} > 0$, where C is the event that $\eta(y) \in [\frac{1}{2} - \frac{\varepsilon}{3}, \frac{1}{2} + \frac{\varepsilon}{3}]$ for all $y \in \{x, x+1, \ldots, x+N\}$. Since B and C are independent, we furthermore get $\mathbb{P}(B \cap C) > 0$, and it is immediate to check that $B \cap C$ implies that x is ε-flat to the right with respect to the sequence $\{\eta_0(y)\}_{y \geq x}$. □

By symmetry, $x \in \mathbb{Z}$ has the same probability of being ε-flat to the left as of being ε-flat to the right, so Lemma 4.2 guarantees that also the former event has strictly positive probability. The same result for two-sided ε-flatness follows almost as effortlessly:

Lemma 4.3 *For any $\varepsilon > 0$ and any $x \in \mathbb{Z}$, we have*

$$\mathbb{P}(x \textit{ is two-sidedly } \varepsilon\textit{-flat}) > 0 .$$

Proof. Fix $\varepsilon > 0$ and consider the three events

$$\begin{aligned} A_1 &= \{\text{site } x-1 \text{ is } \varepsilon\text{-flat to the left}\} \\ A_2 &= \{\eta_0(x) \in [\tfrac{1}{2} - \varepsilon, \tfrac{1}{2} + \varepsilon]\} \\ A_3 &= \{\text{site } x+1 \text{ is } \varepsilon\text{-flat to the right}\}. \end{aligned}$$

Here $\mathbb{P}(A_2) = 2\varepsilon$, while $\mathbb{P}(A_1)$ and $\mathbb{P}(A_3)$ are both nonzero by Lemma 4.2. They are furthermore independent, because they are defined in terms of $\{\eta_0(x)\}_{x\in\mathbb{Z}}$ for disjoint sets of sites. Hence we have for their intersection $A = A_1 \cap A_2 \cap A_3$ that $\mathbb{P}(A) = \mathbb{P}(A_1)\mathbb{P}(A_2)\mathbb{P}(A_3) > 0$.

Next note that for any $m, n \geq 0$ we can rewrite $\frac{1}{m+n+1}\sum_{y=x-m}^{x+n} \eta_0(y)$ as the convex combination of three quantities $\frac{1}{m}\sum_{y=x-m}^{x-1} \eta_0(y)$, $\eta_0(x)$ and $\frac{1}{n}\sum_{y=x+1}^{n} \eta_0(y)$ that, on the event A, are in $[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$, so that $\frac{1}{m+n+1}\sum_{y=x-m}^{x+n} \eta_0(y) \in [\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon]$ and x is two-sidedly ε-flat. Hence

$$\mathbb{P}(x \text{ is two-sidedly } \varepsilon\text{-flat}) \geq \mathbb{P}(A) > 0 .$$

□

5 The Deffuant model for $\theta < 1/2$

Consider the Deffuant model in the $\theta < \frac{1}{2}$ part of the parameter space. With the equipment lain down in Sections 2, 3 and 4, we are now in a position to show that there are edges $e = \langle x, x+1 \rangle$ that are forever blocked from averaging across them. For $x \in \mathbb{Z}$, write B_x for the event $\{|\eta_t(x) - \eta_t(x+1)| > \theta, \ \forall t \geq 0\}$.

Proposition 5.1 *For the Deffuant model with parameters $\mu \in (0, \frac{1}{2}]$ and $\theta \in (0, \frac{1}{2})$, we have for any $x \in \mathbb{Z}$ that $\mathbb{P}(B_x) > 0$.*

Proof. Given $\theta < \frac{1}{2}$, pick ε so that $\theta = \frac{1}{2} - 2\varepsilon$, and define three further events D_x^1, D_x^2 and D_x^3 as

$$\begin{aligned} D_x^1 &= \{\text{site } x-1 \text{ is } \varepsilon\text{-flat to the left}\} \\ D_x^2 &= \{\eta_0(x) > 1 - \varepsilon\} \\ D_x^3 &= \{\text{site } x+1 \text{ is } \varepsilon\text{-flat to the right}\}. \end{aligned}$$

Define $D_x = D_x^1 \cap D_x^2 \cap D_x^3$. By arguing as in the proof of Lemma 4.3 we get that $\mathbb{P}(D_x) = \mathbb{P}(D_x^1)\mathbb{P}(D_x^2)\mathbb{P}(D_x^3) > 0$. The proposition will now follow if we can show that

$$D_x \subseteq B_x \tag{16}$$

so that $\mathbb{P}(B_x) \geq \mathbb{P}(D_x) > 0$. To this end, assume D_x, and note first that $\eta_t(x)$ will sit still (i.e., $\eta_t(x) = \eta_0(x)$) until the first time T that averaging happens across either of the edges $\langle x-1, x\rangle$ or $\langle x, x+1\rangle$. Hence, in order for such averaging to happen, either $\eta_t(x-1)$ or $\eta_t(x+1)$ must exceed $1 - \varepsilon - \theta = \frac{1}{2} + \varepsilon$ before time T.

By the SAD representation of $\eta_t(x+1)$ provided by Lemma 3.1, we can write it as a weighted average

$$\eta_t(x+1) \;=\; \sum_{y\in\mathbb{Z}} \xi_t(y)\eta_0(y) \tag{17}$$

where the weights $\{\xi_t(y)\}_{y\in\mathbb{Z}}$ are nonnegative, sum to 1, and are nonzero for only finitely many y. Before time T we furthermore have $\xi_t(y) = 0$ for all $y \leq x$, so Lemma 2.1 is in force to show that

$$\xi_t(x+1) \geq \xi_t(x+2) \geq \cdots . \tag{18}$$

Define $N = \max\{m : \xi_t(x+m) > 0\}$, and for $k = 1, \ldots, N$ define $\delta_k = \xi_t(x+k) - \xi_t(x+k+1)$ and $c_k = k\delta_k$. By (18) we have $\delta_k \geq 0$ and $c_k \geq 0$ for each k. Since $\xi_t(x+N+1) = 0$, we get $\sum_{n=k}^N \delta_n = \xi_t(x+k)$, so that

$$\begin{aligned}\sum_{n=1}^N c_n &= \sum_{n=1}^N n\delta_n \;=\; \sum_{k=1}^N\sum_{n=k}^N \delta_n \\ &= \sum_{k=1}^N \xi_t(x+k) \;=\; 1\,. \end{aligned} \tag{19}$$

Since $\xi_t(y) = 0$ for all $y \leq x$, the decomposition (17) can be rewritten as

$$\begin{aligned}\eta_t(x+1) &= \sum_{k=1}^N \xi_t(x+k)\eta_0(x+k) \;=\; \sum_{k=1}^N \eta_0(x+k)\sum_{n=k}^N \delta_n \\ &= \sum_{n=1}^N \delta_n \sum_{k=1}^n \eta_0(x+k) \;=\; \sum_{n=1}^N c_n \left(\tfrac{1}{n}\textstyle\sum_{k=1}^n \eta_0(x+k)\right) \end{aligned} \tag{20}$$

Since the c_n's are nonnegative and sum to 1, the last expression is a convex combination of terms of the form $\frac{1}{n}\sum_{k=1}^n \eta_0(x+k)$, each of which, on the event D_x^3 that $x+1$ is ε-flat to the right, is in the interval $[\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon]$. Hence, $\eta_t(x+1)$ will never exceed $\frac{1}{2}+\varepsilon$ before time T, so $|\eta_t(x) - \eta_t(x+1)| > \frac{1}{2} - 2\varepsilon = \theta$ for all $t < T$. Similarly, $|\eta_t(x) - \eta_t(x-1)| > \theta$ for all $t < T$. In conclusion, there will never be an opportunity for site x to average with either of its neighbors, so (16) is established and the proof is complete. □

Next, let I_{B_x} be the indicator function of the event B_x. The random process $\{I_{B_x}\}_{x\in\mathbb{Z}}$ is translation invariant, and a factor of an i.i.d. process $\{Z_x\}_{x\in\mathbb{Z}}$ where each Z_x encodes $\eta_0(x)$ plus all the Poisson firings of the $\langle x, x+1\rangle$. Any such factor is ergodic (see, e.g., [3, p 295, Thm (1.3)]), so we have a.s. that $\lim_{n\to\infty} \frac{1}{n}\sum_{x=1}^n I_{B_x} = \mathbb{P}(B_0)$ and $\lim_{n\to\infty} \frac{1}{n}\sum_{x=1}^n I_{B_{-x}} = \mathbb{P}(B_0)$. In particular, since $\mathbb{P}(B_0) > 0$ (Proposition 5.1), we have the following.

Lemma 5.2 *With probability* 1*, there will be infinitely many sites* x *to the left of* 0 *such that* B_x *happens, and infinitely many to the right.*

In fact, an inspection of the proof of Proposition 5.1 shows that the same thing holds with B_x replaced by the stronger event D_x. This observation allows a straightforward proof of the following result.

Theorem 5.3 *For the Deffuant model with $\theta < \frac{1}{2}$, we have a.s. that for all $x \in \mathbb{Z}$, the limiting value $\eta_\infty(x) = \lim_{t\to\infty} \eta_t(x)$ exists, and that the limiting configuration $\{\eta_\infty(x)\}_{x\in\mathbb{Z}}$ satisfies $\{|\eta_\infty(x) - \eta_\infty(x+1)|\} \in \{0\} \cup [\theta, 1]$ for all $x \in \mathbb{Z}$.*

In other words, what the theorem says about the limiting configuration is that it is piecewise constant, interrupted by jumps of size at least θ.

Proof of Theorem 5.3. Given the initial configuration $\{\eta_0(y)\}_{y\in\mathbb{Z}}$, let y_1 be any vertex such that D_{y_1-1} happens, and define $y_2 = \min\{y > y_1 : D_y\}$. In other words, $\{y_1, y_1+1, \ldots, y_2\}$ are the vertices sitting between two edges that are doomed by $\{\eta_0(y)\}_{y\in\mathbb{Z}}$ never to be crossed by any liquid. Since any $x \in \mathbb{Z}$ is in some such interval, it suffices to prove that the conclusion of the theorem holds for all $x \in \{y_1, y_1+1, \ldots, y_2\}$.

To analyze how η_t evolves on $\{y_1, y_1+1, \ldots, y_2\}$ we borrow the energy idea from the proof of Theorem 2.3. Define

$$W_t = \sum_{x\in\{y_1,y_1+1,\ldots,y_2\}} (\eta_t(y))^2\,,$$

and note that W_t is nonnegative. Note now that (a) no $x \in \{y_1, y_1+1, \ldots, y_2\}$ will ever exchange liquids with any site outside $\{y_1, y_1+1, \ldots, y_2\}$, and (b) each time t that two vertices $x, x+1 \in \{y_1, y_1+1, \ldots, y_2\}$ exchange liquids, W_t drops by $2\mu(1-\mu)|\eta_{t-}(x) - \eta_{t-}(x+1)|^2$. These two observations imply that W_t is decreasing in time. Next, define

$$F_t = \max\{I_{\{|\eta_t(x)-\eta_t(x+1)|\le\theta\}}|\eta_t(x) - \eta_t(x+1)| : x \in \{y_1, y_1+1, \ldots, y_2-1\}\}\,.$$

We need to show that

$$\lim_{t\to\infty} F_t = 0\,. \tag{21}$$

To this end, suppose for contradiction that there is a $\delta > 0$ such that $F_t \ge \delta$ for arbitrarily large t. At each time t that a Poisson clock rings in the interval $\{y_1, y_1+1, \ldots, y_2\}$, check the new value of F_t; if this value exceeds δ, then the probability is at least $\frac{1}{y_2-y_1}$ that an edge $\langle x, x+1\rangle$ with $|\eta_t(x) - \eta_t(x+1)| \in (\delta, \theta]$ will be the next to exchange liquids, in which case the energy W_t will go down by at least $2\mu(1-\mu)\delta^2$. By conditional Borel–Cantelli (see, e.g., Durrett [3, p 207, Corollary (3.2)]), this will happen infinitely often a.s., so that $\lim_{t\to\infty} W_t = -\infty$, contradicting the nonnegativity of W_t, so (21) follows.

Next, we need to show that each edge $\langle x, x+1\rangle$ in the interval satisfies

$$\begin{array}{l}\text{either } |\eta_t(x) - \eta_t(x+1)| > \theta \text{ for all sufficiently large } t,\\ \text{or } \lim_{t\to\infty} |\eta_t(x) - \eta_t(x+1)| = 0.\end{array} \tag{22}$$

In view of (21), the alternative to (22) would be the existence of some $\langle x, x+1\rangle$ such that for any $\delta > 0$, $|\eta_t(x) - \eta_t(x+1)|$ jumps back and forth between $[0, \delta]$ and $(\theta, 1]$ infinitely often. But this is impossible, because the dynamics of the Deffuant model is defined in such a way that a single Poisson event cannot increase $|\eta_t(x) - \eta_t(x+1)|$ by more than $\mu\theta$, which for small enough δ is less than the length of the gap $(\delta, \theta]$ that needs to be crossed. Hence (22) holds.

It remains to prove existence of the limit $\eta_\infty(x) = \lim_{t\to\infty} \eta_t(x)$. Suppose first that no edge $\langle x, x+1\rangle$ in $\{y_1, y_1+1, \ldots, y_2\}$ gets stuck with $|\eta_t(x) - \eta_t(x+1)| > \theta$ for large t. By (22), this implies that $\lim_{t\to\infty} |\eta_t(x) - \eta_t(x+1)| = 0$ for each $\langle x, x+1\rangle$ in the interval. Note now that $\sum_{x\in\{y_1,y_1+1,\ldots,y_2\}} \eta_t(x)$ is preserved over time (because no such x exchanges liquids with any vertex outside the interval). Hence $\eta_t(y_1), \ldots, \eta_t(y_2)$ must all converge, as $t \to \infty$, to their initial average $\frac{1}{y_2-y_1+1} \sum_{x\in\{y_1,y_1+1,\ldots,y_2\}} \eta_0(x)$.

Suppose on the other hand that some $\langle x, x+1\rangle$ in the interval does get stuck with $|\eta_t(x)-\eta_t(x+1)| > \theta$. Then each $x \in \{y_1, y_1+1, \ldots, y_2\}$ still belongs to some subinterval $\{y_1', y_1'+1, \ldots, y_2'\}$ with $y_1 \le y_1' \le y_2' \le y_2$ such that $\lim_{t\to\infty} |\eta_t(x)-\eta_t(x+1)| = 0$ for each $\langle x, x+1\rangle$ in the subinterval, and such that from some time T onwards neither y_1', nor y_2', exchanges liquids with its neighbor outside the subinterval. We then get preservation of $\sum_{x\in\{y_1',y_1'+1,\ldots,y_2'\}} \eta_t(x)$ from time T and onwards, so each $\eta_t(x)$ with x in their subinterval must converge to their average $\frac{1}{y_2'-y_1'+1} \sum_{x\in\{y_1',y_1'+1,\ldots,y_2'\}} \eta_T(x)$, and the proof is complete. □

6 The Deffuant model for $\theta > 1/2$

To analyse the Deffuant model with $\theta > 1/2$, we first need to extend (22) to this part of the parameter space.

Proposition 6.1 *For the Deffuant model with arbitrary threshold parameter $\theta \in (0,1)$, we have a.s. that for each $x \in \mathbb{Z}$,*

$$\begin{array}{l} \textit{either } |\eta_t(x) - \eta_t(x+1)| > \theta \textit{ for all sufficiently large } t, \\ \textit{or } \lim_{t\to\infty} |\eta_t(x) - \eta_t(x+1)| = 0. \end{array} \tag{23}$$

In order to prove this, we proceed again via an energy argument, but since we can no longer assume that x is contained in a finite interval that is forever blocked from exchanging liquids with the outside, we need to go about it somewhat differently. This time, define $W_t(x)$ as the energy $(\eta_t(x))^2$ at site x, and define an auxiliary process $\{W_t^\dagger(x)\}_{x\in\mathbb{Z}}$ as follows. Initially $W_0^\dagger(x) = 0$. Then $W_t^\dagger(x)$ stays constant, except at times when an exchange of liquids happens along $\langle x, x+1\rangle$. At such a time t, $W_t^\dagger(x)$ increases by $2\mu(1-\mu)|\eta_{t-}(x) - \eta_{t-}(x+1)|^2$, which is the exact amount by which $W_t(x) + W_{t+1}(x)$ decreases. Thus, $W_t^\dagger(x)$ measures the amount of energy loss along $\langle x, x+1\rangle$ up to time t, making it intuitively plausible that $\mathbb{E}[W_t(x)] + \mathbb{E}[W_t^\dagger(x)]$ is constant over time. This, however, requires an argument.

Lemma 6.2 *For some (hence any) $x \in \mathbb{Z}$ we have*

$$\mathbb{E}[W_t(x)] + \mathbb{E}[W_t^\dagger(x)] \;=\; \frac{1}{3}\,. \tag{24}$$

Proof. For $t = 0$, (24) is clear, since $\mathbb{E}[W_0(x)] = \int_{[0,1]} r^2\,dr = \frac{1}{3}$ and $\mathbb{E}[W_0^\dagger(x)] = 0$. To handle the case $t > 0$, define $W_t^{tot}(x) = W_t(x) + W_t^\dagger(x)$. By the same ergodicity argument as in the paragraph preceeding Lemma 5.2, we have for fixed t that the process $\{W_t^{tot}(x)\}_{x\in\mathbb{Z}}$ is ergodic, i.e.,

$$\mathbb{P}\left(\lim_{\substack{y_1\to-\infty \\ y_2\to\infty}} \frac{1}{y_2-y_1+1} \sum_{x=y_1}^{y_2} W_t^{tot}(x) \;=\; \mathbb{E}[W_t^{tot}(0)] \right) \;=\; 1\,. \tag{25}$$

In order to show that the limit in (25) is constant over time, note first that for any $t > 0$, we can find some $y_1 < 0$ and some $y_2 > 0$ such that the Poisson clocks at $\langle y_1 - 1, y_1 \rangle$ and $\langle y_2, y_2 + 1 \rangle$ have not rung by time t. By induction over the Poisson rings inside the interval $\{y_1, \ldots, y_2\}$, in chronological order up to time t, we see that $\sum_{x \in \{y_1,\ldots,y_2\}} W_s^{tot}(x)$ is constant as s ranges from 0 to t, so that in particular

$$\sum_{x \in \{y_1,\ldots,y_2\}} W_t^{tot}(x) = \sum_{x \in \{y_1,\ldots,y_2\}} W_0^{tot}(x) .$$

Since y_1 and y_2 could be taken arbitrarily far from 0, the limit in (25) must equal $\lim_{\substack{y_1 \to -\infty \\ y_2 \to \infty}} \frac{1}{y_2 - y_1 + 1} \sum_{x=y_1}^{y_2} W_0^{tot}(x)$, so that

$$\begin{aligned} \mathbb{E}[W_t(x)] + \mathbb{E}[W_t^\dagger(x)] &= \mathbb{E}[W_t^{tot}(x)] = \lim_{\substack{y_1 \to -\infty \\ y_2 \to \infty}} \frac{1}{y_2 - y_1 + 1} \sum_{x=y_1}^{y_2} W_t^{tot}(x) \\ &= \lim_{\substack{y_1 \to -\infty \\ y_2 \to \infty}} \frac{1}{y_2 - y_1 + 1} \sum_{x=y_1}^{y_2} W_0^{tot}(x) \\ &= \mathbb{E}[W_0^{tot}(x)] = \frac{1}{3} . \end{aligned}$$

□

Proof of Proposition 6.1. Fix $x \in \mathbb{Z}$ and $\delta > 0$. We first show that, a.s.,

$$|\eta_t(x) - \eta_t(x+1)| \in [0, \delta] \cup (\theta, 1] \text{ for all sufficiently large } t . \tag{26}$$

To do so, note that each time the Poisson clock of any of the edges $\langle x-1, x \rangle$, $\langle x, x+1 \rangle$ or $\langle x+1, x+2 \rangle$ rings, there is probability $\frac{1}{3}$ that the next such ring will be at $\langle x, x+1 \rangle$. If $|\eta_t(x) - \eta_t(x+1)| \in (\delta, \theta]$, then such a ring at $\langle x, x+1 \rangle$ will cause $W_t^\dagger(x)$ to increase by at least $2\mu(1-\mu)\delta^2$. If (26) fails, then, by conditional Borel–Cantelli as in the proof of Theorem 5.3, such an increase will happen infinitely often, so that $\lim_{t\to\infty} W_t^\dagger(x) = \infty$, an event which must have probability 0 since $\mathbb{E}[W_t^\dagger(x)]$ is bounded by $\frac{1}{3}$ due to Lemma 6.2.

It remains to show for small enough $\delta > 0$ that $|\eta_t(x) - \eta_t(x+1)|$ cannot jump back and forth between $[0, \delta]$ and $(\theta, 1]$ infinitely often. But we already saw in the proof of Theorem 5.3 why this cannot happen: a single Poisson event cannot increase $|\eta_t(x) - \eta_t(x+1)|$ by more than $\mu\theta$, which for small enough δ is less than the length of the gap $(\delta, \theta]$ to be crossed. □

Next comes the final key ingredient in sorting out how the Deffuant model behaves for $\theta > \frac{1}{2}$, namely that certain sites $x \in \mathbb{Z}$ will be predestined by the initial configuration $\{\eta_0(y)\}_{y\in\mathbb{Z}}$ to forever stay close to $\frac{1}{2}$ in their value of $\eta_t(x)$.

Lemma 6.3 *Suppose, given $\varepsilon > 0$, that site $x \in \mathbb{Z}$ is two-sidedly ε-flat for the initial configuration $\{\eta_0(y)\}_{y\in\mathbb{Z}}$. Then, regardless of all future Poisson rings, we have*

$$\eta_t(x) \in [\tfrac{1}{2} - 6\varepsilon, \tfrac{1}{2} + 6\varepsilon] \text{ for all } t \geq 0 . \tag{27}$$

Proof. In order to avoid spending excessive amounts of ink on the additive constant $\frac{1}{2}$, define, for all $t \geq 0$ and all $y \in \mathbb{Z}$, $\zeta_t(y) = \eta_t(y) - \frac{1}{2}$. Fix $\varepsilon > 0$, $x \in \mathbb{Z}$ and

$\{\eta_0(y)\}_{y\in\mathbb{Z}}$ as in the lemma. Without loss of generality we may assume that $x = 0$. Lemma 3.1 tells us that for any $t > 0$ there exists a SAD profile $\{\xi_t(y)\}_{y\in\mathbb{Z}}$ such that

$$\eta_t(0) \;=\; \sum_{y\in\mathbb{Z}} \xi_t(y)\eta_0(y)\,. \tag{28}$$

Since $\sum_{y\in\mathbb{Z}} \xi_t(y) = 1$, we can translate (28) to the ζ notation as

$$\zeta_t(0) \;=\; \sum_{y\in\mathbb{Z}} \xi_t(y)\zeta_0(y)\,. \tag{29}$$

We need to show that

$$\zeta_t(0) \in [-6\varepsilon, 6\varepsilon]\,. \tag{30}$$

Since the SAD profile $\{\xi_t(y)\}_{y\in\mathbb{Z}}$ is nonzero for only finitely many sites y, it takes only a finite collection of values, $a_1, a_2, \ldots, a_n, a_{n+1}$, say, in decreasing order with $a_{n+1} = 0$. Fix $w \in \mathbb{Z}$ such that $\xi_t(w) = a_1$ (the mode) and define for $i = 1, \ldots, n$ the set

$$J_i \;=\; \{y \in \mathbb{Z} :\, \xi_t(y) \geq a_i\}\,.$$

By unimodality of $\{\xi_t(y)\}_{y\in\mathbb{Z}}$ (Lemma 2.2), we have that each J_i is an interval $\{y_i, y_{i+1}, \ldots, y_i'\}$, containing the mode w. Again for $i = 1, \ldots, n$, define $\delta_i = a_i - a_{i+1}$, $|J_i| = y_i' - y_i + 1$ and $c_i = \delta_i|J_i|$. Similarly as the decomposition (20) in the proof of Proposition 5.1, we can rewrite (29) as

$$\begin{aligned}
\zeta_t(0) \;=\; \sum_{y\in\mathbb{Z}} \xi_t(y)\zeta_0(y) \;&=\; \sum_{i=1}^{n} \delta_i \sum_{y\in J_i} \zeta_0(y) &(31)\\
&=\; \sum_{i=1}^{n} c_i \left(\frac{1}{|J_i|} \sum_{y\in J_i} \zeta_0(y) \right)\,. &(32)
\end{aligned}$$

The coefficients $c_1, \ldots, c_n$ sum to 1 (this follows using the property $\sum_{y\in\mathbb{Z}} \xi_t(y) = 1$ of the SAD profile, similarly as in (19)), so if we knew that each of the intervals $J_1, \ldots, J_n$ contained site 0, we would also know from two-sided ε-flatness of 0 that (32) is a convex combination of numbers in $[-\varepsilon, \varepsilon]$, and thus itself take a value in $[-\varepsilon, \varepsilon]$, completing the proof. Alas, as we saw in Section 2, the mode w does not need to coincide with 0, so we need to do a bit more.

We may assume without loss of generality that $w > 0$. By the nesting property $J_1 \subset J_2 \subset \cdots \subset J_n$, there is an $m \in \{1, 2, \ldots, n+1\}$ such that each of $J_m, J_{m+1}, \ldots, J_n$ contains either site 0 or site $2w$ (or both), whereas each of $J_1, \ldots, J_{m-1}$ contains neither. With a hybrid of (31) and (32) we may write $\zeta_t(0)$ as

$$\zeta_t(0) \;=\; \sum_{i=1}^{m-1} \delta_i \sum_{y\in J_i} \zeta_0(y) + \sum_{i=m}^{n} c_i \left(\frac{1}{|J_i|} \sum_{y\in J_i} \zeta_0(y) \right)\,. \tag{33}$$

To estimate the first sum $\sum_{i=1}^{m-1} \delta_i \sum_{y\in J_i} \zeta_0(y)$ in (33), note that for such i, the interval $J_i = \{y_i, \ldots, y_i'\}$ has endpoints y_i and y_i' strictly between 0 and $2w$. By two-sided ε-flatness of 0, we get

$$\begin{aligned}
\left| \sum_{y\in J_i} \zeta_0(y) \right| \;=\; \left| \sum_{y=y_i}^{y_i'} \zeta_0(y) \right| \;&\leq\; \left| \sum_{y=0}^{y_i'} \zeta_0(y) \right| + \left| \sum_{y=0}^{y_i-1} \zeta_0(y) \right| \\
&\leq\; (y_i'+1)\varepsilon + y_i\varepsilon \;\leq\; 2w\varepsilon + w\varepsilon \;=\; 3w\varepsilon
\end{aligned}$$

so that

$$\left|\sum_{i=1}^{m-1}\delta_i\sum_{y\in J_i}\zeta_0(y)\right| \;\le\; \sum_{i=1}^{m-1}\delta_i\left|\sum_{y\in J_i}\zeta_0(y)\right| \tag{34}$$

$$\le\; 3w\varepsilon\sum_{i=1}^{m-1}\delta_i \;\le\; 3w\varepsilon\sum_{i=1}^{n}\delta_i$$

$$=\; 3w\varepsilon\xi_t(w) \;\le\; \frac{3w\varepsilon}{w+1} \;<\; 3\varepsilon \tag{35}$$

where the fourth inequality is due to Theorem 2.3.

For the second sum $\sum_{i=m}^{n} c_i\left(\frac{1}{|J_i|}\sum_{y\in J_i}\zeta_0(y)\right)$ in (33), the average $\frac{1}{|J_i|}\sum_{y\in J_i}\zeta_0(y)$ can be estimated as follows. If $0\in J_i$, then $\left|\frac{1}{|J_i|}\sum_{y\in J_i}\zeta_0(y)\right| < \varepsilon$ by two-sided ε-flatness of 0. Otherwise $2w\in J_i=\{y_i,\dots,y_i'\}$, so that $|J_i|\ge w$ and $|J_i|\ge\frac{y_i'+1}{2}$, so that

$$\left|\frac{1}{|J_i|}\sum_{y\in J_i}\zeta_0(y)\right| = \left|\frac{1}{|J_i|}\sum_{y=y_i}^{y_i'}\zeta_0(y)\right| \;\le\; \left|\frac{1}{|J_i|}\sum_{y=0}^{y_i'}\zeta_0(y)\right| + \left|\frac{1}{|J_i|}\sum_{y=0}^{y_i-1}\zeta_0(y)\right|$$

$$\le\; \left|2\frac{1}{y_i'+1}\sum_{y=0}^{y_i'}\zeta_0(y)\right| + \left|\frac{1}{y_i}\sum_{y=0}^{y_i-1}\zeta_0(y)\right|$$

$$\le\; 2\varepsilon+\varepsilon \;=\; 3\varepsilon\,,$$

again by two-sided ε-flatness of 0. Since the weights $c_m, c_{m+1},\dots,c_n$ are nonnegative and sum to at most 1, we get

$$\left|\sum_{i=m}^{n} c_i\left(\frac{1}{|J_i|}\sum_{y\in J_i}\zeta_0(y)\right)\right| \;\le\; \sum_{i=m}^{n} c_i\left|\frac{1}{|J_i|}\sum_{y\in J_i}\zeta_0(y)\right|$$

$$\le\; 3\varepsilon\sum_{i=m}^{n} c_i \;\le\; 3\varepsilon\,. \tag{36}$$

Plugging (35) and (36) into (33) yields (30), so the proof is complete. □

With Lemma 6.3 in our hands, we quickly get the following refinement of Proposition 6.1.

Proposition 6.4 *For the Deffuant model with $\theta>\frac{1}{2}$, we have a.s. for all $x\in\mathbb{Z}$ that* $\lim_{t\to\infty}|\eta_t(x)-\eta_t(x-1)|=0$.

Proof. Fix the parameter $\theta>\frac{1}{2}$, and pick an $\varepsilon>0$ such that $\theta>\frac{1}{2}+6\varepsilon$. In view of Proposition 6.1, all we need in order to prove the lemma is that, for any x,

$$\mathbb{P}(|\eta_t(x)-\eta_t(x+1)|>\theta \text{ for all sufficiently large } t) \;=\; 0\,. \tag{37}$$

Suppose for contradiction that the event in (37) has positive probability. Then by ergodicity it happens for infinitely many $x\in\mathbb{Z}$ a.s., and we can follow the proof of Theorem 5.3 in order to show that $\eta_\infty(y)=\lim_{t\to\infty}\eta_t(y)$ exists for each $y\in\mathbb{Z}$, and that the limiting configuration $\{\eta_\infty(y)\}_{y\in\mathbb{Z}}$ is piecewise constant interrupted by jumps of size at least θ.

We know from Lemma 4.3 plus ergodicity that there will a.s. exist some $z \in \mathbb{Z}$ that is two-sidely ε-flat for the initial configuration $\{\eta_0(y)\}_{y\in\mathbb{Z}}$. By Lemma 6.3, we thus have $\eta_t(z) \in [\frac{1}{2} - 6\varepsilon, \frac{1}{2} + 6\varepsilon]$ for all t, so that $\eta_\infty(z) \in [\frac{1}{2} - 6\varepsilon, \frac{1}{2} + 6\varepsilon]$. By Proposition 6.1, $\eta_\infty(z+1)$ must either exceed $\eta_\infty(z)$ by at least θ, or be less than $\eta_\infty(z)$ by at least θ, or equal $\eta_\infty(z)$. But by choice of ε, the first option causes $\eta_\infty(z+1)$ to exceed 1, while the second causes it to be negative. Both are impossible so we are stuck with $\eta_\infty(z+1) = \eta_\infty(z)$. By the same token, $\eta_\infty(z-1) = \eta_\infty(z)$, and by iterating we get $\eta_\infty(y) = \eta_\infty(z)$ for all $y \in \mathbb{Z}$. This contradicts positivity of (37), as desired. □

One more application of Lemma 6.3 will suffice to establish our last result:

Theorem 6.5 *For the Deffuant model with $\theta > \frac{1}{2}$, we have a.s. for all $x \in \mathbb{Z}$ that $\lim_{t\to\infty} \eta_t(x) = \frac{1}{2}$.*

Proof. Fix $x \in \mathbb{Z}$ and $\varepsilon > 0$. As in the proof of Proposition 6.4, we know that there will a.s. exist some $z \in \mathbb{Z}$ that is two-sidedly ε-flat for the initial configuration $\{\eta_0(y)\}_{y\in\mathbb{Z}}$. By Lemma 6.3, we thus have $\eta_t(z) \in [\frac{1}{2} - 6\varepsilon, \frac{1}{2} + 6\varepsilon]$ for all t. By applying Proposition 6.4 to each of the (finitely many) edges on the interval between z and x, we get that $\eta_t(x)$ must be in $[\frac{1}{2} - 7\varepsilon, \frac{1}{2} + 7\varepsilon]$ for all sufficiently large t. But $\varepsilon > 0$ was arbitrary, so $\lim_{t\to\infty} \eta_t(x) = \frac{1}{2}$ as desired. □

A glaring omission so far is the lack of discussion of the critical case $\theta = \frac{1}{2}$. Depending on whether $\mathbb{P}(B_x) > 0$ or not (with B_x as in Section 5), the qualitative limiting behavior will be the same as for $\theta < \frac{1}{2}$ or $\theta > \frac{1}{2}$, respectively. In either case, we have the a.s. existence of the limiting configuration $\{\eta_\infty(x)\}_{x\in\mathbb{Z}}$, but to determine which of the two cases we get remains open.

Acknowledgement. I am grateful to a referee, an editor and Timo Hirscher for helpful comments and corrections to an earlier draft.

References

[1] Castellano, C., Fortunato, S. and Loreto, V. (2009) Statistical physics of social dynamics, *Reviews of Modern Physics* **81**, 591–646.

[2] Deffuant, G., Neau, D., Amblard, F. and Weisbuch, G. (2001) Mixing beliefs among interacting agents, *Adv. Complex Syst.* **3**, 87–98.

[3] Durrett, R. (1991) *Probability: Theory and Examples*, Wadsworth & Brooks/Cole, Pacific Grove, CA.

[4] Häggström, O. and Jonasson, J. (2006) Uniqueness and non-uniqueness in percolation theory, *Probab. Surveys* **3**, 289–344.

[5] Lanchier, N. (2011) The critical value of the bounded confidence Deffuant model equals one half, `http://math.asu.edu/~lanchier/articles/2011h_lanchier.pdf`

[6] Liggett, T.M. (1985) *Interacting Particle Systems*, Springer, New York.