Nonmonotonic behavior in hard-core and Widom–Rowlinson models

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Abstract

We give two examples of nonmonotonic behavior in symmetric systems, exhibiting more than one critical point at which spontaneous symmetry-breaking appears or disappears. The two systems are the hard-core model and the Widom–Rowlinson model, and both examples take place on a variation of the Cayley tree (Bethe lattice) devised by Schonmann and Tanaka. We obtain similar, though less constructive, examples of non-monotonicity via certain local modifications of any graph, e.g. the square lattice, which is known to have a critical point for either model. En route we discuss the critical behavior of the Widom–Rowlinson model on the ordinary Cayley tree. Some results about monotonicity of the phase transition phenomenon relative to graph structure are also given.

Keywords. phase transition, symmetry breaking, Widom–Rowlinson model, hard-core model, critical points, Gibbs measures, Bethe lattice, monotonicity

1 Introduction

This paper is concerned with the phase transition phenomenon for two well-known Gibbs systems living on graphs: the hard-core model and the Widom–Rowlinson model. A common feature of these two models is that they both exhibit so-called *hard constraints*, meaning that their properties arise by forbidding certain configurations, as opposed e.g. to the Ising model where undesirable configurations are merely discouraged. Although phase transition takes place only on infinite graphs, we start for simplicity by describing the models in the case where they live on a finite graph structure.

In the following, G will denote a (finite or infinite) locally finite connected graph with vertex set V and edge set E. For two vertices $x, y \in V$, we write $x \sim y$ to indicate the existence of an edge $e \in E$ between x and y.

The hard-core model. Call $\omega \in \{0,1\}^V$ *H-feasible* if for each pair of vertices $x, y \in V$ such that $x \sim y$ we have $\omega(x)\omega(y) = 0$. In other words, a configuration of 0's and 1's on V is feasible if no two 1's sit next to each other in the graph. For $\lambda > 0$ and G finite, define the Gibbs measure μ_G^λ for the hard-core model on G with activity parameter λ to be the probability measure on $\{0,1\}^V$ which to each $\omega \in \{0,1\}^V$ assigns probability

$$\mu_G^\lambda(\omega) = \frac{1}{Z_G^\lambda} \prod_{x \in V} \lambda^{\omega(x)} \mathbf{1}_{\{\omega \text{ is H-feasible}\}} \, .$$

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Here Z_G^λ is a normalizing constant making μ_G^λ a probability measure (Z with various sub- and superscripts will always denote such a normalizing constant). One way to think of μ_G^λ is as the distribution of the $\{0,1\}^V$-valued random variable X which arises by first letting $\{X(x)\}_{x\in V}$ be i.i.d. with probability $\frac{\lambda}{\lambda+1}$ of having a 1 at a given vertex, and then conditioning on the event that X is H-feasible. We may interpret 1's as particles and 0's as empty locations; we thus have a crude model for a gas whose particles have non-negligible radii and would overlap if they sat at adjacent vertices. Combinatorially, this is a model for a "random independent set" in the graph G.

The Widom–Rowlinson model. Call $\eta \in \{-1,0,1\}^V$ *W-feasible* if for each pair of vertices $x, y \in V$ such that $x \sim y$ we have $\eta(x)\eta(y) \neq -1$. If we think of 1's and -1's as two different types of particles, and of 0's as empty locations, then the feasibility condition means that two particles of different type are not allowed to sit next to each other in G. For $\lambda > 0$ and G finite, define the Gibbs measure ν_G^λ for the Widom–Rowlinson model on G with activity parameter λ by letting

$$\nu_G^\lambda(\eta) = \frac{1}{\tilde{Z}_G^\lambda} \prod_{x\in V} \lambda^{|\eta(x)|} \mathbf{1}_{\{\eta \text{ is W-feasible}\}}.$$

As in the hard-core model, we can think of ν_G^λ as arising by first letting the values at different vertices be i.i.d., chosen according to the probability vector $(\frac{\lambda}{2\lambda+1}, \frac{1}{2\lambda+1}, \frac{\lambda}{2\lambda+1})$, and then conditioning on W-feasibility.

For infinite graphs, we will consider the standard DLR (Dobrushin–Lanford–Ruelle) approach to infinite-volume Gibbs measures. This means, roughly speaking, that probability measures on $\{0,1\}^V$ or $\{-1,0,1\}^V$, where V is infinite, are said to be Gibbs measures if their conditional distributions on finite sets agree with the conditional distributions that arise from the above "finite-volume" definitions. For the hard-core model, a formal definition is given below; the case of the Widom–Rowlinson model is completely analogous. If V_1 and V_2 are disjoint subsets of V, and ω_1 and ω_2 are elements of $\{0,1\}^{V_1}$ and $\{0,1\}^{V_2}$, respectively, then we write $\omega_1 * \omega_2$ for the element of $\{0,1\}^{V_1\cup V_2}$ which agrees with ω_1 on V_1 and with ω_2 on V_2.

Definition 1.1 *Let $G = (V, E)$ be infinite, let μ be a probability measure on $\{0,1\}^V$, and let X be a $\{0,1\}^V$-valued random variable with distribution μ. We call μ a Gibbs measure for the hard-core model with activity parameter $\lambda > 0$ if it is concentrated on H-feasible elements of $\{0,1\}^V$ and admits conditional probabilities such that for all finite $V' \subset V$, all $\omega' \in \{0,1\}^{V'}$ and μ-a.e. $\omega \in \{0,1\}^{V\setminus V'}$ we have*

$$\mu(X(V') = \omega' \mid X(V \setminus V') = \omega) \;=\; \frac{1}{Z_{V',\omega}^\lambda} \prod_{x\in V'} \lambda^{\omega'(x)} \mathbf{1}_{\{\omega*\omega' \textit{ is H-feasible}\}} \,. \tag{1}$$

For both the hard-core and the Widom–Rowlinson model, the existence (for any given $\lambda > 0$ and infinite G) of some Gibbs measure follows from standard Gibbs theory; see e.g. Georgii [10]. The issue considered here is that of (non-)uniqueness. Can there be more than one Gibbs measure? The answer is sometimes yes, and when this happens, we speak of a phase transition.

By far the most studied choice of G in Gibbs theory is the cubic lattice $\mathbb{Z}^d$, $d \geq 1$. With a harmless abuse of notation, we write $\mathbb{Z}^d$ for the graph whose vertex set is $\mathbb{Z}^d$ and whose edge set consists of those pairs of vertices that sit at (Euclidean) distance 1 from each other. In pioneering work, Dobrushin [9] demonstrated that the hard-core model on $\mathbb{Z}^d$, $d \geq 2$, has a unique Gibbs measure provided that λ is taken to be sufficiently small, but multiple Gibbs measures when λ is sufficiently large. An intuitive picture of what happens in the phase transition regime is the following. Since λ is large, the system "wants" to pack particles as closely to each other as possible. Since $\mathbb{Z}^d$ is a bipartite graph, there exist, in some sense, exactly two optimal packings: one where all $x \in \mathbb{Z}^d$ of even parity contain a particle, and one where all odd x contain a particle. When a phase transition occurs, we can find two particular Gibbs measures μ_{even} and μ_{odd}, where in μ_{even} we see a.s. the "even checkerboard pattern" with only some perturbations, whereas in μ_{odd} we see a.s. a perturbed "odd checkerboard pattern". The situation for the Widom–Rowlinson model on $\mathbb{Z}^d$, $d \geq 2$ is similar. For small λ, we have a unique Gibbs measure; this follows e.g. from Dobrushin's uniqueness condition (see [9] or [10]). For λ sufficiently large, we have phase transition, as shown by Lebowitz and Gallavotti [17]. This time, we find in the phase transition regime two different Gibbs measures μ_+ and μ_-, where under μ_+ we a.s. see a configuration whose density of +1-particles is greater than the density of −1-particles, and the other way around for μ_-.

These are two examples of what in statistical mechanics is called a symmetry breaking. For the hard-core model on $\mathbb{Z}^d$, the translation invariance of the model is not always inherited by the Gibbs measures, whereas in the Widom–Rowlinson model on $\mathbb{Z}^d$, the Gibbs measures do not need to exhibit the ± 1 symmetry that is possessed by the model itself.

Confronted with the above results, it is extremely tempting to make the following (widely believed) conjectures. For the hard-core model on $\mathbb{Z}^d$, $d \geq 2$, there ought to exist a critical value $\lambda_c = \lambda_c(d)$ such that for $\lambda < \lambda_c$ we have a unique Gibbs measure, whereas for $\lambda > \lambda_c$ we have multiple Gibbs measures. A similar critical value should also exist for the Widom–Rowlinson model in $d \geq 2$ dimensions. These conjectures are tantamount to saying that the occurrence of multiple Gibbs measures is increasing in the activity parameter.

Support for such conjectures comes mainly from intuition and by analogy with other models and other graph structures. For the symmetric (zero external field) Ising model, the occurrence of multiple Gibbs measures is increasing in the reciprocal temperature parameter. Let us replace $\mathbb{Z}^d$ by the d-branching Cayley tree (sometimes called "Bethe lattice") T^d, defined as the infinite tree in which each vertex has exactly $d+1$ nearest neighbors. Then the desired monotonicity does indeed hold both for the hard-core and the Widom–Rowlinson model. This is well known in the hard-core case (see, for instance, Kelly [16] or Brightwell and Winkler [6]). In the Widom–Rowlinson case, the result seems to have appeared only implicitly in the literature before, and the theorem stated below as Theorem 3.1 is a strong form of a result of Wheeler and Widom [22].

The main purpose of the present paper is to show that the desired monotonicity result *fails* for general graphs. In other words, there exist $0 < \lambda_1 < \lambda_2$ and an infinite graph G with the property that the hard-core model on G has multiple Gibbs measures at activity λ_1, and a unique Gibbs measure at activity λ_2. The corresponding statement is true also for the Widom–Rowlinson model. The following results tell us this and more.

Theorem 1.2 *There exist $0 < \lambda_1 < \lambda_2$ and an infinite graph G, such that the hard-core model on G with activity λ has a unique Gibbs measure for $\lambda \in (0, \lambda_1] \cup [\lambda_2, \infty)$, and multiple Gibbs measures for $\lambda \in (\lambda_1, \lambda_2)$.*

Theorem 1.3 *There exist $0 < \lambda_1 < \lambda_2 < \lambda_3$ and an infinite graph G, such that the Widom–Rowlinson model on G with activity λ has a unique Gibbs measure for $\lambda \in (0, \lambda_1] \cup [\lambda_2, \lambda_3]$, and multiple Gibbs measures for $\lambda \in (\lambda_1, \lambda_2) \cup (\lambda_3, \infty)$.*

Even more complex critical behavior can be obtained, e.g. by applying our methods recursively; see also Corollary 5.1 at the end of this paper.

The examples used to prove Theorems 1.2 and 1.3 are trees of the following form. To each vertex of the $(d+1)$-regular tree T^d, add n pendant edges, each terminating in a single "leaf"

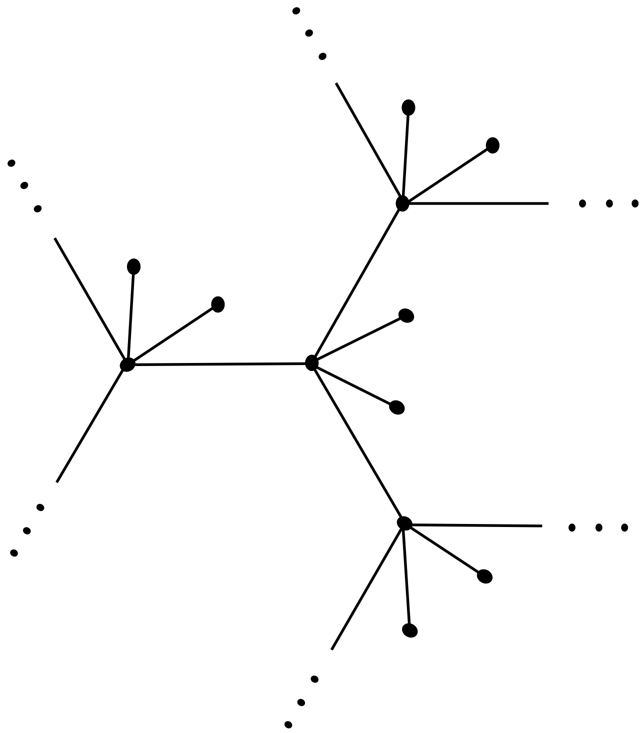

Figure 1: A piece of the modified Cayley tree T_2^2

vertex. Write T_n^d for the resulting tree (Fig. 1 illustrates a piece of T_2^2). This class of trees has recently been exploited by Schonmann and Tanaka [20] to demonstrate certain nonmonotonicities in the Ising model with an external field, and in fact the Schonmann–Tanaka paper was a main source of inspiration for the present work. To get an example of a graph G with the behavior indicated in Theorem 1.2, one may take $d = 13$ and $n = 2$, whereas $d = 40$ and $n = 7$ does the trick for Theorem 1.3.

These results show that if the above conjectures about the behavior on $\mathbb{Z}^d$ of hard-core and Widom–Rowlinson models are true, then any proof will have to make essential use of some aspects of the graph structure of $\mathbb{Z}^d$. Neither of the two proofs that we are aware of for the monotonicity of phase transition in the symmetric Ising model (namely, Griffiths inequalities and random-cluster representations, see e.g. [19] and [14], respectively) make any use of the particular graph structure, and thus go through for any G. We therefore fear that they are of little or no use for giving any insight into how to prove the conjectures about hard-core and Widom–Rowlinson models. Perhaps our results even cast a slight doubt over whether the conjectures are true.

It is worth noting that the counterexamples used to prove Theorems 1.2 and 1.3 belong to the class of quasi-transitive (sometimes also called almost transitive, or almost homogeneous) graphs, which is generally believed to be "well-behaved" in various senses; see e.g. some of the conjectures in Benjamini and Schramm [1]. A quasi-transitive graph is a graph whose vertex set V can be partitioned into finitely many sets $(V_1, \dots, V_k)$ in such a way that for any $j \in \{1, \dots, k\}$ and any $x, y \in V_j$, there exists a graph automorphism of G mapping x to y.

Before closing this section, let us mention some other work on these and related models. A nice probabilistic discussion of the hard-core model on $\mathbb{Z}^d$ and other graphs is given in Van den Berg and Steif [4]. A natural generalization is to allow the activity to be inhomogeneous, and for bipartite graphs a case of particular interest is when the activity at a vertex only depends on its parity; this thread is taken up in [4] and continued in [13]. The hard-core model also has a continuum analogue where point particles live in $\mathbb{R}^d$ and have a hard-core exclusion of fixed radius R; see e.g. Tanemura [21]. The Widom–Rowlinson model can also be defined in a continuum setting, as in the original paper of Widom and Rowlinson [23]; see also Chayes, Chayes and Kotecký [8] for a modern approach to this model. An attempt to alter the Gibbs potential of the (discrete) Widom–Rowlinson model in such a way that the conjectured monotonicity can be proved (but without changing the "spirit" of the model) is made in [15].

Other generalizations of the discrete Widom–Rowlinson model on the Bethe lattice have been studied: Lebowitz, Mazel, Nielaba and Šamaj [18] allow more than two types of particle, while Wheeler and Widom [22] consider a model where different types of particle are discouraged rather than banned on adjacent vertices. In both cases, critical values for phase transition (in various sense) are obtained, and it seems very likely that our results can be extended to cover these situations.

The rest of the paper is organized as follows. Section 2 is devoted to the hard-core model, and contains a proof of Theorem 1.2. Section 3 contains a description of the behavior of the Widom–Rowlinson model on a regular tree, which is then exploited to give a proof of Theorem 1.3. In Section 4, we discuss (non-)monotonicity of the phase transition phenomenon from a different viewpoint, where G (rather than λ) plays the role of the "parameter". Finally, in Section 5, we make a simple observation relating hard-core and Widom–Rowlinson models to each other.

A slightly more detailed version of this paper is available as a preprint [5].

2 The hard-core model: Proof of Theorem 1.2

We begin by noting a useful characterization of Gibbs measures for the hard-core model, showing that it is enough to check the Gibbs condition (1) for sets V' consisting of a single vertex. This is an instance of a result from [7] characterizing the hard-constraint systems for which such a "one-site" condition implies the full Gibbs condition. It is not hard to prove the result directly.

Lemma 2.1 *Let $G = (V, E)$ be finite or infinite, and let μ be a probability measure on $\{0,1\}^V$. Then μ is a Gibbs measure for the hard-core model on G with activity λ if and only if it admits conditional probabilities such that*

$$\mu(X(v) = 1 \mid X(V \setminus v) = \omega) = \begin{cases} \frac{\lambda}{1+\lambda} & \text{if } \omega(y) = 0 \text{ for all } y \sim x \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

for all $v \in V$ and μ-a.e. $\omega \in \{0,1\}^{V \setminus \{v\}}$.

Now, let $G = (V, E)$ be an infinite graph, and let $G_n = (V_n, E_n)$ be the graph obtained from G by adding n leaves to each $v \in V$ (in particular, if $G = \mathsf{T}^d$, then $G_n = \mathsf{T}^d_n$).

Our main tool is the following simple lemma, showing that the hard-core model on G_n, with activity λ, can be viewed as deriving from a hard-core model on G, with a different activity parameter λ^*. As we shall see, the important feature is that the function $\lambda \mapsto \lambda^*$ is not monotonic (see Fig. 2, for instance). Once we have this, then it is already clear that, whenever we have a phase transition (in any sense whatsoever) for the hard-core model on G, occurring at a suitable value of the activity, then we will get our desired nonmonotonicity behavior in the hard-core model on G_n. To exhibit explicit examples will require only a little more work.

Lemma 2.2 *Let μ_{G_n} be a Gibbs measure for the hard-core model on G_n with activity $\lambda > 0$. Then the projection of μ_{G_n} on $\{0,1\}^V$ is a Gibbs measure for the hard-core model on G with activity*

$$\lambda^* = \frac{\lambda}{(\lambda+1)^n} \ .$$

Conversely, let μ_G be a Gibbs measure for the hard-core model on G with activity λ^*, and obtain $X \in \{0,1\}^{V_n}$ by first picking $X(V)$ according to μ_G and then, independently for each $v \in V_n \setminus V$, set $X(v) = 1$ with probability

$$\begin{cases} \frac{\lambda}{1+\lambda} & \text{if } X(w) = 0 \text{ for the (unique) neighbor } w \text{ of } v \text{ in } G_n \\ 0 & \text{otherwise.} \end{cases}$$

Then the distribution of X is a Gibbs measure for the hard-core model on G_n with activity λ. These two mappings give a one-to-one correspondence between Gibbs measures for the hard-core model on G_n with activity λ, and Gibbs measures for the hard-core model on G with activity λ^*.

Proof. Pick a vertex $v \in V$, let $v_1, \ldots, v_n$ be the n vertices of $V_n \setminus V$ that are adjacent to v, and write W for the vertex set $\{v, v_1, \ldots, v_n\}$. Let ω be some H-feasible element of $\{0,1\}^{V \setminus \{v\}}$. Pick $Y \in \{0,1\}^{V_n}$ according to μ_{G_n}. By a direct application of (1) with $V' = W$, we get

$$\mu_{G_n}(Y(v) = 1 \,|\, Y(V \setminus \{v\}) = \omega) \;=\; 0$$

if $Y(u) = 1$ for some $u \in V$ such that $u \sim v$, and

$$\begin{aligned} \mu_{G_n}(Y(v) = 1 \,|\, Y(V \setminus \{v\} = \omega) &= \frac{\lambda}{\lambda + \sum_{k=0}^{n} \binom{n}{k} \lambda^k} \\ &= \frac{\lambda}{\lambda + (\lambda+1)^n} \\ &= \frac{\lambda^*}{\lambda^* + 1} \end{aligned}$$

otherwise. Since $v \in V$ and ω were arbitrary, we have from Lemma 2.1 that the projection of μ_{G_n} on $\{0,1\}^V$ is a Gibbs measure for the hard-core model on G with activity λ^*.

The converse is equally straightforward, and we observe also that the two mappings described above are inverse to each other, so that we indeed have a one-to-one correspondence between Gibbs measures for the two models. ∎

The desired nonmonotonicity behavior on T^d_n will be shown by combining Lemma 2.2 with the following result about the hard-core model on T^d, known from [16] and [6].

Theorem 2.3 *The hard-core model with activity λ on the regular tree T^d, $d \geq 2$, has a unique Gibbs measure if and only if $\lambda \leq d^d(d-1)^{-d-1}$.*

The next result provides the example which proves Theorem 1.2.

Proposition 2.4 *For $d \geq 13$, there exist $0 < \lambda_1 < \lambda_2$ (depending on d) such that the hard-core model on T^d_2 with activity λ has a unique Gibbs measure for $\lambda \in (0, \lambda_1] \cup [\lambda_2, \infty)$, and multiple Gibbs measures for $\lambda \in (\lambda_1, \lambda_2)$.*

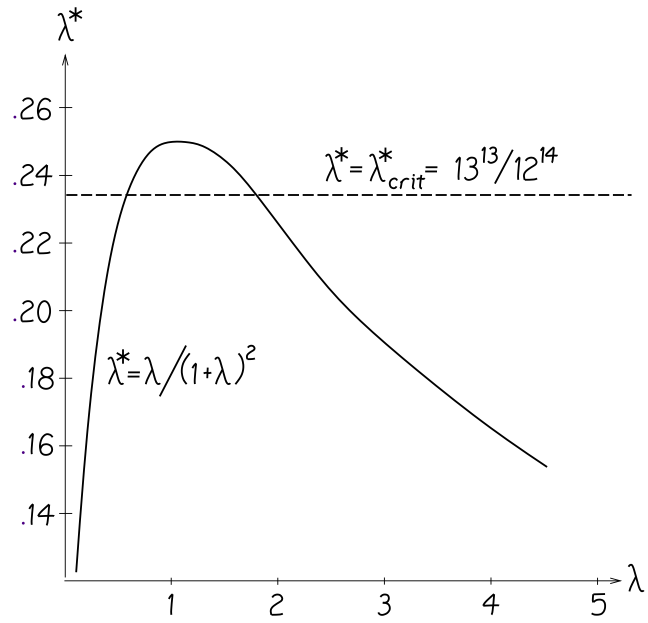

Figure 2: Hard-core activity on T^{13} as a function of activity on T_2^{13}

Proof. We have from Lemma 2.2 that uniqueness of Gibbs measures for the hard-core model on T_2^d with activity λ is equivalent to uniqueness of Gibbs measures for the hard-core model on T^d with activity $\lambda^* = f(\lambda)$, where $f(\lambda) = \frac{\lambda}{(\lambda+1)^2}$. Differentiating, we see that f is increasing on $[0,1]$ and decreasing on $[1,\infty)$. Also, $f(0) = 0$, $f(1) = \frac{1}{4}$ and $\lim_{\lambda\to\infty} f(\lambda) = 0$. The desired behavior on T_2^d thus occurs whenever the critical value for the hard-core model on T^d is less than $\frac{1}{4}$. By Theorem 2.3, we thus need

$$\frac{d^d}{(d-1)^{d+1}} < \frac{1}{4}\ ,$$

which holds for $d \geq 13$ (see Fig. 2). ∎

Our intuition for this behavior is something like this: as λ rises from 0, trying to pack more 1's into the interior nodes of T_2^{13} results eventually in symmetry-breaking as it does on T^{13}. However, the largest independent set in T_2^{13} is the set of leaves. When λ becomes very large, at least one from nearly every pair of leaves will map to 1 and thus the interior nodes are denuded of their 1's, relieving the pressure and restoring odd-even symmetry.

It is evident that the idea of replacing G with G_2 will give other examples of non-monotonicity for hard-core models on infinite graphs. Unfortunately, there is a distinct lack of exact results about critical behavior on graphs more complicated than trees, so it is not possible to be as precise in other cases.

What is known is that, for any graph G of bounded degree, there is some $\lambda_0 > 0$ such that, for any $\lambda < \lambda_0$, the hard-core model on G with activity λ has a unique Gibbs measure. For instance, this follows readily using the "disagreement percolation" argument of Van den Berg [2] (see the proof of Proposition 4.1). This gives us the following consequence of Lemma 2.2, proved in exactly the same way as Proposition 2.4.

Theorem 2.5 *Let G be any graph of bounded degree such that there is some $\lambda < 1/4$ for which the hard-core model on G with activity λ has multiple Gibbs measures. Then there are activities $\lambda_1 < \lambda_2 < \lambda_3$ such that the hard-core model on G_2 with activity λ has a unique Gibbs measure for $\lambda = \lambda_1$ and $\lambda = \lambda_3$, but multiple Gibbs measures for $\lambda = \lambda_2$.*

In particular, we expect that Theorem 2.5 applies to $\mathbb{Z}^d$, for suitably large d, so that the hard-core model on $\mathbb{Z}_2^d$ exhibits nonmonotonicity. Indeed, it is generally believed that the (conjectured) critical value $\lambda_c(d)$ for the existence of phase transition in the hard-core model on $\mathbb{Z}^d$ tends to 0 as $d \to \infty$, so that $\mathbb{Z}^d$ satisfies the hypotheses of Theorem 2.5 for sufficiently large d.

3 The Widom–Rowlinson model: Proof of Theorem 1.3

The Widom–Rowlinson model on T^d was first considered by Wheeler and Widom in [22]. They consider (implicitly) only Gibbs measures that are *invariant* under all symmetries of the tree, and *simple*, i.e. conditioned on the value at the root, the configurations on different branches from the root are independent. (Simple invariant Gibbs measures for general hard-constraint systems are investigated in [6].) In [22], Wheeler and Widom show that the model with activity λ has a unique simple invariant Gibbs measure if and only if

$$\lambda \leq \frac{1}{d-1}\left(\frac{d+1}{d}\right)^d.$$

One way to construct a simple invariant Gibbs measure for the Widom–Rowlinson model on T^d is, very loosely speaking, to condition on the boundary being $+1$. More precisely, for $k \geq 1$, let $\mathsf{T}^d(k)$ denote the subtree of T^d consisting of those vertices at distance at most k from the fixed root, and let $\nu_{+1}^\lambda(k)$ be the Gibbs measure on the finite graph $\mathsf{T}^d(k)$ obtained from $\nu_{T^d(k)}^\lambda$ by conditioning on all the leaves x having $\eta(x) = +1$. The measure $\nu_{+1}^\lambda(k)$ can be viewed as a measure on the infinite tree T^d, and then the sequence $\nu_{+1}^\lambda(k)$ converges to a simple invariant Gibbs measure ν_{+1}^λ for the Widom–Rowlinson model on T^d. The simple invariant Gibbs measure ν_{-1}^λ is constructed in the same way: what Wheeler and Widom prove is effectively that these two measures are equal if and only if λ is at or below the critical value $((d+1)/d)^d/(d-1)$. Above the critical value, these two measures are different, and there is also a third simple invariant Gibbs measure preserving the symmetry between $+1$ and -1.

In fact, below this critical value, there is only one Gibbs measure of any type for this model. This follows from the result of Wheeler and Widom, using standard monotonicity arguments (see e.g. Giacomin, Lebowitz and Maes [11]) which show that equality of ν_{+1}^λ and ν_{-1}^λ is sufficient for uniqueness of Gibbs measures in the Widom–Rowlinson model. A self-contained proof of the result below also appears in an earlier version of the present paper [5].

Theorem 3.1 *The Widom–Rowlinson model with activity λ on the regular tree T^d, $d \geq 2$, has a unique Gibbs measure if and only if*

$$\lambda \leq \frac{1}{d-1}\left(\frac{d+1}{d}\right)^d.$$

For λ above the critical value, there is an abundance of Gibbs measures, only three of which are simple and invariant.

The other main ingredient of the proof of Theorem 1.3 is the following Widom–Rowlinson analogue of Lemma 2.2. Recall the definition of G_n prior to Lemma 2.2.

Lemma 3.2 *Fix $G = (V, E)$, and let ν_{G_n} be a Gibbs measure for the Widom–Rowlinson model on $G_n = (V_n, E_n)$ with activity $\lambda > 0$. Then the projection of ν_{G_n} on $\{0,1\}^V$ is a Gibbs measure for the Widom–Rowlinson model on G with activity*

$$\lambda^* = \lambda \left(\frac{1+\lambda}{1+2\lambda} \right)^n .$$

Conversely, let ν_G be a Gibbs measure for the Widom–Rowlinson model on G with activity λ^*, and obtain $X \in \{0,1\}^{V_n}$ by first picking $X(V)$ according to ν_G and then, independently for each $v \in V_n \setminus V$, pick $X(v)$ according to appropriate conditional distribution (under the Widom–Rowlinson model with parameter λ) given the value of its (unique) nearest neighbor in G_n. Then the distribution of X is a Gibbs measure for the Widom–Rowlinson model on G_n with activity λ. These two mappings give a one-to-one correspondence between Gibbs measures for the Widom–Rowlinson model on G_n with activity λ, and Gibbs measures for the Widom–Rowlinson model on G with activity λ^*.

Proof. This is completely analogous to the proof of Lemma 2.2. ∎

The next result immediately implies Theorem 1.3.

Proposition 3.3 *The Widom–Rowlinson model on T_7^{40} behaves as follows. There are three critical values $\lambda_1 \approx 0.2179$, $\lambda_2 \approx 0.4013$ and $\lambda_3 \approx 4.5519$ such that if the activity is λ then we get a unique Gibbs measure for $\lambda \in (0, \lambda_1] \cup [\lambda_2, \lambda_3]$, and multiple Gibbs measures for $\lambda \in (\lambda_1, \lambda_2) \cup (\lambda_3, \infty)$.*

We remark that there are many (in fact, infinitely many) other values of d and n for which T_n^d exhibits the desired behavior. However, $d = 40$ and $n = 7$ are the smallest for which this happens.

Proof. Lemma 3.2 tells us that uniqueness of Gibbs measures for the Widom–Rowlinson model on T_n^d with activity λ is equivalent to uniqueness of Gibbs measures for the Widom–Rowlinson model on T^d with activity $\lambda^* = f_n(\lambda)$, where $f_n(\lambda) = \lambda \left(\frac{1+\lambda}{1+2\lambda}\right)^n$. Differentiating, we get

$$f_n'(\lambda) = \frac{(1+\lambda)^{n-1}}{(1+2\lambda)^{n+1}} [2\lambda^2 + (3-n)\lambda + 1] .$$

For $n \leq 5$, we deduce that $f_n'(\lambda) > 0$ throughout $[0, \infty)$ so that f_n is increasing on $[0, \infty)$. For such n and any d, we thus have by Theorem 3.1 that nonuniqueness of Gibbs measures is increasing in λ. If we take $n \geq 6$, however, the situation becomes different: the equation $f_n(\lambda) = 0$ then has two positive solutions $\lambda_{n,1}$ and $\lambda_{n,2}$ given by

$$\lambda_{n,1} = \frac{n-3}{4} - \sqrt{\left(\frac{n-3}{4}\right)^2 - \frac{1}{2}}$$

and

$$\lambda_{n,2} = \frac{n-3}{4} + \sqrt{\left(\frac{n-3}{4}\right)^2 - \frac{1}{2}} .$$

We conclude that $f_n(\lambda)$ is increasing on $[0, \lambda_{n,1}]$, decreasing on $[\lambda_{n,1}, \lambda_{n,2}]$ and increasing again on $[\lambda_{n,2}, \infty]$. We thus get the desired nonmonotonicity if we can find a d such that the critical value for the Widom–Rowlinson model on T^d is strictly between $f_n(\lambda_{n,2})$ and $f_n(\lambda_{n,1})$. For $n = 6$, it

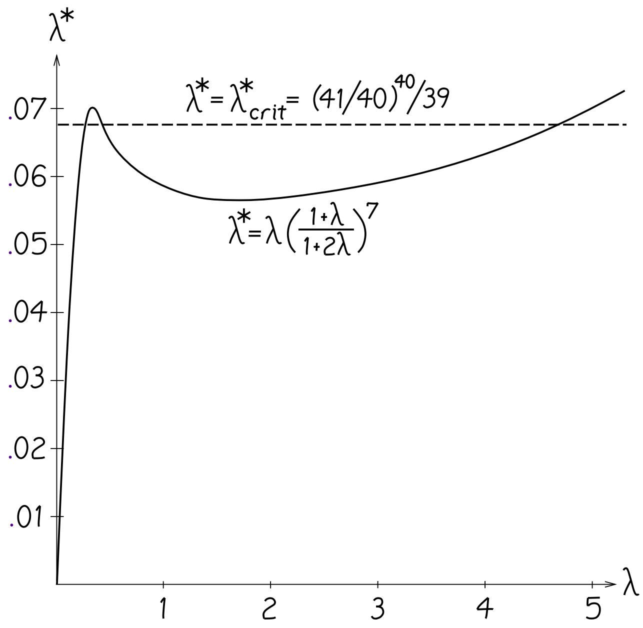

Figure 3: Widom–Rowlinson activity on T^{40} as a function of activity on T^{40}_7

turns out that no such d exists, but for $n = 7$, taking $d = 40$ (or indeed any of $d = 41, 42, \ldots, 49$) does the job (see Fig. 3). The numerical values for λ_1, λ_2 and λ_3 pop out as the solutions to the equation

$$f_n(\lambda) = \frac{1}{d-1}\left(\frac{d+1}{d}\right)^d$$

with $n = 7$ and $d = 40$ (see Fig. 3 again). ∎

Our intuition for the triple-critical-point behavior is similar to the hard-core case but with an extra twist. For low λ the random configuration is mostly 0, but as λ rises, either 1's or -1's tend to take over the interior vertices as in T^{40}. Then comes the third interval, where the septuplets of leaves, wanting to contain both 1's and -1's, force more 0's on the interior vertices, relieving the pressure and restoring $\{-1, 1\}$-symmetry. Finally the activity becomes so large that the random configuration is willing to give up $\{-1, 1\}$-variety among the septuplets in order to kill 0's among the interior vertices, and symmetry-breaking appears once again.

As in the hard-core case, the method above can be used to show that we get nonmonotonicity on many graphs of the form G_n. Indeed, calculations reveal that the ranges $[\lambda_{n,1}, \lambda_{n,2}]$ overlap for $n \geq 7$, so that, if G is any graph of bounded degree for which there is some $\lambda < 0.070135\ldots$ such that the Widom–Rowlinson model on G with activity λ has multiple Gibbs measures, then there is some n such that the Widom–Rowlinson model on G_n exhibits nonmonotonicity of phase transition. Again, we expect this to apply to $\mathbb{Z}^d$, for large enough d, but we are a long way from a proof.

However, given any bounded-degree graph G which is known to exhibit phase transition above some λ_0 for the hard-core or Widom–Rowlinson model, we can construct a variation kG of G with a critical point as close to 0 as we wish. Then for some n, the graph $(kG)_n$ will have multiple critical points. If G (unlike T^d) has the properties of amenability (large finite subraphs with relatively small boundary) and well-connectedness (no finite deletion creates multiple infinite components), then so have kG and $(kG)_n$. Hence the nonmonotonicity occurs in graphs which are kin to lattices in Euclidean space.

The construction is simple: let the vertex set of $kG = (kV, kE)$ be given by

$$kV := \{u_i : \ u \in V, \ i \in \{1, \ldots, k\}\}$$

with $u_i \sim v_j$ iff either $u \sim v$, or $u = v$ and $i \neq j$. In the hard-core model, the clique $[u] := \{u_1, \ldots, u_k\}$ either has precisely one occupied site (in which case no neighboring clique may have one) or no occupied sites; this dichotomy induces a correspondence between Gibbs measures with activity λ on kG and Gibbs measures with activity $\lambda^* = k\lambda$ on G.

In the Widom–Rowlinson model, there is a trichotomy: the clique $[u]$ either has one or more positive sites, one or more negative sites, or all sites zero (unoccupied). Again the classification influences a neighboring clique exactly as in neighboring sites of G, but here the relation between activities is given by $\lambda^* = (1+\lambda)^k - 1$. We thus have:

Theorem 3.4 *Let G be a graph on which the hard-core or Widom–Rowlinson model has a critical activity $\lambda_0 > 0$, and let $\epsilon > 0$. Then for sufficiently large k the graph kG has a critical point (for the same model) in the interval $(0, \epsilon)$.*

Even if G (and thus kG) already has multiple critical points, choosing k so as to drive some critical activity λ_0 for kG below 0.070135 will enable us to choose n such that $(kG)_n$ will have two (hard-core) or three (Widom–Rowlinson) critical points corresponding to λ_0. Hence,

Corollary 3.5 *With G, λ_0 and k as in Theorem 3.4, there is an n such that $(kG)_n$ has multiple critical points.*

As an example, we try $G = \mathbb{Z}^2$ in the Widom–Rowlinson model. Here the random-cluster representation (see the proof of Theorem 4.2 below) dominates independent percolation at

$$p = \frac{2^{-3}\lambda}{1 + 2^{-3}\lambda}$$

and is dominated by independent percolation at

$$p = \frac{2^{1}\lambda}{1 + 2^{1}\lambda}$$

since the occupation of a new site can reduce the number of connected components by at most 3 and increase it by at most 1. (See e.g. [8] or the proof of Theorem 4.2 below for other examples of this reasoning.) From [24] we have the (rigorous) upper bound 0.679492 for the critical probability of site percolation on $\mathbb{Z}^2$, and from [3] the lower bound 0.556.

It follows that there is at least one critical point λ_0^* for the Widom–Rowlinson model on $\mathbb{Z}^2$ with

$$0.626 < \lambda_0^* < 16.961$$

and taking $k = 43$, we have a critical point λ_0 for $k\mathbb{Z}^2$ with

$$f_{10}(\lambda_{10,2}) < 0.011369 \leq \lambda_0 \leq 0.06995 < 0.070135 \ .$$

It follows that for some n between 7 and 10, $(43\mathbb{Z}^2)_n$ has multiple critical points for the Widom–Rowlinson model.

4 Monotonicity in G

In this section we will address the question of whether the occurrence of phase transition is monotone increasing as we vary the graph G rather than the activity parameter λ. We write, with some abuse of notation, $G \subseteq G'$ if $G = (V, E)$, $G' = (V', E')$, $V \subseteq V'$ and $E \subseteq E'$. If the hard-core or the Widom–Rowlinson model exhibits phase transition on G, is the same true on G' ?

For fixed λ, this is certainly not the case for either of the models. Take for instance $G = \mathsf{T}^2$, $G' = \mathsf{T}^2_2$ and $\lambda = 5$. Then $G \subseteq G'$ while the results in Section 2 show that the hard-core model on G exhibits a phase transition at activity $\lambda = 5$, whereas the hard-core model on G' does not. The same counterexample works for the Widom–Rowlinson model, as is readily shown using Theorem 3.1 and Lemma 3.2.

One can, however, ask for a weaker form of monotonicity in the graph structure. Call a graph G *live* with respect to the hard-core (resp. Widom–Rowlinson) model if there exists some $\lambda > 0$ such that the hard-core (resp. Widom–Rowlinson) model on G with activity λ has multiple Gibbs measures. Does liveness of G imply liveness of G' for $G' \supseteq G$?

For the hard-core model, the example $G = \mathsf{T}^2$, $G' = \mathsf{T}^2_2$ again provides a counterexample, showing that liveness is not increasing in the graph structure. For the Widom–Rowlinson model, a more complicated counterexample shows that liveness fails to be increasing in the graph structure; this is demonstrated in Proposition 4.1 below. However, we can prove a result of this kind for the Widom–Rowlinson model if we make the assumption that G' has bounded degree; see Theorem 4.2. Summarizing what we know, we get the following trichotomy. Assume that $G \subseteq G'$ and that the graphs have bounded degree.

1. Suppose that the (symmetric) Ising model on G at reciprocal temperature $\beta > 0$ exhibits phase transition. Then the same thing holds with G replaced by G' . As with the monotonicity property of the Ising model quoted in the introduction, this follows from Griffiths inequalities [19], or, alternatively, random-cluster methods [14].

2. For the Widom–Rowlinson model with activity $\lambda > 0$, having phase transition on G does not imply phase transition on G' with the same activity. However, liveness of G does imply liveness of G'.

3. For the hard-core model, not even liveness is increasing in the graph structure.

Here the assumption of bounded degree is essential, because of the following result.

Proposition 4.1 *There exists a pair of graphs G and G' such that*

(i) $G \subseteq G'$,

(ii) G is live for the Widom–Rowlinson model, while

(iii) G' is not live for the Widom–Rowlinson model.

Proof. Our candidates for G and G' are T^2 and T^2_{dist}, where T^2_{dist} is defined as follows. Take T_2, designate one of its vertices as the root, and attach to each vertex a number of leaves. This is done in the same manner as in the construction of T^d_n, except that this time the number of leaves depends on where in the tree we are. Specifically, each vertex v of T^2 is assigned exactly $\mathrm{dist}(v, r)$ leaves, where $\mathrm{dist}(v, r)$ is the graph-theoretical distance between v and r. Clearly $\mathsf{T}^2 \subseteq \mathsf{T}^2_{dist}$, and we know from Theorem 2.3 that T^2 is live.

It only remains to show that T^2_{dist} is not live. Fix $\lambda > 0$. First note that Lemma 3.2 easily extends to show that phase transition for the Widom–Rowlinson model on T^2_{dist} with activity λ is equivalent to phase transition for the "inhomogeneous" Widom–Rowlinson model (defined in the obvious way) on T^2, in which each vertex $v \in V$ (where V is the vertex set of T^2) has activity

$$\lambda_v^* = \lambda \left(\frac{1+\lambda}{1+2\lambda} \right)^{\mathrm{dist}(v,r)} . \tag{3}$$

Now let ν and ν' be two (a priori possibly different) Gibbs measures for such an inhomogeneous Widom–Rowlinson model on T^2, and write Y and Y' for two $\{-1,0,1\}$-valued random objects picked according to the product measure $\mathbf{P} = \nu \times \nu'$. Define, for each vertex v of T^2,

$$p_v = \sup_{\eta,\eta'} \mathbf{P}(Y(v) \neq Y'(v) \,|\, Y(V \setminus \{v\}) = \eta \,,\, Y'(V \setminus \{v\}) = \eta') \,,$$

where the supremum is over all feasible elements of $\{-1,0,1\}^{V\setminus\{v\}}$. A direct calculation shows that

$$p_v \,\leq\, 1 - \left(\frac{1}{1+2\lambda_v^*} \right)^2 . \tag{4}$$

Next, we want to apply the uniqueness condition of Van den Berg [2], and to this end we need to recall some percolation theory (see Grimmett [12] for a general introduction to percolation). In standard site percolation, each vertex of a graph G is independently assigned value 1 with probability p and value 0 with probability $1-p$. One then asks whether there exists some infinite open cluster (i.e. some infinite connected component of 1's). It is a standard fact that there exists a critical value $p_c = p_c(G)$ such that the probability of having some infinite open cluster is 0 if $p < p_c$ and 1 if $p > p_c$. It is also well known, and follows from an easy branching process argument, that $p_c(\mathsf{T}^2) = 1/2$ and that a.s. there is no infinite open cluster when $p = 1/2$. We now want to compare standard site percolation on T^2 with parameter $p = 1/2$ to the inhomogeneous percolation process where each $v \in V$ independently is assigned value 1 with probability p_v and value 0 with probability $1 - p_v$. For a positive integer N, define

$$V_N = \{v \in V : \mathrm{dist}(v,r) \leq R\}.$$

By (3) and (4), we have for any λ that $p_v \to 0$ as $\mathrm{dist}(v,r)$ goes to infinity. Hence, we can find an N (depending on λ) such that $p_v \leq 1/2$ for all $v \in V \setminus V_N$. It follows that the inhomogeneous percolation process restricted to $V \setminus V_N$ is stochastically dominated by the i.i.d. ($p = 1/2$) percolation process on the same vertex set. Hence, there is a.s. no infinite open cluster in the inhomogeneous percolation process restricted to $V \setminus V_N$, and since V_N is finite there is no infinite cluster in the unrestricted inhomogeneous percolation process either. But this is exactly what is needed in Corollary 2 of Van den Berg [2] in order to conclude that $\nu = \nu'$. Since λ was arbitrary, the proof is complete. ∎

Theorem 4.2 *Suppose that $G \subseteq G'$, that G' has bounded degree, and that G is live for the Widom–Rowlinson model. Then there exists a $\lambda_0 > 0$ such that the Widom–Rowlinson model on G' exhibits phase transition for all $\lambda > \lambda_0$.*

Proof. We first need to introduce the *random-cluster representation* of the Widom–Rowlinson model. If $G_\Box = (V_\Box, E_\Box)$ is a finite graph, then we define for $\lambda > 0$ the probability measure $\phi^\lambda_{G_\Box}$

on $\{0,1\}^{V_\Box}$ by

$$\phi^\lambda_{G_\Box}(\omega) = \frac{1}{\tilde{Z}^\lambda_{G_\Box}} \prod_{x \in V_\Box} \lambda^{\omega(x)} 2^{k(\omega)} \tag{5}$$

to each $\omega \in \{0,1\}^{V_\Box}$; here $k(\omega)$ denotes the number of connected components of the vertex set $\{x \in V_\Box : \omega(x) = 1\}$. Suppose now that we pick $X \in \{0,1\}^{V_\Box}$ according to $\phi^\lambda_{G_\Box}$, and generate $Y \in \{-1,0,1\}^{V_\Box}$ from X by first setting $Y(x) = 0$ whenever $X(x) = 0$, and then flipping an independent fair coin for each connected component of the set of 1's in X to determine whether to set $Y \equiv 1$ or $Y \equiv -1$ on that connected component. A direct calculation shows that Y is distributed according to the Gibbs measure $\nu^\lambda_{G_\Box}$ for the Widom–Rowlinson model on $G_\Box$. The measure $\phi^\lambda_{G_\Box}$ is in fact identical to the random-cluster representation of the so-called Iceberg model that was introduced in Section 7 of Häggström [14]; see also [15] for related ideas.

By a direct application of (5), we have

$$\phi^\lambda_{G_\Box}(X(x) = 1 \,|\, X(V_\Box \setminus \{x\}) = \omega) \;=\; \frac{\lambda 2^{1-\kappa(x,\omega)}}{\lambda 2^{1-\kappa(x,\omega)} + 1} \tag{6}$$

for any $x \in V_\Box$ and any $\omega \in \{0,1\}^{V_\Box \setminus \{x\}}$; here $\kappa(x,\omega)$ is the number of connected components of the set of 1's in ω that contain some vertex incident to x. Writing $d(G_\Box)$ for the maximum degree in $G_\Box$, we have that the right hand side of (6) satisfies

$$\frac{\lambda 2^{1-d(G_\Box)}}{\lambda 2^{1-d(G_\Box)} + 1} \;\le\; \frac{\lambda 2^{1-\kappa(x,\omega)}}{\lambda 2^{1-\kappa(x,\omega)} + 1} \;\le\; \frac{2\lambda}{2\lambda + 1} \tag{7}$$

for any $x \in V_\Box$ and $\omega \in \{0,1\}^{V_\Box \setminus \{x\}}$.

Next, for $p \in [0,1]$, let $\pi^p_{G_\Box}$ be the product probability measure on $\{0,1\}^{V_\Box}$ where each $x \in V_\Box$ takes value 1 independently with probability p. By (6) and (7), we have, using e.g. Holley's Theorem (see [19] or [14]), that

$$\pi^{p_1}_{G_\Box} \;\preceq\; \phi^\lambda_{G_\Box} \;\preceq\; \pi^{p_2}_{G_\Box} \tag{8}$$

where p_1 and p_2 denote the upper and lower bounds in (7), and $\preceq$ denotes stochastic domination.

Moving on to the infinite graph $G = (V,E)$, we may copy the arguments in [14] or [15] to see that phase transition for the Widom–Rowlinson model on G with activity λ is equivalent to having an infinite cluster in certain limits of random-cluster measures for finite subgraphs of G converging to G in the sense that each $x \in V$ is in all but finitely many such subgraphs. Since G is live, there exists some $\lambda > 0$ for which we have phase transition. We then have an infinite cluster in the corresponding limiting random-cluster representation, and by (8) also in i.i.d. site percolation on G with parameter $2\lambda/(2\lambda+1)$. Hence the critical value $p_c(G)$ for site percolation on G satisfies

$$p_c(G) \;\le\; \frac{2\lambda}{2\lambda+1} \;<\; 1\,.$$

Since $G' \supseteq G$, we also have $p_c(G') \le p_c(G)$ so that in particular $p_c(G') < 1$. We can then find a λ_0 sufficiently large so that

$$\frac{\lambda' 2^{1-d(G')}}{\lambda' 2^{1-d(G')} + 1} \;>\; p_c(G')$$

for all $\lambda' > \lambda_0$. By another application of (8), we get infinite clusters in the limiting random-cluster measures for G' for all such λ', and hence also phase transition for the Widom–Rowlinson model on G' with such a choice of the activity. ∎

Note that the above proof shows that if a graph G has bounded degree, then G is live for the Widom–Rowlinson model if and only if the critical value for standard site percolation on G is strictly less than 1. A similar statement is true for the Ising model (even without the bounded degree assumption) and can be shown similarly.

5 A hard-core representation of the Widom–Rowlinson model

Let us finally indicate an alternative route to proving nonmonotonicity of phase transition in the hard-core model, via the Widom–Rowlinson model. Let $G = (V, E)$ be infinite, and let $G^* = (V^*, E^*)$ be the graph which is obtained from G as follows. Let $V^* = V \times \{0, 1\}$, and let two vertices (x, i) and (y, j) in V^* be linked by an edge if

(a) $i \neq j$, and

(b) either $x \sim y$ or $x = y$.

Now let X be a $\{0,1\}^{V^*}$-valued random variable distributed according to a Gibbs measure for the hard-core model on G^* with activity λ, and let Y be the $\{-1, 0, 1\}^V$-valued random variable obtained by letting

$$Y(x) = \begin{cases} -1 & \text{if } X(x,0) = 1 \\ 1 & \text{if } X(x,1) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

for each $x \in V$. Then it turns out that Y is distributed according to a Gibbs measure for the Widom–Rowlinson model on G with activity λ, and moreover it is not hard to show that this gives a one-to-one correspondence between Gibbs measures for the hard-core model on G^* and Gibbs measures for the Widom–Rowlinson model on G. Theorem 1.3 thus implies the following additional result for the hard-core model, slightly different from Theorem 1.2.

Corollary 5.1 *There exist $0 < \lambda_1 < \lambda_2 < \lambda_3$ and an infinite graph G, such that the hard-core model on G with activity λ has a unique Gibbs measure for $\lambda \in (0, \lambda_1] \cup [\lambda_2, \lambda_3]$, and multiple Gibbs measures for $\lambda \in (\lambda_1, \lambda_2) \cup (\lambda_3, \infty)$.*

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