Biased random walk in a one-dimensional percolation model

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June 2, 2009

Abstract

We consider random walk with a nonzero bias to the right, on the infinite cluster in the following percolation model: take i.i.d. bond percolation with retention parameter p on the so-called infinite ladder, and condition on the event of having a bi-infinite path from $-\infty$ to ∞. The random walk is shown to be transient, and to have an asymptotic speed to the right which is strictly positive or zero depending on whether the bias is below or above a certain critical value which we compute explicitly.

Mathematics Subject Classifications: 60J10 60K35 60K37

Key words: percolation, random walk, asymptotic speed

1 Introduction

Random walk on a percolation cluster has received considerable attention in recent years. For simple random walk on the infinite cluster of i.i.d. percola-

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tion on the $\mathbb{Z}^d$ lattice, several authors such as De Masi et al. [7], Berger and Biskup [5] and Mathieu and Piatnitski [12] focused on invariance principles. Others have considered the recurrence-transience problem; see Grimmett et al. [9] for the basic result showing that the usual recurrence-transience dichotomy between random walks on $\mathbb{Z}^2$ and in higher dimensions is inherited by random walks on the infinite clusters, and [3], [10] and [1] for some subsequent developments.

Introducing a bias by letting the walk favor moves in a pre-specified direction turns out to make the walk transient regardless of the dimension d, but a very interesting dichotomy of a different kind was established by Berger, Gantert and Peres [6] and Sznitman [14] concerning the asymptotic speed. Namely, when the bias is small enough, the walk exhibits a positive asymptotic speed in the direction of the bias, while for large bias the asymptotic speed vanishes. At first sight it may seem somewhat counterintuitive that the asymptotic speed should go from being strictly positive to zero when the bias is increased, but what happens is actually not so strange: when the bias becomes too large, the walk starts spending huge amounts of time in "dead end" regions (taking the shape of peninsulas stretching out in the direction of the bias) before eventually backtracking and continuing its march off to infinity; once the walker finds himself in such a dead end region, the larger the bias is the more reluctant is he to perform the necessary backtrack.

The Berger–Gantert–Peres–Sznitman (BGPS) result suggests that there should exist a critical value $\beta_c = \beta_c(d,p)$ for the bias parameter β, where p is the retention probability of the underlying percolation process, such that the asymptotic speed is positive when $\beta < \beta_c$ and zero when $\beta > \beta_c$. For this, we need to know that the speed cannot go from zero to strictly positive as β increases, but proving such a monotonicity result appears to be difficult.

The purpose of the present paper is to establish the asked-for critical phenomenon in a different percolation setting that turns out to be more tractable. A similar result was obtained by Lyons, Pemantle and Peres [11] in the setting of Galton–Watson trees. The setting we opt for here is the one introduced in our recent paper [2], where we considered a dependent one-dimensional percolation model that arises by taking i.i.d. bond percolation on the so-called infinite ladder, and conditioning, in a specific sense, on the existence of a bi-infinite path from $-\infty$ to ∞. For biased random walk on the infinite cluster arising in this model, we recover the corresponding phenomenon as in the BGPS result, but also find an explicit expression for a critical value $\beta_c = \beta_c(p)$ separating the positive speed region from the zero speed region; this is Theorem 3.2 below, which is our main result.

Our paper is organized as follows. In Section 2, we define the percolation model and recall from [2] some key tools for analyzing it. In Section 3, we introduce the random walk and state our main result. To prove it, we employ an electrical network analysis a la Doyle and Snell [8] in Section 4, together with coupling and ergodicity arguments in Section 5.

2 The percolation model

Write $\mathcal{L} = (V, E)$ for the graph with vertex set $V = \mathbb{Z} \times \{0, 1\}$ and edge set E consisting of pairs of vertices at Euclidean distance 1 from each other; for obvious reasons, $\mathcal{L}$ is known as the infinite ladder.

In i.i.d. bond percolation with parameter p, each edge of an infinite connected graph G is, independently of all others, removed with probability $1-p$ and retained with probability p. Retained and removed edges are also called open and closed, and the resulting subgraph can be identified with an element of $\{0,1\}^E$, where 0 denotes "closed" and 1 "open". The critical value $p_c(G)$ is defined as the infimum of all $p \in [0,1]$ such that i.i.d. bond percolation on G a.s. produces an infinite connected component. Unlike for instance the standard $\mathbb{Z}^d$ lattice in $d \geq 2$ dimensions, the infinite ladder has no nontrivial critical value: a simple Borel–Cantelli argument shows that $p_c(\mathcal{L}) = 1$. This gives reason to dismiss percolation on $\mathcal{L}$ as uninteresting, but in [2] we made an attempt to resurrect the topic by introducing a dependent percolation model involving conditioning i.i.d. bond percolation with retention parameter $p \in (0,1)$ on the event that there is an open path from $-\infty$ to ∞. Since that event has probability zero, some explanation is needed:

For $N_1, N_2 > 0$, let B_{N_1,N_2} be the event that there exists an open path from some vertex with x-coordinate $-N_1$ to some vertex with x-coordinate N_2, and for $p \in (0,1)$ let P_{p,N_1,N_2} be the probability measure on $\{0,1\}^E$ that arises by conditioning i.i.d. bond percolation with parameter p on the event B_{N_1,N_2}. Let $B = \cap_{i=1}^{\infty} B_{i,i}$, so that informally speaking B is the event of having an open path from $-\infty$ to ∞. In [2] we established the following result, where convergence is in the product topology, meaning that the probability of any cylinder event converges.

Theorem 2.1 *For any $p \in (0,1)$, the probability measures P_{p,N_1,N_2} converge weakly to a probability measure P_p on $\{0,1\}^E$ as $N_1, N_2 \to \infty$. P_p is translation invariant and assigns probability 1 to the event B.*

Obviously the resulting measure P_p gives rise to dependence between edges. In fact, it turns out that edges arbitrarily far from each other are correlated (though the correlation does decay to 0 with the distance), and sometimes they are correlated conditionally on the status of all other edges. The last observation indicates a kind of non-Markovianity of the model. In spite of this, we found in [2] the following Markovian representation of it, which will be heavily exploited in later sections. Theorems 2.2 and 2.3 below are proved in [2].

For fixed $i \in \mathbb{Z}$, define $E^{i,-} \subset E$ as the set of edges both of whose endpoints have x-coordinates not exceeding i. Given the percolation process $X \in \{0,1\}^E$, we say that a vertex $\{i,j\}$ is **backwards-communicating** if it is connected to $-\infty$ via a path that is completely contained in $E^{i,-}$ and all of whose edges are open in X. Define the $\{\texttt{00}, \texttt{01}, \texttt{10}, \texttt{11}\}$-valued process $\{T_i\}_{i\in\mathbb{Z}}$ by setting, for each $i \in \mathbb{Z}$,

$$T_i = \begin{cases} \texttt{00} & \text{if neither } \{i,0\} \text{ nor } \{i,1\} \text{ is backwards-communicating} \\ \texttt{01} & \text{if } \{i,1\} \text{ but not } \{i,0\} \text{ is backwards-communicating} \\ \texttt{10} & \text{if } \{i,0\} \text{ but not } \{i,1\} \text{ is backwards-communicating} \\ \texttt{11} & \text{if both } \{i,0\} \text{ and } \{i,1\} \text{ are backwards-communicating.} \end{cases} \quad (1)$$

Note that the $P_p(B) = 1$ part of Theorem 2.1 implies that $P_p(T_i = \texttt{00}) = 0$ for any i; the first line of (1) is included for completeness only.

Theorem 2.2 *$\{T_i\}_{i\in\mathbb{Z}}$ is a time-homogeneous Markov chain.*

The next result shows that the percolation process X has a very simple distribution conditional on the Markov chain. Define $E^i = E^{i,-} \setminus E^{i-1,-}$, or in other words

$$E^i = \{\langle\{i-1,0\},\{i,0\}\rangle, \langle\{i-1,1\},\{i,1\}\rangle, \langle\{i,0\},\{i,1\}\rangle\}. \quad (2)$$

Note that given T_{i-1} and $X(E^i)$ we can read off for each of the vertices $\{i,0\}$ and $\{i,1\}$ whether it is backwards-communicating, and thus we also know

T_i. Given $T_{i-1} = \mathtt{ab} \in \{\mathtt{01}, \mathtt{10}, \mathtt{11}\}$ and $T_i = \mathtt{cd} \in \{\mathtt{01}, \mathtt{10}, \mathtt{11}\}$, we call a configuration $\eta \in \{0,1\}^{E^i}$ T_{i-1}, T_i**-compatible** if $T_{i-1} = \mathtt{ab}$ and $X(E^i) = \eta$ yields $T_i = \mathtt{cd}$. Note that for $T_{i-1} = \mathtt{01}$ and $T_i = \mathtt{10}$ or vice versa there is no T_{i-1}, T_i-compatible $\eta \in \{0,1\}^{E^i}$. For all other choices, define the measure P_{p,i,T_{i-1},T_i} on $\{0,1\}^{E^i}$ by setting, for each $\eta \in \{0,1\}^{E^i}$,

$$P_{p,i,T_{i-1},T_i}(\eta) = \frac{I_{\{\eta \text{ is } T_{i-1},T_i-\text{compatible}\}}}{Z_{p,i,T_{i-1},T_i}} \prod_{e \in E^i} p^{\eta(e)}(1-p)^{1-\eta(e)} \tag{3}$$

where Z_{p,i,T_{i-1},T_i} is a normalizing constant making P_{p,i,T_{i-1},T_i} a probability measure.

Theorem 2.3 *The conditional distribution of the percolation process $X \in \{0,1\}^E$ given the Markov chain $\{T_i\}_{i\in\mathbb{Z}}$ is*

$$\prod_{j\in\mathbb{Z}} P_{p,j,T_{j-1},T_j} \, . \tag{4}$$

In other words, given $\{T_i\}_{i\in\mathbb{Z}}$, it is for each j the case that $X(E^j)$ has distribution P_{p,j,T_{j-1},T_j}, with independence for different j's. Theorems 2.2 and 2.3 immediately imply the following.

Corollary 2.4 *For any i, we have that $X(E^{i,-})$ and $X(E \setminus E^{i,-})$ are conditionally independent given T_i.*

We mention that extensions of Theorems 2.2 and 2.3 were obtained in [2] for a wider class of one-dimensional periodic lattices.

The next ingredient that will be crucial to our random walk analysis is to have explicit expressions for the transition matrix and stationary distribution for $\{T_i\}_{i\in\mathbb{Z}}$. Obtaining those in the larger generality of one-dimensional periodic lattices seems cumbersome, but we did obtain them in [2] for the case of $\mathcal{L}$. Ignoring the state $\mathtt{00}$ which, as remarked above, has probability 0 of showing up, we may view $\{T_i\}_{i\in\mathbb{Z}}$ as a 3-state Markov chain. Its transition matrix turns out to equal

$$\mathbf{p} = \begin{pmatrix} p_{\mathtt{01},\mathtt{01}} & p_{\mathtt{01},\mathtt{10}} & p_{\mathtt{01},\mathtt{11}} \\ p_{\mathtt{10},\mathtt{01}} & p_{\mathtt{10},\mathtt{10}} & p_{\mathtt{10},\mathtt{11}} \\ p_{\mathtt{11},\mathtt{01}} & p_{\mathtt{11},\mathtt{10}} & p_{\mathtt{11},\mathtt{11}} \end{pmatrix} = \begin{pmatrix} 1-p_{\mathtt{01},\mathtt{11}} & 0 & p_{\mathtt{01},\mathtt{11}} \\ 0 & 1-p_{\mathtt{01},\mathtt{11}} & p_{\mathtt{01},\mathtt{11}} \\ p_{\mathtt{11},\mathtt{01}} & p_{\mathtt{11},\mathtt{01}} & 1-2p_{\mathtt{11},\mathtt{01}} \end{pmatrix} . \tag{5}$$

where

$$p_{01,11} = \frac{1}{2p}\left(2p^2 - 1 + \sqrt{1 + 4p^2 - 8p^3 + 4p^4}\right)$$

and

$$p_{11,01} = \frac{1}{4(1-p)}\left(2(1-p) - (3-2p)\left(1 + 2p - 2p^2 - \sqrt{1 + 4p^2 - 8p^3 + 4p^4}\right)\right). \quad (6)$$

These expressions for $p_{01,11}$ and $p_{11,01}$ can be seen to be strictly positive for $p \in (0,1)$, making the chain irreducible. It therefore has a unique stationary distribution, which is easily calculated as

$$\pi = \{\pi_{01}, \pi_{10}, \pi_{11}\} = \left\{\frac{p_{11,01}}{2p_{11,01} + p_{01,11}}, \frac{p_{11,01}}{2p_{11,01} + p_{01,11}}, \frac{p_{01,11}}{2p_{11,01} + p_{01,11}}\right\}.$$

3 The random walk: main result

Let $\mathcal{L} = (V, E)$ be the infinite ladder as in the previous section, fix $p \in (0,1)$ and generate a percolation configuration $X \in \{0,1\}^E$ according to P_p. With X thus fixed, a random walk $Y = (Y_0, Y_1, \ldots)$, with each $Y_i \in V$, is defined as follows. First fix the so-called drift parameter $\beta \geq 0$. The random walk is taken to begin at the origin $\mathbf{0}$, i.e. we set $Y_0 = \mathbf{0} = (0,0)$. For each $i \geq 1$, Y_i is generated from Y_{i-1} as follows. A candidate value Y_{cand} is chosen among the three nearest-neighbors of $Y_{i-1} = (x, y)$ with distribution given by

$$Y_{cand} = \begin{cases} (x-1, y) & \text{with probability } \frac{\beta^{-1}}{\beta^{-1}+1+\beta} \\ (x, 1-y) & \text{with probability } \frac{1}{\beta^{-1}+1+\beta} \\ (x+1, y) & \text{with probability } \frac{\beta}{\beta^{-1}+1+\beta} \end{cases} \quad (7)$$

conditional on X and $(Y_0, \ldots, Y_{i-1})$. The vertex Y_{cand} is where the random walker “wants” to move, but only moves along edges that are open in X are allowed, so if $X(\langle Y_{i-1}, Y_{cand}\rangle) = 0$, the move is suppressed. We thus set

$$Y_i = \begin{cases} Y_{cand} & \text{if } X(\langle Y_{i-1}, Y_{cand}\rangle) = 1 \\ Y_{i-1} & \text{otherwise.} \end{cases}$$

(An alternative, and perhaps more common, way to set up the random walk is to let Y_i have the distribution indicated in (7) conditioned on the event that the chosen vertex is among those that are linked to Y_{i-1} via an edge that is open in X. This would make no difference to our main results (Proposition 3.1 and Theorem 3.2 below), but would have the downside of requiring separate treatment of the somewhat boring case when the random walk happens to start at an isolated (in X) vertex.)

By left-right reflection invariance of $\mathcal{L}$ and of the percolation model, changing the drift parameter from β to β^{-1} makes no essential difference. We therefore restrict to $\beta \geq 1$ without loss of generality.

The recurrence-transience problem for our random walk is of course only interesting when the starting point $(0,0)$ happens to be in the infinite connected component of the percolation configuration X; write A for the event that this happens. The case $\beta = 1$ corresponds to simple random walk, for which recurrence holds for the following reason. By Rayleigh's monotonicity principle (see, e.g., Doyle and Snell [8]), removing edges from a graph cannot transform simple random walk from recurrence to transience. The subgraph of $\mathcal{L}$ corresponding to X is of course also a subgraph of the $\mathbb{Z}^2$ lattice, for which recurrence is well-known.

Taking instead $\beta > 1$ changes the situation:

Proposition 3.1 *For any $p \in (0,1)$ and any $\beta > 1$, we have a.s. on the event A that the random walk $(Y_0, Y_1, \ldots)$ is transient.*

This will follow from some very simple electrical considerations in the next section. A much more subtle issue, once the random walk is shown to be transient, is to ask how fast it moves away from its starting point. Specifically, writing $x(Y_i)$ for the x-coordinate of Y_i, we may ask whether the asymptotic speed $\lim_{i\to\infty} \frac{x(Y_i)}{i}$ exists. In the following result, which is the main result of this paper, we find that the speed is indeed well-defined, and moreover that it is a.s. constant on the event A. Furthermore, we find that the asymptotic speed is strictly positive when $\beta \in (1, \beta_c)$, and 0 when $\beta \geq \beta_c$, where $\beta_c = \beta_c(p)$ is the critical value given explicitly in (8). Deriving an explicit expression for the actual speed in the positive speed regime would require a more refined analysis that we haven't been able to push through: crude upper and lower bounds are easily extracted from our analysis in the following sections, but not an exact value.

Theorem 3.2 *For any fixed $p \in (0,1)$ and $\beta > 1$, we have that the asymptotic speed $\lim_{i\to\infty} \frac{x(Y_i)}{i}$ is well-defined a.s., and is an a.s. constant $\theta(p,\beta)$ on the event A. Furthermore,*

$$\theta(p,\beta) \begin{cases} > 0 & \text{for } \beta \in (1,\beta_c) \\ = 0 & \text{for } \beta \geq \beta_c\,, \end{cases}$$

where the critical value $\beta_c = \beta_c(p)$ is given by

$$\beta_c \;=\; \sqrt{2/(1+2p-2p^2-\sqrt{1+4p^2-8p^3+4p^4})}\,. \tag{8}$$

It is worth noting that the critical value β_c tends to ∞ both as $p \to 0$ and as $p \to 1$.

4 Electrical analysis

An alternative description of the random walk introduced in the previous section is as follows. Assign to each edge $e = \langle (x,y),(x',y') \rangle$ in $\mathcal{L}$ a weight $C(e)$, given by

$$C(e) \;=\; \beta^{x+x'}\,.$$

A random walker standing at vertex Y_{i-1} picks an edge at random among the three edges incident to Y_{i-1} with probabilities proportional to their weights; if the chosen edge is open in the percolation configuration X then the walker traverses it, while otherwise he stays where he is for one more time unit. It is immediate that this gives the same transition kernel, and thus the same model, as the one defined in connection with (7).

The weights $\{C(e)\}_{e\in E}$ are also known as conductances – a terminology stemming from the identity between random walks and electrical networks outlined beautifully in the monograph by Doyle and Snell [8]. From that theory, we will make particular use of the notion of effective conductance to infinity. This requires a few preliminary definitions.

Let $G = (V,E)$ be any infinite connected graph with nonnegative edge conductances $\{C(e)\}_{e\in E}$. By a **flow** on G we mean a function $F : V^2 \to \mathbf{R}$ satisfying

(a) $F(u,v)=0$ unless u and v share an edge in E, and

(b) $F(u,v)=-F(v,u)$,

for all $u,v\in V$. (We should think of $F(u,v)$ as the amount of current flowing from u to v through the edge $\langle u,v\rangle$.) For $v\in V$, a **unit current from** v **to** ∞ is defined as a flow F on G such

$$\sum_{w\in V} F(u,w) = \begin{cases} 1 & \text{if } u=v \\ 0 & \text{otherwise.} \end{cases}$$

The **energy** $W(F)$ of a flow F on G is defined as

$$W(F) = \sum_{e\in E} W(F,e)$$

where for any $e=\langle u,w\rangle\in E$ we set $W(F,e)=\frac{F(u,w)^2}{C(e)}$. Finally, the **effective conductance** $C_{eff}(v,\infty)$ between v and ∞ is defined as the infimum of $W(F)$ over all unit currents from v to ∞. The following result on random walk is well-known (see, e.g. [8]). By the **escape probability** of a random walk from its starting point $v\in V$, we mean the probability that once it leaves v it never comes back.

Lemma 4.1 *Consider random walk on an infinite connected graph* $G=(V,E)$ *with edge weights* $\{C(e)\}_{e\in E}$ *starting at* $v\in V$. *The probability that the random walk never returns to* v *equals* $\frac{C_{eff}(v,\infty)}{C(v)}$, *where* $C(v)$ *is the sum of the conductances of all edges incident to* v.

For the proof of Proposition 3.1 and also later, one more definition, this time pertaining to the percolation subgraph X of the infinite ladder $\mathcal{L}=(V,E)$, is convenient. Given $X\in\{0,1\}^E$, a vertex $v\in V$ is said to be **good** if x contains an open path from v to $+\infty$ that never visits any vertex whose x-coordinate is smaller than that of v. Vertices that are not good are called **bad**.

Proof of Proposition 3.1. Transience is equivalent to the random walk having a nonzero escape probability from the starting point $\mathbf{0}$. Thus, in view

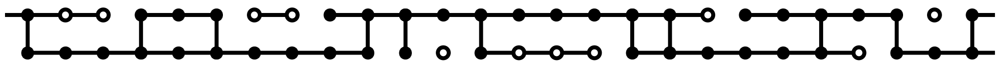

Figure 1: Good vertices are filled circles, and bad vertices are unfilled. Note that all bad vertices in the infinite cluster sit in "dead ends" extending rightwards.

of Lemma 4.1, we need to show that if $\mathbf{0}$ is in the infinite cluster of X, then there exists a finite energy unit current from $\mathbf{0}$ to ∞.

Assume first that $\mathbf{0}$ is good. Then we can find a infinite self-avoiding path S from $\mathbf{0}$ to $+\infty$ in the infinite cluster of $\mathbf{0}$ that never visits a vertex with negative x-coordinate. Fix such a path S. It follows from the geometry of $\mathcal{L}$ that S never "backtracks" (in the sense of taking a step to the left). We now simply define F by pushing a unit current through S. By the non-backtracking property of S, we have for each $k \geq 0$ that S contains at most one edge e with $C(e) = \beta^k$. Hence the energy of the unit flow through S is at most $\sum_{k=0}^{\infty} \beta^{-k} < \infty$, settling the case of $\mathbf{0}$ being good.

Assume next that $\mathbf{0}$ is bad and part of the infinite cluster. Let m be the smallest $n \geq 0$ such that the vertical edge at x-coordinate $-n$ is open. Then $\{-m, 0\}$ is a good vertex, and furthermore all of the edges

$$\langle \{0,0\}, \{-1,0\} \rangle, \langle \{-1,0\}, \{-2,0\} \rangle, \ldots, \langle \{-m+1,0\}, \{-m,0\} \rangle \qquad (9)$$

are open. Let S' be the path consisting of precisely these edges (i.e., going straight from $\mathbf{0}$ to $\{-m, 0\}$), let S'' be some non-backtracking self-avoiding path from $\{-m, 0\}$ to $+\infty$, and let S be the concatenation of S' and S''. Again define a unit current F from $\mathbf{0}$ by pushing the whole current through the path S. We get

$$\begin{aligned} W(F) &= \sum_{e \in S'} W(F,e) + \sum_{e \in S''} W(F,e) \\ &= \sum_{e \in S'} C(e)^{-1} + \sum_{e \in S''} C(e)^{-1} \\ &\leq \sum_{k=1}^{m} \beta^{2k-1} + \sum_{k=-2m}^{\infty} \beta^{-k} < \infty \end{aligned} \qquad (10)$$

as desired. ■

It turns out to that as a preliminary step towards proving our main result – Theorem 3.2 – on the asymptotic speed of the random walk, it is useful to have a further refinement of the classification of good and bad vertices. To this end, define, for $m \geq 0$, a vertex v of the infinite cluster in X to be m-**bad** if the smallest number of steps that a path from v to $+\infty$ along open edges needs to backtrack is precisely m. Note that 0-bad vertices are in fact good, while m-bad vertices for $m \geq 1$ are bad.

Lemma 4.2 *Assume for some $m \geq 0$ that $\mathbf{0}$ is m-bad for the percolation configuration X. Then the effective conductance $C_{eff}(\mathbf{0}, \infty)$ in X satisfies*

$$\left(\sum_{k=1}^{m} \beta^{2k-1} + \sum_{k=-2m}^{\infty} \beta^{-k} \right)^{-1} \leq C_{eff}(\mathbf{0}, \infty) \leq \left(\sum_{k=1}^{m} \beta^{2k-1} \right)^{-1} . \qquad (11)$$

Proof of Lemma 4.2. To obtain the lower bound in (11), it suffices to construct a unit current F from $\mathbf{0}$ to ∞ in X whose energy $W(F)$ does not exceed $\sum_{k=1}^{m} \beta^{2k-1} + \sum_{k=-2m}^{\infty} \beta^{-k}$. But this is exactly what the quantitative estimate in (10) gave us.

We turn now to the upper bound. For this, we need to show that for *any* unit current F from $\mathbf{0}$ to ∞ in X, the energy $W(F)$ must be at least $\sum_{k=1}^{m} \beta^{2k-1}$. To see this, note that any path from $\mathbf{0}$ to ∞ must go via every one of the edges in (9). Hence the current through each of these edges must be exactly 1, and the energy at the edge $e = \langle \{k-1, 0\}, \{k, 0\} \rangle$ becomes $C(e) = \beta^{2k-1}$. Summing over the edges in (9) yields $W(F) \geq \sum_{k=1}^{m} \beta^{2k-1}$, as desired. ∎

Our next step is to consider the number $Z_{\mathbf{0}}(\mathbf{0}) = \sum_{i=0}^{\infty} I_{\{Y_i = \mathbf{0}\}}$ of visits to $\mathbf{0}$ that the random walk makes. (We will later, in Section 5, allow random walks starting from vertices other than $\mathbf{0}$, and let $Z_v(w)$ denote the number of visits the random walk starting from v makes to w.) We first consider the expected value of $Z_{\mathbf{0}}(\mathbf{0})$ conditional on X, assuming as usual that $\mathbf{0}$ is in the infinite cluster of X. By the Markov property of the random walk (conditioned on X), the number of such visits is geometrically distributed with mean

$$1/\mathbf{P}(Y_i \neq \mathbf{0} \text{ for all } i \geq 1 \,|\, X) . \qquad (12)$$

We now claim that Lemma 4.1 implies

$$\mathbf{P}(Y_i \neq \mathbf{0} \text{ for all } i \geq 1 \,|\, X) = \frac{C_{eff}(\mathbf{0}, \infty)}{\beta^{-1} + 1 + \beta} . \qquad (13)$$

In the case where all three edges incident to $\mathbf{0}$ are open, this is just the $\frac{C_{eff}(v,\infty)}{C(v)}$ formula of the lemma. Closing one or more of the edges reduces $C(\mathbf{0})$, but this is exactly compensated for by the probability that the walk stays put at time 1; the best way to see this is that closing an edge makes it act as a self-loop for the random walk, thus leaving $C(\mathbf{0})$ unchanged. Hence (13), which combined with (12) gives

$$\mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) \mid \mathbf{0} \text{ is } m\text{-bad}] \;=\; \frac{\beta^{-1}+1+\beta}{C_{eff}(\mathbf{0},\infty)}\,.$$

Plugging in the conclusions of Lemma 4.2 yields upper and lower bounds for $\mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) \mid \mathbf{0} \text{ is } m\text{-bad}]$ given by

$$\begin{aligned}\mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) \mid \mathbf{0} \text{ is } m\text{-bad}] &\geq (\beta^{-1}+1+\beta)\left(\sum_{k=1}^{m}\beta^{2k-1}\right) \\ &= (\beta^{-1}+1+\beta)\frac{\beta(\beta^{2m}-1)}{\beta^2-1} \end{aligned} \qquad (14)$$

and

$$\begin{aligned}\mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) \mid \mathbf{0} \text{ is } m\text{-bad}] &\leq (\beta^{-1}+1+\beta)\left(\sum_{k=1}^{m}\beta^{2k-1}+\sum_{k=-2m}^{\infty}\beta^{-k}\right) \\ &= (\beta^{-1}+1+\beta)\left(\frac{\beta(\beta^{2m}-1)}{\beta^2-1}+\frac{\beta^{2m}}{\beta-1}\right). \end{aligned} \qquad (15)$$

We next turn to estimating $\mathbf{E}[Z_{\mathbf{0}}(\mathbf{0})]$ without conditioning, i.e., averaged over all possible X. This allows for the possibility that $\mathbf{0}$ is in a finite connected component of X, in which case $Z_{\mathbf{0}}(\mathbf{0}) = \infty$; since this has positive probability, we get $\mathbf{E}[Z_{\mathbf{0}}(\mathbf{0})] = \infty$ for a trivial and irrelevant reason (irrelevant since the result we are trying to prove – Theorem 3.2 – deals only with the case where $\mathbf{0}$ is in the infinite connected component). We therefore switch to studying a truncated variant $Z_{\mathbf{0}}^{trunc}(\mathbf{0})$ of $Z_{\mathbf{0}}(\mathbf{0})$, defined by

$$Z_{\mathbf{0}}^{trunc}(\mathbf{0}) = \begin{cases} Z_{\mathbf{0}}(\mathbf{0}) & \text{if } \mathbf{0} \text{ is in the infinite connected component of } X \\ 0 & \text{otherwise.} \end{cases}$$

Note that by Proposition 3.1, $Z_{\mathbf{0}}^{trunc}(\mathbf{0})$ is a.s. finite. Its expected value $\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})]$ might of course nevertheless be infinite, and determining when this happens will turn out to be the key to deciding when the asymptotic speed is zero, i.e., to proving Theorem 3.2.

We may decompose $\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})]$ according to the value of m for which $\mathbf{0}$ is m-bad, getting

$$\begin{aligned}\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})] &= \sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad})\, \mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad}] \\ &= \sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad})\, \mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad}]\,. \qquad (16)\end{aligned}$$

Inequalities (14) and (15) provide us with good enough estimates for the latter factor in the summands of (16), and it remains to deal with the former factor $\mathbf{P}(\mathbf{0}$ is m-bad$)$. With some work, this can be calculated exactly using the Markov chain representation of X discussed in Section 2. To save work, we will settle for estimating $\mathbf{P}(\mathbf{0}$ is m-bad$)$ to within a constant factor independent of m; this is obviously enough for determining whether the sum in (16) is finite or infinite.

For $m \geq 1$, the event $\{\mathbf{0}$ is m-bad$\}$ is equivalent to the intersection of all of the events $A_{-m}, B_{-m+1}, B_{-m+2}, \ldots, B_{-1}, B_0, C_0$ defined by

$$A_k = \text{the vertical edge } \langle\{k,0\},\{k,1\}\rangle \text{ is open},$$

$$\begin{aligned}B_k = \{&\text{the horizontal edges } \langle\{k-1,0\},\{k,0\}\rangle \text{ and } \langle\{k-1,1\},\{k,1\}\rangle \\ &\text{are both open, while the vertical edge } \langle\{k,0\},\{k,1\}\rangle \text{ is closed}\}\,,\end{aligned}$$

and

$$\begin{aligned}C_k = \{&\text{There exists an open path from } \{k,1\} \text{ to } +\infty \text{ visiting no other} \\ &\text{vertices with } x\text{-coordinate} \leq k, \text{ but no such path from } \{k,0\}\}\,.\end{aligned}$$

We can thus decompose the probability that $\mathbf{0}$ is m-bad as

$$\begin{aligned}&\mathbf{P}(\mathbf{0} \text{ is } m\text{-bad}) \\ &= \mathbf{P}(A_{-m}, B_{-m+1}, B_{-m+2}, \ldots, B_{-1}, B_0, C_0) \\ &= \mathbf{P}(A_{-m}) \left(\prod_{k=-m+1}^{0} \mathbf{P}\left(B_k \,\middle|\, A_{-m}, \bigcap_{j=-m+1}^{k-1} B_j \right) \right) \\ &\quad \cdot \mathbf{P}(C_0 \,|\, A_{-m}, B_{-m+1}, B_{-m+2}, \ldots, B_0)\,. \qquad (17)\end{aligned}$$

By translation invariance of P_p (Theorem 2.1), $\mathbf{P}(A_{-m})$ does not depend on m; let us denote it by $\alpha(p)$.

Next, by Corollary 2.4, we have $\mathbf{P}(B_k \,|\, A_{-m}, B_{-m+1}, B_{-m+2}, \ldots, B_{k-1}) = \mathbf{P}(B_k \,|\, T_{k-1} = \mathtt{11})$. Note that if $T_{k-1} = \mathtt{11}$ and B_k happens, then $T_k = \mathtt{11}$ as well, so

$$\begin{aligned} &\mathbf{P}(B_k \,|\, A_{-m}, B_{-m+1}, B_{-m+2}, \ldots, B_{k-1}) \\ &\quad = \mathbf{P}(B_k \,|\, T_{k-1} = \mathtt{11}) \\ &\quad = \mathbf{P}(T_k = \mathtt{11} \,|\, T_{k-1} = \mathtt{11})\mathbf{P}(B_k \,|\, T_{k-1} = \mathtt{11}, T_k = \mathtt{11}) \\ &\quad = p_{\mathtt{11},\mathtt{11}}\mathbf{P}(B_k \,|\, T_{k-1} = \mathtt{11}, T_k = \mathtt{11})\,. \end{aligned} \tag{18}$$

From (5) and (6) we get

$$\begin{aligned} p_{\mathtt{11},\mathtt{11}} &= 1 - 2p_{\mathtt{11},\mathtt{01}} \\ &= \frac{(3-2p)\left(1 + 2p - 2p^2 - \sqrt{1 + 4p^2 - 8p^3 + 4p^4}\right)}{2(1-p)}\,, \end{aligned} \tag{19}$$

while a direct application of Theorem 2.3 gives

$$\begin{aligned} \mathbf{P}(B_k \,|\, T_{k-1} = \mathtt{11}, T_k = \mathtt{11}) &= \frac{p^2(1-p)}{p^3 + 3p^2(1-p)} \\ &= \frac{1-p}{3-2p}\,. \end{aligned} \tag{20}$$

Inserting (19) and (20) into (18) gives

$$\begin{aligned} &\mathbf{P}(B_k \,|\, A_{-m}, B_{-m+1}, B_{-m+2}, \ldots, B_{k-1}) \\ &\quad = \tfrac{1}{2}\left(1 + 2p - 2p^2 - \sqrt{1 + 4p^2 - 8p^3 + 4p^4}\right)\,. \end{aligned}$$

A similar application of Corollary 2.4 gives

$$\mathbf{P}(C_0 \,|\, A_{-m}, B_{-m+1}, B_{-m+2}, \ldots, B_0) \;=\; \mathbf{P}(C_0 \,|\, T_0 = \mathtt{11})$$

which we note does not depend on n; let us denote this probability by $\gamma(p)$.

Now we have expressions, or at least notation, for all of the factors in (17), and we get

$$\begin{aligned} &\mathbf{P}(\mathbf{0} \text{ is } m\text{-bad}) \\ &\quad = \alpha(p)\gamma(p)\left(\tfrac{1}{2}\left(1 + 2p - 2p^2 - \sqrt{1 + 4p^2 - 8p^3 + 4p^4}\right)\right)^m\,. \end{aligned}$$

Plugging this into (16) together with inequalities (14) and (15) gives

$$\begin{aligned}\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})] &= \sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad})\, \mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad}] \\ &\geq \sum_{m=0}^{\infty} \alpha(p)\gamma(p) \left(\tfrac{1}{2}\left(1+2p-2p^2-\sqrt{1+4p^2-8p^3+4p^4}\right)\right)^m \\ &\quad \cdot(\beta^{-1}+1+\beta)\frac{\beta(\beta^{2m}-1)}{\beta^2-1} \end{aligned} \tag{21}$$

and

$$\begin{aligned}\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})] &= \sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad})\, \mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad}] \\ &\leq \sum_{m=0}^{\infty} \alpha(p)\gamma(p) \left(\tfrac{1}{2}\left(1+2p-2p^2-\sqrt{1+4p^2-8p^3+4p^4}\right)\right)^m \\ &\quad \cdot(\beta^{-1}+1+\beta)\left(\frac{\beta(\beta^{2m}-1)}{\beta^2-1}+\frac{\beta^{2m}}{\beta-1}\right). \end{aligned} \tag{22}$$

Note now that the sum in (21) diverges if

$$\beta^2\left(\tfrac{1}{2}\left(1+2p-2p^2-\sqrt{1+4p^2-8p^3+4p^4}\right)\right) \geq 1,$$

and similarly that the sum in (22) converges if

$$\beta^2\left(\tfrac{1}{2}\left(1+2p-2p^2-\sqrt{1+4p^2-8p^3+4p^4}\right)\right) < 1.$$

We thus have the following.

Lemma 4.3 *The truncated expected number of returns* $\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})]$ *to the origin satisfies*

$$\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})] \begin{cases} < \infty & \text{for } \beta \in (1, \beta_c) \\ = \infty & \text{for } \beta \geq \beta_c, \end{cases}$$

where $\beta_c = \beta_c(p)$ *is given by*

$$\beta_c = \sqrt{2/(1+2p-2p^2-\sqrt{1+4p^2-8p^3+4p^4})}.$$

This strongly suggests Theorem 3.2. The main task in the next and final section will be to show how Lemma 4.3 implies Theorem 3.2.

5 Coupling from the past construction

We begin this section with a concrete, but perhaps somewhat lavish, construction of the random walk $(Y_0, Y_1, \ldots)$ given the percolation configuration $X \in \{0,1\}^E$. Namely, we introduce an array $\{Y_{cand}(v,i)\}_{v\in V, i\in\{1,2,\ldots\}}$ of $\{-1,0,1\}$-valued random variables, independent also of X, such that each

$$Y_{cand}(v,i) = \begin{cases} -1 & \text{with probability } \frac{\beta^{-1}}{\beta^{-1}+1+\beta} \\ 0 & \text{with probability } \frac{1}{\beta^{-1}+1+\beta} \\ 1 & \text{with probability } \frac{\beta}{\beta^{-1}+1+\beta}\,. \end{cases} \qquad (23)$$

The variable $Y_{cand}(v,i)$ instructs the random walker in which direction to attempt the next step upon the i'th visit to vertex v. More precisely, if $Y_{cand}(v,i) = -1$, then the walker attempts to take a step to the left; if $Y_{cand}(v,i) = 0$, then it attempts to take a vertical step; and if $Y_{cand}(v,i) = 1$, then the walker attempts to take a step to the right. If the chosen edge is open in X, then the step is taken, while if the edge is closed, then the walker stays at v for one more time unit. It is obvious that this gives $(Y_0, Y_1, \ldots)$ the desired distribution as defined in Section 3.

The point of this construction is that it provides a useful way to construct random walks from different starting points simultaneously. Namely, given the percolation $X \in \{0,1\}^E$ and the array $\{Y_{cand}(v,i)\}_{v\in V, i\in\{1,2,\ldots\}}$, we can define for any $u \in V$ a random walk $(Y_0^u, Y_1^u, \ldots)$ starting from $Y_0^u = u$ and governed by the candidate jump array $\{Y_{cand}(v,i)\}_{v\in V, i\in\{1,2,\ldots\}}$.

In order to be able to exploit this construction, we begin with a couple of lemmas. Write $\mathcal{C}(X)$ for for the infinite cluster of X, and define **regeneration point** $v = \{x, 0\}$ to be a vertex such that

(a) vertex $\{x, 1\}$ is isolated in X (so that in particular $v \in \mathcal{C}(X)$), and

(b) the random walk $(Y_0^v, Y_1^v, \ldots)$ never returns to v.

Lemma 5.1 *With probability* 1, X *and* $\{Y_{cand}(v,i)\}_{v\in V, i\in\{1,2,\ldots\}}$ *give rise to infinitely many regeneration points.*

Proof of Lemma 5.1. We first claim that it suffices to show that there a.s. exists at least one regeneration point. To see this, note that, by translation invariance of X and $\{Y_{cand}(v,i)\}_{v\in V, i\in\{1,2,\dots\}}$, the random set of regeneration points is also translation invariant. This means that there cannot exist a leftmost regeneration point, because conditional the existence of such a point its x-coordinate must be uniformly distributed on $\mathbb{Z}$, which is a contradiction.

Call v a **pre-regeneration point** if v satisfies condition (a) above. It follows easily from Theorems 2.2 and 2.3, plus the irreducibility of $\{T_i\}_{i\in\mathbb{Z}}$, that there exist a.s. infinitely many pre-regeneration points $V_1, V_2, \dots$ to the right of $\mathbf{0}$. Consider the first pre-regeneration point V_1 to the right of $\mathbf{0}$. By the definition of pre-regeneration points, V_1 is a 0-bad vertex, so Lemma 4.2 in combination with (13) implies that

$$\mathbf{P}(Y_0^{V_1}, Y_1^{V_1}, \dots \text{ never returns to } V_1) \geq \left((\beta^{-1}+1+\beta) \sum_{k=0}^{\infty} \beta^{-k} \right)^{-1} . \quad (24)$$

Now we would like to apply this not just to V_1 but to $V_2, V_3, \dots$ as well, and apply Borel–Cantelli to deduce that with probability 1, at least one of them will be a regeneration point. A problem with this approach, however, is that knowledge of V_1 not being a regeneration point may affect the conditional probability that V_2 is a regeneration point downwards. To handle this problem, consider the following scheme.

1. Set $i = 1$.
2. Is V_i a regeneration point? If YES, then we are done. If NO, then inspect the trajectory of the random walk $(Y_0^{V_i}, Y_1^{V_i}, \dots, Y_R^{V_i})$ where R is the time of its first return to V_i.
3. Set j to be the index of the first (i.e. leftmost) pre-regeneration point to the right of V_i that is not visited by $(Y_0^{V_i}, Y_1^{V_i}, \dots, Y_R^{V_i})$. Then set $i = j$ and go back to Step 2.

This scheme has the property that each time we ask whether a pre-regeneration point V_i is a regeneration point, the probability of it being so (an event that depends on X and $\{Y_{cand}(v,i)\}_{v\in V, i\in\{1,2,\dots\}}$ only via their values at x-coordinates at or to the right of V_i) is unaffected by any information gathered about previously inspected pre-regeneration points and the corresponding first-excursion trajectories. Hence, the right hand side of (24) gives a lower bound for the conditional probability, each time we reach Step 2 of the above scheme, of getting a YES answer. Hence, by conditional Borel–Cantelli, we get with probability one a YES answer eventually, as desired. ■

For later convenience, we record separately the following result, which is immediate from Lemma 5.1 in conjunction with the translation invariance argument at the beginning of the proof of Lemma 5.1.

Corollary 5.2 *With probability* 1, X *and* $\{Y_{cand}(v,i)\}_{v\in V, i\in\{1,2,\ldots\}}$ *give rise to infinitely many regeneration points to the left of* $\mathbf{0}$.

Next, we introduce **hitting event** $H(u,v)$ from one vertex to another, defined as the event that the random walk $(Y_0^u, Y_1^u, \ldots)$ ever hits v. In other words, the indicator $I_{\{H(u,v)\}}$ of the hitting event $H(u,v)$ is given by

$$I_{\{H(u,v)\}} = \begin{cases} 1 & \text{if } \exists i \geq 0 \text{ such that } Y_i^u = v \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that the notion of **hitting event from minus infinity** of a vertex $v \in V$, makes sense. This event is denoted $H(-\infty, v)$ and is defined through its indicator function $I_{\{H(-\infty,v)\}}$ by setting

$$I_{\{H(-\infty,v)\}} = \lim_{\substack{\{x,y\}\in\mathcal{C}(X) \\ x\to-\infty}} I_{\{H(\{x,y\},v)\}} \tag{25}$$

where, as it turns out, the limit does exist:

Lemma 5.3 *For any* $v \in V$ *and* P_p*-a.e.* $X \in \{0,1\}^E$, *the random variable* $I_{\{H(-\infty,v)\}}$ *in (25) is well-defined.*

Proof of Lemma 5.3. Fix v, and define w as the first regeneration point to the left of v; the existence of such a w is guaranteed by Corollary 5.2. Now consider random walk from some vertex $u \in \mathcal{C}$ to the left of w. This walk will a.s. reach w eventually, and any visit to v will have to happen after this visit to w. But the trajectory of the random walk from the time of the first visit to w and onwards will be just a time-delay of the walk $(Y_0^w, Y_1^w, \ldots)$; this is because, due to the definition of regeneration point, the entire trajectory after the first visit to w will take place to the right of w, which at the time of that first visit to w is still virgin territory, so that the jump candidate variables

$\{Y_{cand}(v,i)\}_{v\in V, i\in\{1,2,\ldots\}}$ will give all the same instructions as for $(Y_0^w, Y_1^w, \ldots)$. Hence, the random walk $(Y_0^u, Y_1^u, \ldots)$ hits v if and only if $(Y_0^w, Y_1^w, \ldots)$ does. Since this holds for all $u \in \mathcal{C}$ to the left of w, we get that the limit in (25) exists (and equals $I_{\{H(w,v)\}}$). ■

The key thing that happens in the above proof is that the random walks starting from arbitrary vertices in $\mathcal{C}$ to the left of the regeneration point w all coalesce at w. This is reminiscent of the Propp–Wilson coupling-from-the-past algorithm [13], and also of the way that particles wander in from infinity in models of diffusion limited aggregation (see, e.g., [4]).

In fact, not only does the hitting or not of a given vertex v from $-\infty$ make sense; we can even define an entire random walk trajectory from $-\infty$. We call this the **two-sided random walk**, and denote it by

$$(\ldots, Y_{-2}^{-\infty}, Y_{-1}^{-\infty}, Y_0^{-\infty}, Y_1^{-\infty}, \ldots)\,. \tag{26}$$

To achieve this, let $W_1, W_2, \ldots$ be the regeneration points to the left of $\mathbf{0}$, enumerated from right to left. There is a certain arbitrariness in the time indexing of the two-sided walk, but for definiteness we choose to define time 0 as the first time that the walk visits either of the two vertices $\mathbf{0} = \{0,0\}$ and $\{0,1\}$ that have x-coordinate 0. For each regeneration point W_i to the left of $\mathbf{0}$, we define $T_0^{W_i}$ as the time taken for the random walk $(Y_0^{W_i}, Y_1^{W_i}, \ldots)$ to hit x-coordinate 0, i.e.,

$$T_0^{W_i} \;=\; \min\{j : Y_j^{W_i} \in \{\{0,0\},\{0,1\}\}\}\,.$$

Note that

$$T_0^{W_1} < T_0^{W_2} < \cdots\,,$$

and that for any $i \geq 1$ we have

$$(Y^{W_{i+1}}_{T_0^{W^{i+1}} - T_0^{W^i}}, Y^{W_{i+1}}_{T_0^{W^{i+1}} - T_0^{W^i}+1}, \ldots) \;=\; (Y_0^{W_i}, Y_1^{W_i}, \ldots)\,, \tag{27}$$

again by the definition of regeneration points. The two-sided random walk in (26) is defined by setting, for $i = 1, 2, \ldots$,

$$(Y^{-\infty}_{-T_0^{W_i}}, Y^{-\infty}_{-T_0^{W_i}+1}, \ldots) \;=\; (Y_0^{W_i}, Y_1^{W_i}, \ldots)\,.$$

Note first that, for any k, $Y_k^{-\infty}$ is defined eventually, i.e. for large enough i; this is because $\lim_{i\to\infty} T_0^{W_i} = \infty$. Note secondly that each $Y_k^{-\infty}$ gets multiply defined, but that (27) guarantees that this causes no conflict.

Another crucial observation is the following. The origin $\mathbf{0}$ appears explicitly – as a kind of "anchor" – in the definition of the two-sided random walk. Nevertheless, choosing a different vertex $v \in V$ as the anchor would have given the very same two-sided random walk trajectory, modulo only a possible time delay. This (together with translation invariance of X and $\{Y_{cand}(v,i)\}_{v\in V, i\in\{1,2,\ldots\}}$) gives the walk a certain kind of translation invariance, which can intuitively be described as follows. Suppose that, like spectators of the *Tour de France*, we position ourselves next to $\mathcal{L}$, at some x-coordinate x, in order to watch the two-sided walk pass by. Then the time at which the walk passes by depends on our choice of x, but apart from this time lag the behavior of the walk we expect to see in the window we observe ($\mathcal{L}$ restricted to x-coordinates in $[x-10, x+10]$, say) does not.

It follows that the random process $Z = \{Z(v)\}_{v\in V}$, defined by

$$Z(v) \;=\; \#\{i : Y_i^{-\infty} = v\}\,,$$

is translation invariant. This process will be the key to completing the proof of Theorem 3.2. The following lemma is reminiscent of Lemma 4.3, but takes us one step closer to Theorem 3.2.

Lemma 5.4 *The number of visits $Z(\mathbf{0})$ by the two-sided random walk to the origin satisfies*

$$\mathbf{E}[Z(\mathbf{0})] \begin{cases} < \infty & \text{for } \beta \in (1, \beta_c) \\ = \infty & \text{for } \beta \geq \beta_c\,, \end{cases}$$

where β_c is given by (8).

Proof of Lemma 5.4. Since $Z(\mathbf{0})$ is non-negative integer-valued, we have, provided $\mathbf{P}(Z(v) \geq 1) > 0$, that

$$\begin{aligned} \mathbf{E}[Z(\mathbf{0})|X] &= \mathbf{P}(Z(\mathbf{0}) \geq 1 \,|\, X)\mathbf{E}[Z(\mathbf{0}) \,|\, Z(\mathbf{0}) \geq 1, X] \\ &= \mathbf{P}(H(-\infty, \mathbf{0}) \,|\, X)\mathbf{E}[Z(\mathbf{0}) \,|\, H(-\infty, \mathbf{0}), X]\,. \end{aligned}$$

We calculate

$$
\begin{aligned}
&\mathbf{E}[Z(\mathbf{0})] \\
&= \mathbf{P}(\mathbf{0} \notin \mathcal{C})\mathbf{E}[Z(\mathbf{0}) \,|\, \mathbf{0} \notin \mathcal{C}] + \sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad})\mathbf{E}[Z(\mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad}] \\
&= \sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad})\mathbf{E}[Z(\mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad}] \\
&= \sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad})\mathbf{P}(H(-\infty, \mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad})\mathbf{E}[Z(\mathbf{0}) \,|\, H(-\infty, \mathbf{0}), \mathbf{0} \text{ is } m\text{-bad}] \\
&= \sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad})\mathbf{P}(H(-\infty, \mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad})\mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) | \, \mathbf{0} \text{ is } m\text{-bad}] \,, \qquad (28)
\end{aligned}
$$

where, as in Section 4, $Z_{\mathbf{0}}(\mathbf{0})$ is the number of visits to $\mathbf{0}$ by the one-sided random walk $(Y_0, Y_1, \ldots)$ starting from $Y_0 = \mathbf{0}$. The sum in (28) is obviously bounded by

$$
\sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad}) \, \mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad}] \,,
$$

which is precisely the decomposition of $\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})]$ obtained in (16). So if $\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})] < \infty$, then $\mathbf{E}[Z(\mathbf{0})] < \infty$ as well. For $\beta \in (1, \beta_c)$, Lemma 4.3 tells us precisely that $\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})] < \infty$, so we conclude that $\mathbf{E}[Z(\mathbf{0})] < \infty$, and the case $\beta \in (1, \beta_c)$ is settled.

It remains to consider the case $\beta \geq \beta_c$. Again (28) will help, and if we can find a lower bound $\delta > 0$ for $\mathbf{P}(H(-\infty, \mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad})$ (uniformly in m), then we know, due to (16), that $\mathbf{E}[Z(\mathbf{0})] \geq \delta \mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})]$, and since Lemma 4.3 tells us that $\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})] = \infty$ when $\beta > \beta_c$, we will be done.

To execute this program, consider first the case $m = 0$, i.e. where $\mathbf{0}$ is good. Given X, there will a.s. be vertices with arbitrarily small hitting probabilities from $-\infty$ (namely those sitting far out on dead ends extending to the left), but every such vertex nevertheless has positive probability of being hit from $-\infty$, so

$$
\mathbf{P}(H(-\infty, \mathbf{0}) \,|\, \mathbf{0} \text{ is } 0\text{-bad}) \, > \, 0 \,, \qquad (29)
$$

which is all we need to know about the case $m = 0$.

Next, consider the case where $\mathbf{0}$ is m-bad for some $m \geq 1$. Then, by the detailed description of the event $\{\mathbf{0} \text{ is } m\text{-bad}\}$ following (16), we have that the vertical edge $\langle\{-m,0\},\{-m,1\}\rangle$ is open, that the horizontal edges $\langle\{i-1,0\},\{i,0\}\rangle$ are open for $i=-m+1,-m+2,\ldots,0$, and that the vertical edges $\langle\{i,0\},\{i,1\}\rangle$ are closed for $i=-m+1,-m+2,\ldots,0$. We first estimate $\mathbf{P}(H(-\infty,\{-m,0\})\,|\,\mathbf{0}\text{ is }m\text{-bad})$. The walk from $-\infty$ must eventually hit x-coordinate $-m$. If this happens at $\{-m,0\}$, then we are happy, while if it happens at $\{-m,1\}$ then the probability that the walk immediately takes a step from there to $\{-m,0\}$ is $(\beta^{-1}+1+\beta)^{-1}$. Hence,

$$\mathbf{P}(H(-\infty,\{-m,0\})\,|\,\mathbf{0}\text{ is }m\text{-bad}) \;\geq\; (\beta^{-1}+1+\beta)^{-1}\,. \tag{30}$$

Next, once $\{-m,0\}$ is hit, the walker has probability $\beta(\beta^{-1}+1+\beta)^{-1}$ of immediately taking a step to $\{-m+1,0\}$, so

$$\begin{aligned}&\mathbf{P}(H(-\infty,\{-m+1,0\})\,|\,\mathbf{0}\text{ is }m\text{-bad})\\ &\quad\geq\; \beta(\beta^{-1}+1+\beta)^{-1}\mathbf{P}(H(-\infty,\{-m,0\})\,|\,\mathbf{0}\text{ is }m\text{-bad})\,.\end{aligned} \tag{31}$$

And once it has reached $\{-m+1,0\}$, the probability that it reaches $\mathbf{0}$ before going back to $\{-m,0\}$ is easily seen (for instance by electrical analysis) to equal $\frac{1-\beta^{-1}}{1-\beta^{-m}}$, so

$$\mathbf{P}(H(-\infty,\mathbf{0})\,|\,\mathbf{0}\text{ is }m\text{-bad}) \;\geq\; \frac{1-\beta^{-1}}{1-\beta^{-m}}\mathbf{P}(H(-\infty,\{-m+1,0\})\,|\,\mathbf{0}\text{ is }m\text{-bad})\,. \tag{32}$$

Multiplying (30), (31) and (32) gives

$$\mathbf{P}(H(-\infty,\mathbf{0})\,|\,\mathbf{0}\text{ is }m\text{-bad}) \;\geq\; (\beta^{-1}+1+\beta)^{-2}\beta(1-\beta^{-1})(1-\beta^{-m})^{-1}$$

which in turn exceeds

$$(\beta^{-1}+1+\beta)^{-2}\beta(1-\beta^{-1})\,.$$

The point is that this bound is strictly positive and independent of m. Combining with (29) gives

$$\min\{(\beta^{-1}+1+\beta)^{-2}\beta(1-\beta^{-1}),\mathbf{P}(H(-\infty,\mathbf{0})\,|\,\mathbf{0}\text{ is 0-bad})\} \;>\; 0\,. \tag{33}$$

We now take δ to be the left hand side of (33); for this $\delta>0$ we know that

$$\mathbf{P}(H(-\infty,\mathbf{0})\,|\,\mathbf{0}\text{ is }m\text{-bad}) \;\geq\; \delta$$

for all m. This allows us to continue the calculation (28) with the estimate

$$
\begin{aligned}
&\mathbf{E}[Z(\mathbf{0})] \\
&= \sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad}) \mathbf{P}(H(-\infty, \mathbf{0}) \,|\, \mathbf{0} \text{ is } m\text{-bad}) \mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) | \, \mathbf{0} \text{ is } m\text{-bad}] \\
&\geq \delta \sum_{m=0}^{\infty} \mathbf{P}(\mathbf{0} \text{ is } m\text{-bad}) \mathbf{E}[Z_{\mathbf{0}}(\mathbf{0}) | \, \mathbf{0} \text{ is } m\text{-bad}] \\
&= \delta \mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})]
\end{aligned}
$$

where the last equality comes from (16). Since $\mathbf{E}[Z_{\mathbf{0}}^{trunc}(\mathbf{0})] = \infty$ when $\beta \geq \beta_c$ (Lemma 4.3), we get $\mathbf{E}[Z(\mathbf{0})] = \infty$ for the same range of β, and the proof is complete. ■

Moving on, we now define the process $\{Z^*(i)\}_{i \in \mathbb{Z}}$ by setting

$$
Z^*(i) \;=\; Z(\{i,0\}) + Z(\{i,1\})
$$

for each i, so that in other words $Z^*(i)$ is the total time spent by the two-sided walk at x-coordinate i. The translation invariance of $\{Z(v)\}_{v \in V}$ is of course inherited by $\{Z^*(i)\}_{i \in \mathbb{Z}}$. Furthermore

$$
\begin{aligned}
\mathbf{E}[Z^*(i)] &= \mathbf{E}[Z(\{i,0\})] + \mathbf{E}[Z(\{i,1\})] \\
&= 2\mathbf{E}[Z(\mathbf{0})] \begin{cases} < \infty & \text{for } \beta \in (1, \beta_c) \\ = \infty & \text{for } \beta \geq \beta_c\,, \end{cases}
\end{aligned}
$$

by Lemma 5.4. In the finite expectation case $\beta \in (1, \beta_c)$, the pointwise ergodic theorem tells us that the limit

$$
\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} Z^*(i)
$$

exists a.s. In the infinite expectation case $\beta \geq \beta_c$, we have, since the process is non-negative, that the limit still exists, but may take value $+\infty$. In fact we have the following.

Lemma 5.5 *The process $\{Z^*(i)\}_{i \in \mathbb{Z}}$ is ergodic, so*

$$
\mathbf{P}\left(\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} Z^*(i) \;=\; \mathbf{E}[Z^*(0)] \right) \;=\; 1\,.
$$

For the proof we need some basic facts from ergodic theory. These are collected in the following lemma; they can be found, e.g., in Walters [15].

Lemma 5.6 *Some ergodic-theoretic facts:*

(a) *If $\{U_i\}_{i\in\mathbb{Z}}$ is an irreducible and aperiodic stationary finite-state Markov chain, then it is ergodic.*

(b) *Suppose that $\{U_i\}_{i\in\mathbb{Z}}$ is ergodic and that $\{U_i'\}_{i\in\mathbb{Z}}$ is an i.i.d. process independent of $\{U_i\}_{i\in\mathbb{Z}}$, and define $U_i'' = (U_i, U_i')$. Then $\{U_i''\}_{i\in\mathbb{Z}}$ is ergodic.*

(c) *Suppose that $\{U_i\}_{i\in\mathbb{Z}}$ is ergodic and that there is a function f such that for each i, $U_i' = f(U_i, U_{i+1}, U_{i+2}, \ldots; U_{i-1}, U_{i-2}, U_{i-3}, \ldots)$. Then $\{U_i'\}_{i\in\mathbb{Z}}$ is ergodic.*

Proof of Lemma 5.5. Note first that, by Lemma 5.6 (a), $\{T_i\}_{i\in\mathbb{Z}}$ is ergodic.

To show that also the percolation process $X \in \{0,1\}^E$ is ergodic, we make the way in which X is obtained from $\{T_i\}_{i\in\mathbb{Z}}$ slightly more concrete. (To place X in the context of $\mathbb{Z}$-indexed processes, we should think of X as $X = \{X(E_i)\}_{i\in\mathbb{Z}}$, where E_i is the triplet of edges defined in (2).) Independently for each i and each pair $(\mathtt{ab}, \mathtt{cd})$ such that the transition probability $p_{\mathtt{ab},\mathtt{cd}}$ defined in (5) is nonzero, define a $\{0,1\}^3$-valued random variable $X^{cand}_{i,\mathtt{ab},\mathtt{cd}}$ with distribution $P_{p,i,\mathtt{ab},\mathtt{cd}}$ given by (3). Then construct X by for each i setting $X(E_i) = X^{cand}_{i,T_{i-1},T_i}$. By Lemma 5.6 (b) the process $\{(T_i, X^{cand}_{i,\mathtt{01},\mathtt{01}}, \ldots, X^{cand}_{i,\mathtt{11},\mathtt{11}})\}_{i\in\mathbb{Z}}$ is ergodic, whence by Lemma 5.6 (c) X is ergodic.

Similarly, $\{(\{X(E_i), \{Y_{cand}(\{i,j\},k)\}_{j\in\{1,2\},k\in\{1,2,\ldots\}}\})\}_{i\in\mathbb{Z}}$ is ergodic by another application of Lemma 5.6 (b). This in combination with Lemma 5.6 (c) implies ergodicity of $\{Z^*(i)\}_{i\in\mathbb{Z}}$. ∎

Next, define the process $\{R_i\}_{i\in\mathbb{Z}}$ by setting

$$R_i = \begin{cases} 1 & \text{if } \{i,0\} \text{ is a regeneration point} \\ 0 & \text{otherwise.} \end{cases}$$

for each i. Also define $\pi_p = \mathbf{P}(\mathbf{0} \text{ is a regeneration point})$, and note that $\mathbf{E}[R_i] = \pi_p$ for each i.

Lemma 5.7 *The process $\{R_i\}_{i\in\mathbb{Z}}$ is ergodic, so*

$$\mathbf{P}\left(\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}R_i = \pi_p\right) = 1\,.$$

Proof of Lemma 5.7. This follows from ergodicity of

$$\{(\{X(E_i), \{Y_{cand}(\{i,j\},k)\}_{j\in\{1,2\},k\in\{1,2,\dots\}}\})\}_{i\in\mathbb{Z}}$$

by the same argument as in the punchline of the proof of Lemma 5.5. ■

Equipped with Lemmas 5.5 and 5.7, we are finally prepared to wrap up the proof of Theorem 3.2.

Proof of Theorem 3.2. We will prove the statement of the theorem with

$$\theta(p,\beta) = \frac{1}{\mathbf{E}[Z^*(0)]}$$

which in the $\mathbf{E}[Z^*(0)] = \infty$ case $\beta \geq \beta_c$ means of course that $\theta(p,\beta) = 0$.

Write $\rho_1, \rho_2, \dots$ for the x-coordinates of the regeneration points at or to the right of $\mathbf{0}$, enumerated from left to right. Due to Lemma 5.7, we have a.s. that

$$\lim_{j\to\infty}\frac{\rho_j}{j} = \frac{1}{\pi_p}\,. \tag{34}$$

For each i, define τ_j as the first (and only) time that $(Y_0, Y_1, \dots)$ hits the regeneration point $\{\rho_j, 0\}$, and define $\tau_j^{-\infty}$ as the first (and only) time that $(Y_0^{-\infty}, Y_1^{-\infty}, \dots)$ hits the regeneration point $\{\rho_j, 0\}$. Define the random variable $\Delta = \tau_1 - \tau_1^{-\infty}$. Note that after hitting the first regeneration point $\{\rho_1, 0\}$ the two random walk trajectories are identical, i.e.,

$$(Y_{\tau_1}, Y_{\tau_1+1}, \dots) = \left(Y^{-\infty}_{\tau_1^{-\infty}}, Y^{-\infty}_{\tau_1^{-\infty}+1}, \dots\right)\,.$$

This implies in particular that

$$\tau_j - \tau_j^{-\infty} = \Delta \tag{35}$$

for all j. Next define Δ^* as the total time spent by $(Y_0^{-\infty}, Y_1^{-\infty}, \ldots)$ at strictly negative x-coordinates; since the random walk is transient to the right, we have $\Delta^* > \infty$ a.s. Note that for any j we have

$$\tau_j^{-\infty} = \Delta^* + \sum_{n=0}^{\rho_j - 1} Z^*(n)$$

whence by (35) we get

$$\tau_j = \Delta + \Delta^* + \sum_{n=0}^{\rho_j - 1} Z^*(n) . \tag{36}$$

Lemma 5.5 (ergodicity of $\{Z^*(n)\}_{n\in\mathbb{Z}}$) gives

$$\lim_{j\to\infty} \frac{1}{\rho_j} \sum_{n=0}^{\rho_j - 1} Z^*(n) = \mathbf{E}[Z^*(0)] \text{ a.s.,}$$

which in combination with (36) yields

$$\lim_{j\to\infty} \frac{\tau_j}{\rho_j} = \mathbf{E}[Z^*(0)] \text{ a.s.,} \tag{37}$$

or in other words that

$$\lim_{j\to\infty} \frac{\rho_j}{\tau_j} = \theta(p, \beta) \text{ a.s.}$$

This goes a long way towards proving Theorem 3.2, because it means that the desired limiting behavior holds along a subsequence: in the language of the theorem, where $x(Y_i)$ denotes the x-coordinate of Y_i,

$$\lim_{i\to\infty} \frac{x(Y_i)}{i} = \theta(p, \beta) \text{ a.s. along the subsequence } i = \tau_1, \tau_2, \ldots. \tag{38}$$

It remains to fill in the full sequence. Given an arbitrary time point $i \geq \tau_1$, define i^- as the last time before i at which $(Y_0, Y_1, \ldots)$ visits a regeneration point, and let i^+ be the *first* time *after* i at which $(Y_0, Y_1, \ldots)$ visits a regeneration point. Note that, since Y_{i^-} and Y_{i^+} are regeneration points, (38) tells us that

$$\lim_{i\to\infty} \frac{x(Y_{i^-})}{i^-} = \theta(p, \beta) \tag{39}$$

and

$$\lim_{i\to\infty} \frac{x(Y_{i^+})}{i^+} = \theta(p,\beta)\,.$$

Because of existence of the limit in (34) we have

$$\lim_{j\to\infty} \frac{\rho_j}{\rho_{j-1}} = 1\,. \tag{40}$$

This implies that

$$\lim_{i\to\infty} \frac{x(Y_{i^+})}{x(Y_{i^-})} = 1\,,$$

and since Y_i is sandwiched between $x(Y_{i^-})$ and $x(Y_{i^+})$, we get

$$\lim_{i\to\infty} \frac{x(Y_i)}{x(Y_{i^-})} = 1 \quad \text{and} \quad \lim_{i\to\infty} \frac{x(Y_i)}{x(Y_{i^+})} = 1\,. \tag{41}$$

Consider now the case $\beta \in (1,\beta_c)$ where $\mathbf{E}[Z^*(0)] < \infty$. From existence of the limit in (37), in combination with (40), we conclude that

$$\lim_{j\to\infty} \frac{\tau_j}{\tau_{j-1}} = 1\,. \tag{42}$$

Hence

$$\lim_{i\to\infty} \frac{i^+}{i^-} = 1$$

and since i is sandwiched between i^- and i^+ we get

$$\lim_{i\to\infty} \frac{i}{i^-} = 1 \quad \text{and} \quad \lim_{i\to\infty} \frac{i}{i^+} = 1\,. \tag{43}$$

This gives

$$\begin{aligned}
\lim_{i\to\infty} \frac{x(Y_i)}{i} &= \lim_{i\to\infty} \left(\frac{x(Y_{i^-})}{i^-} \frac{i^-}{i} \frac{x(Y_i)}{x(Y_{i^-})} \right) \\
&= \lim_{i\to\infty} \left(\frac{x(Y_{i^-})}{i^-} \right) \lim_{i\to\infty} \left(\frac{i^-}{i} \right) \lim_{i\to\infty} \left(\frac{x(Y_i)}{x(Y_{i^-})} \right) \\
&= \lim_{i\to\infty} \frac{x(Y_{i^-})}{i^-} = \theta(p,\beta)
\end{aligned}$$

using (43), (41) and (39).

Finally we consider the case $\beta \geq \beta_c$ where $\mathbf{E}[Z^*(0)] = \infty$. Here we can just combine (39) with (41) and the observation that $i^- \leq i$ to get

$$\begin{aligned}\lim_{i\to\infty} \frac{x(Y_i)}{i} &= \lim_{i\to\infty} \left(\frac{x(Y_{i^-})}{i^-} \frac{i^-}{i} \frac{x(Y_i)}{x(Y_{i^-})} \right) \\ &= \lim_{i\to\infty} \left(\frac{x(Y_{i^-})}{i^-} \right) \lim_{i\to\infty} \left(\frac{i^-}{i} \frac{x(Y_i)}{x(Y_{i^-})} \right) \\ &\leq \lim_{i\to\infty} \frac{x(Y_{i^-})}{i^-} = \theta(p,\beta) = 0\,,\end{aligned}$$

and the proof is complete. ■

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