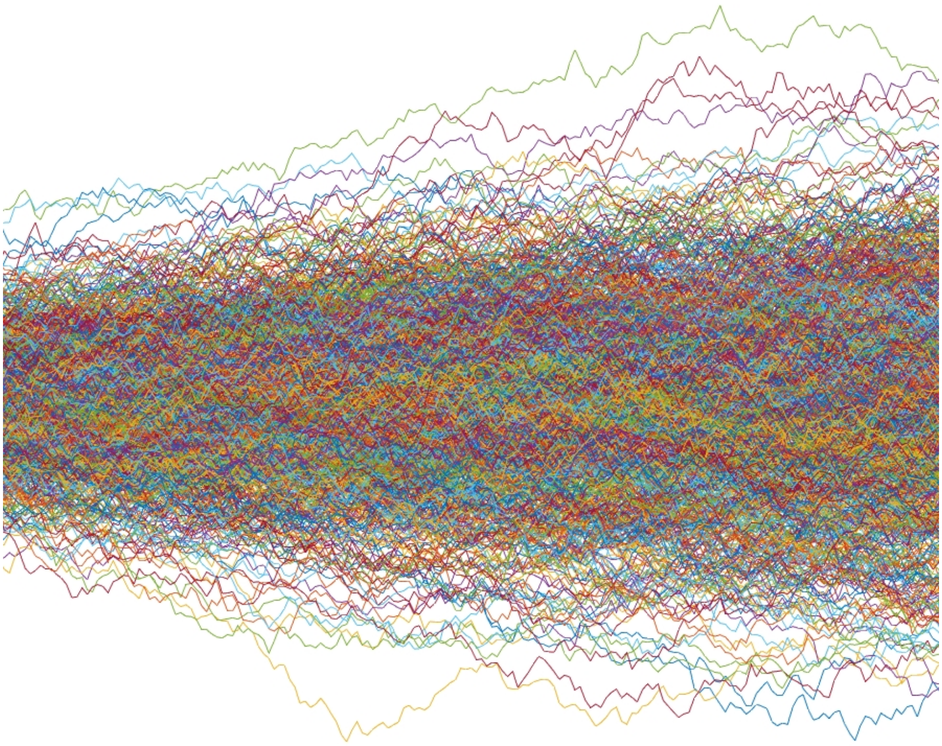

Simulation of Portfolio Strategies

using Heston Stochastic Volatility and Hull-White models

Master's thesis in Financial Mathematics

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Gothenburg, Sweden 2019

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Cover: Stock Market noise as Brownian Motion paths generated in MATLAB.

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Abstract

In this thesis, two portfolio strategies are compared. The Option Based Portfolio Insurance (OBPI) is a static strategy. The investor buys the stock and a European put option written on the stock. The rest of the investment is used to buy the risk free asset. The amount invested in the assets remains constant until maturity. The Constant Proportion Portfolio Insurance (CPPI), is a dynamic strategy, where the investor reallocates the capital invested in the risk free and the risky asset in every time step. In general, OBPI performs better in a falling market, while CPPI takes a better advantage of a sharp increase in the stock price. To generate paths for the stock and the put, the Heston stochastic volatility model is used, calibrated from real market data. To price the risk free asset, in this case a Zero Coupon Bond, the Hull-White one factor model is employed.

Keywords: CPPI, OBPI, Heston, Hull-White, stochastic volatility, Options, Zero Coupon Bond, Monte Carlo Simulation.

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Christos Saltapidas, Gotheburg, October 2019

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1
Introduction

In this thesis we will compare two simulated portfolio strategies, using two different models. Heston model will be used to simulate the stock and therefore the put option paths. The Hull-White model will be used to generate interest rates and price the Zero Coupon Bond. Chapter 1, is a quick introduction to the problem and a glance at the goal of the thesis. Next chapters include the theory used, calibration of the models and simulation (chapters 2 and 3 respectively). Last but not least, in chapter 4, the results will be presented followed by conclusions.

1.1 Background

The Black-Scholes-Merton (BSM) model was introduced in 1973 and led to a boom of the Options trading and opened many doors in Mathematical finance. It is widely used by professionals in the industry until today. However, BSM is based on some strong assumptions. One of them is constant volatility.

Heston, in 1993, proposed to model volatility as a stochastic quantity and introduced his famous stochastic volatility model. With the performance of volatility being uncertain, the evolution of financial products is now, more realistic.

The Hull-White model was first introduced by John Hull and Alan White in 1990. It is a mean reverting Ornstein–Uhlenbeck process and for this thesis, the one-factor model will be employed, meaning that only one of the parameters is time dependent. This model is also very popular in the industry and a common tool for practitioners, as it takes into account the term structure of interest rates.

In the end, two portfolio strategies are compared. The Option Based Portfolio Insurance (OBPI) is a static strategy and it was firstly introduced in 1976 by Leland and Rubinstein. The Constant Proportion Portfolio Insurance (CPPI), is a dynamic strategy, introduced by Perold (1986).

1.2 Aim

The goal of the thesis is to compare the performance of two portfolio strategies, following the steps below:

- Calibrate the models using real market data.
- Use the Heston model to simulate stocks and price a put option.

- Use the Hull-White model to simulate interest rates and price a Zero Coupon Bond.
- Compare the two portfolio strategies using the simulated stock, put and bond paths.

2

Theory

THis chapter equips the reader with the basic theory behind the calculations and results. Firstly, the Heston model will be presented along with some discussion about its parameters and the property of providing closed form solutions for European vanilla calls using characteristic functions. This property will be used later for calibration purposes. Secondly, the Hull-White model is presented followed by the the portfolio strategies and their structure.

2.1 The Heston model

As stated in the introduction, in 1993, Heston introduced the Heston stochastic volatility model [8]. The evolution of the underlying asset $S(t)$ is determined by the following dynamics (risk-neutral):

$$dS(t) = rS(t)dt + \sqrt{V(t)}S(t)dW(t)_S \tag{2.1}$$

$$dV(t) = a(\bar{V} - V(t))dt + \eta\sqrt{V(t)}dW(t)_V, \tag{2.2}$$

with

$$dW(t)_S \, dW(t)_V = \rho \, dt.$$

The parameters in the equations above are:

- $S(t)$ is the asset's price (more likely, the stock price at time t)
- r is the risk free rate of interest
- $V(t)$ is the variance at time t
- $\bar{V}$ is the long run average variance
- a is the speed at which $V(t)$ reverts to $\bar{V}$
- η is the volatility of variance
- $dW(t)_S, dW(t)_V$ are correlated Brownian motions, with correlation ρ.

The Heston model, describes volatility as a CIR process which is a mean-reverting process. This phenomenon is observed in the market, where if that was not the case, assets could experience volatility explosion or going to zero. A mean reverting process helps to avoid such cases.

To get an essence of time series with stochastic volatility, a simulated future path for the S&P 500 index is illustrated below in Figure 2.1:

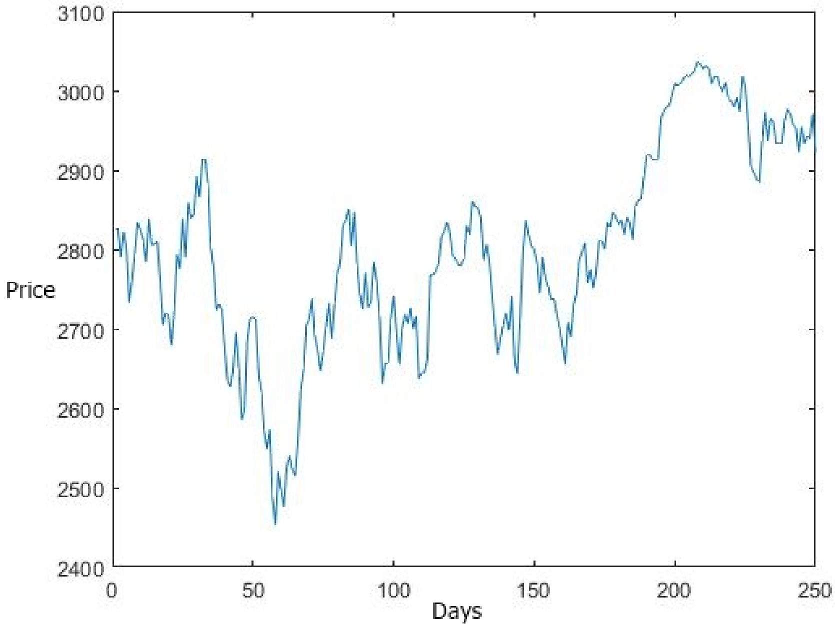

Figure 2.1: One year Simulation of the S&P 500 index using the Heston model and the 1-year LIBOR risk free rate. Simulations will be discussed later.

Another nice property of the model, bringing the analysis closer to the real world, is that the stock price and the volatility are not independent. Usually, when the market plunges, volatility tends to rise. The VIX index is a descent example. It is also called the Fear Index and it measures the volatility in the market. VIX tends to increase fast when major indexes such as S&P 500 fall. This is a reason to expect negative correlation between the stock and the variance.

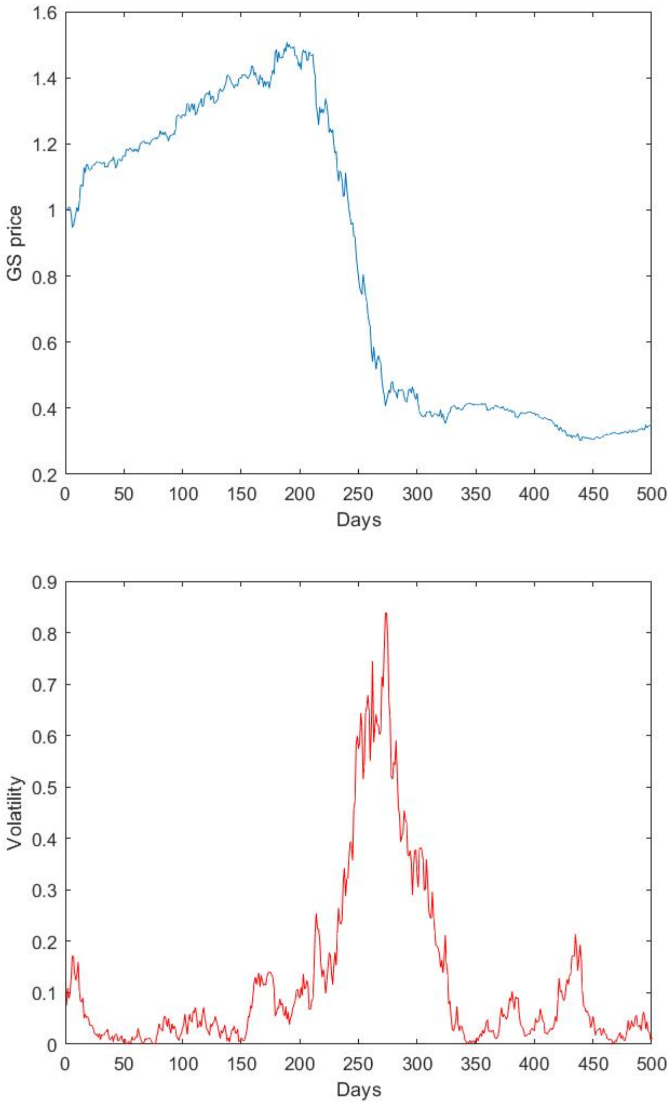

Figure 2.2: The negative correlation between the stock price and market volatility. This 2-year simulation, is based on the Goldman Sachs stock. We assume that the initial price is 1. Most of the time, for a declining market, one can observe spikes in volatility.

As stated before, equations (2.1) and (2.2) are in the risk neutral probability world and not the real probability world. This means that we are not able to infer whether the price of an asset will rise or fall, so everyone has the same expectations and values derivatives with the same price. For the interested reader, the transition of the real world to the risk-neutral world dynamics, can by found in [4].

In order to give a more intuitively meaning to some of the parameters, a quick definition of the logarithmic returns is given below:

Definition 2.1.1. *A logarithmic return $X(t)$ of a stock $S(t)$ in a discrete time interval t_i for $i = 1, ..., n$ is defined by:*

$$X(t_i) = \ln\left(\frac{S(t_i)}{S(t_{i-1})}\right),$$

where $X(t_1) = 0$.

While working with log returns, one can observe the lack of symmetry in the distribution, namely the skewness of the distribution, which is in our case represented by the correlation ρ. Another feature of logarithmic returns distribution, the kurtosis, is controlled by the parameter η from the Heston model.

2.1.1 Characteristic functions and options prices

A very convenient result in the Heston model framework is the existence of closed form solutions for call options. That means that there is a formula to compute the fair price of such an option given a specific set of parameters. In the following subsections the method to derive the formula will be described.

Definition 2.1.2. *The* ***risk-neutral*** *price (or fair price) at time $t = 0$ of the European call option with strike K, payoff $Y_T = (S_T - K)_+$ and maturity $T > 0$ is given by:*

$$C(t_0, K, T) = e^{-rT} \int_0^\infty Y_T \, p(S_T) \, dS_T,$$

where $p(S_T)$ is the risk-neutral probability density of the asset at time T.[3]

A common problem is that there exist some processes for which the density function is hard to obtain, as it may be not available in closed form. The idea here is to use the logarithm of the stock price as a process and secure a formula for the characteristic function:

Definition 2.1.3. *Let X be a random variable. The function $\phi_X : \mathbb{R} \to \mathbb{C}$ given by*

$$\phi_X(u) = \mathbb{E}[e^{iuX}],$$

is called ***the characteristic function of X*** *.*

Now, if the variable X admits the density f_X, then

$$\phi_X(u) = \int_{\mathbb{R}} e^{iuX} \, f_X(x) \, dx \tag{2.3}$$

which means that the characteristic function, is basically the inverse Fourier transform of the density [3].

The density function of X can then be recovered using its characteristic function, by applying the Inverse Fourier transform and get:

$$f_X(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-iux}\ \phi_X(u)\, du.$$

2.1.2 The closed-form solution

As seen in Heston (1993), the fair price of a European call option is given by:

$$C_0 = S_0\,\Delta_1 - e^{-rT}\,K\,\Delta_2, \tag{2.4}$$

where $\Delta_{1,2}$ are obtained with the following formulas:

$$\Delta_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\Big[\frac{e^{-iu\,\ln(K)}\ \phi_{\ln S_T}(u-i)}{i\,u\,\phi_{\ln S_T}(-i)}\Big]\,du \tag{2.5}$$

and

$$\Delta_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^{\infty} \Re\Big[\frac{e^{-iu\,\ln(K)}\ \phi_{\ln S_T}(u)}{i\,u}\Big]\,du. \tag{2.6}$$

The proof of the formulas can be found on the Appendix A of [5] .

The expression for Δ_1 can be interpreted as the Delta of the call option, while Δ_2 is seen by traders as the probability of the option expiring in the money (ITM), meaning that the stock price at maturity is greater than the strike K.
The only thing missing now, in order to have a closed form solution for the call as seen in equation (2.4), is a formula for the characteristic function.

A formula for the Heston characteristic function was introduced by Gatheral in 2006 [7] and it is the following:

$$\phi_{\ln S_t}(u) = \exp\Big(c(t,u)\,\bar{V} + D(t,u)\,V_0 + i\,u\,\ln(S_0\,e^{rt})\Big), \tag{2.7}$$

where

$$c(t,u) = a\Big[q_- \cdot t - \frac{2}{\eta^2}\,\ln\Big(\frac{1-G\,e^{-\lambda t}}{1-G}\Big)\Big],$$

$$D(t,u) = q_- \cdot \frac{1-e^{-\lambda t}}{1-G\,e^{-\lambda t}},$$

$$q_\pm = \frac{\beta \pm \lambda}{\eta^2},$$

$$\lambda = \sqrt{\beta^2 - 4\bar{\alpha}\frac{\eta^2}{2}},$$

$$G = \frac{q_-}{q_+},$$

$$\bar{\alpha} = -\frac{u(u+i)}{2},$$

$$\beta = a - \rho\eta i u.$$

Now, we can plug equation (2.7) in the equations (2.5) and (2.6) in order to calculate Δ_1 and Δ_2. Finally we use them to obtain the fair value of the European call option under the risk-neutral measure, given by (2.4).

A last observation is that C_0 is a function of all the Heston model parameters $[a, \eta, V_0, \bar{V}, \rho]$, a result that later will help us with the calibration of the model.

2.2 The Hull-White one factor model

In this thesis, the interest rates are modeled using the Hull-White model [2]. Hull and White (1990) introduced the following no-arbitrage mean reversion model for the evolution of interest rates [2]:

$$dr(t) = [\theta(t) - ar(t)]\,dt + \sigma\,dW(t) \tag{2.8}$$

$$= a\left[\frac{\theta(t)}{a} - r(t)\right]dt + \sigma\,dW(t).$$

One can say that this is similar to a Vasicek model but now, the long term mean level is time dependent. In fact, it is called a mean-reverting Ornstein–Uhlenbeck process. The other parameters are speed of reversion a and volatility σ.

Given that the parameter θ is time dependent, using Itô's Lemma and (2.8) yields the following expression:

$$r(t) = e^{-at}\,r_0 + \int_0^t e^{a\,(s-t)}\,\theta(s)\,ds + \sigma\,e^{-at}\int_0^t e^{a\,s}\,dW(s).$$

Therefore, $r(t)$ is normally distributed with mean and variance given by:

$$\mathbb{E}[r(t)] = e^{-at}\, r_0 + \int_0^t e^{a\,(s-t)}\;\theta(s)\,ds$$

and

$$Var[r(t)] = \frac{\sigma^2}{2a}\,(1 - e^{-2at}),$$

respectively. As seen in [2] and [10], $\theta(t)$ can be described by the following expression:

$$\theta(t) = F_t(0,t) + a\,F(0,t) + \frac{\sigma^2}{2\,a}(1 - e^{-2\,a\,t}). \tag{2.9}$$

The term $F(0,t)$ is called the Instantaneous forward rate or just the Forward rate. The Forward rate is calculated based on interest rates for various maturities. These rates are typically plotted on a graph forming the so called, yield curve which illustrates the yield of the bonds with different maturity dates but equal credit score.

As seen in later chapters, in this thesis, the Forward rate will be taken to be polynomial in nature [12]. This polynomial is just a function of time, so $F_t(0,t)$ in equation (2.9) is the derivative of that polynomial with respect to time t.

A common expression to derive the Forward rates $F(0,t)$ using the rates from the yield curve $R(0,t)$, is the following [9]:

$$F(0,t) = t\,\frac{\partial R(0,t)}{\partial t} + R(0,t). \tag{2.10}$$

Below, a plot of the yield curve data which were downloaded from the US Department of the Treasury web page and the derived Forward curve based on these data and equation (2.10) are illustrated:

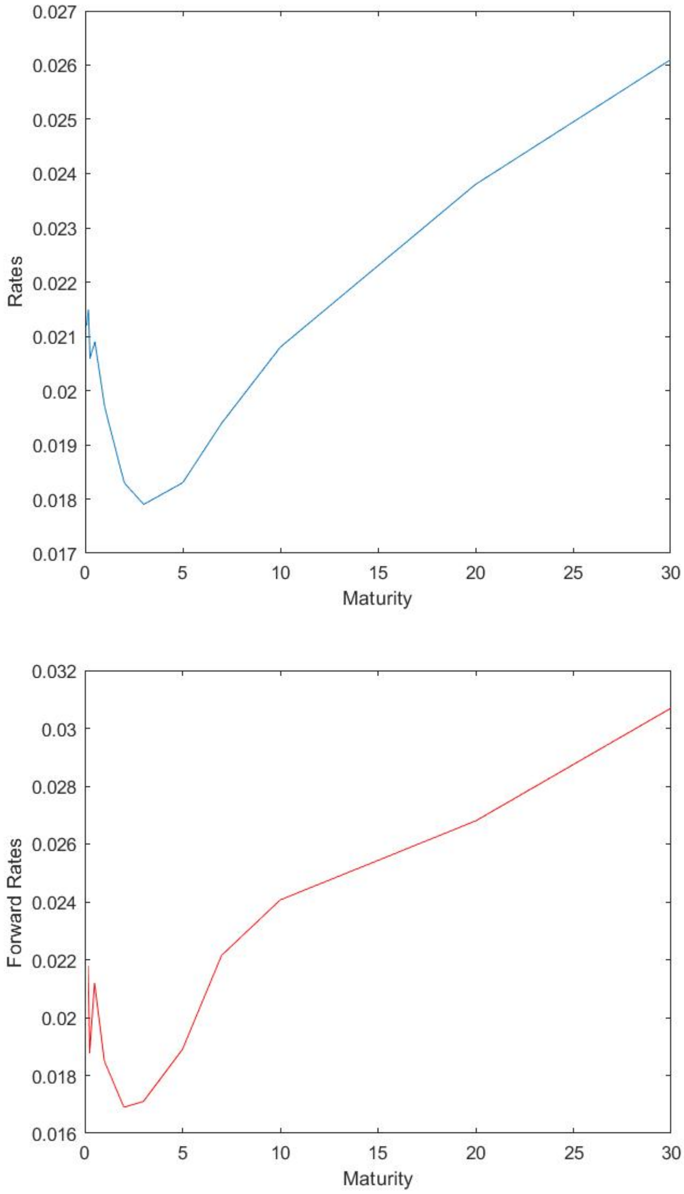

Figure 2.3: Yield curve in blue and the derived Forward curve in red, as seen on July 24th, 2019. Maturities span from 1 month to 30 years.

Later in the thesis, the bond will be used in the portfolio strategies, so we need to know its price at time t using the interest rates from the Hull-White dynamics (2.8).

In general, the price at time t of a Zero Coupon Bond, with a face value of 1 at maturity time T is given by the following expectation:

$$P(t,T) = \mathbb{E}\Big[e^{\int_t^T e^{-r(s)ds}}\Big].$$

This expectation can in fact, be computed under the dynamics (2.8) and gives:

$$P(t,T) = A(t,T)\, e^{-B(t,T)r(t)}, \tag{2.11}$$

where

$$B(t,T) = \frac{1}{a}\Big[1 - e^{-a(T-t)}\Big]$$

and

$$A(t,T) = \frac{P(0,T)}{P(0,t)} \exp\Big[- B(t,T)\frac{\partial \log(P(0,t))}{\partial t} - \frac{\sigma^2}{4a^3}\Big(e^{-aT} - e^{-at}\Big)^2 \Big(e^{2at} - 1\Big)\Big].$$

Formulas from [2]. (2.11) will be used later to price the bonds given the dynamics of the interest rates.

Below, in Figure 2.4, a simulated path of the interest rates using the Hull-White model and the corresponding bond price paying 1 at maturity T=2 years, using equation (2.11) can be seen:

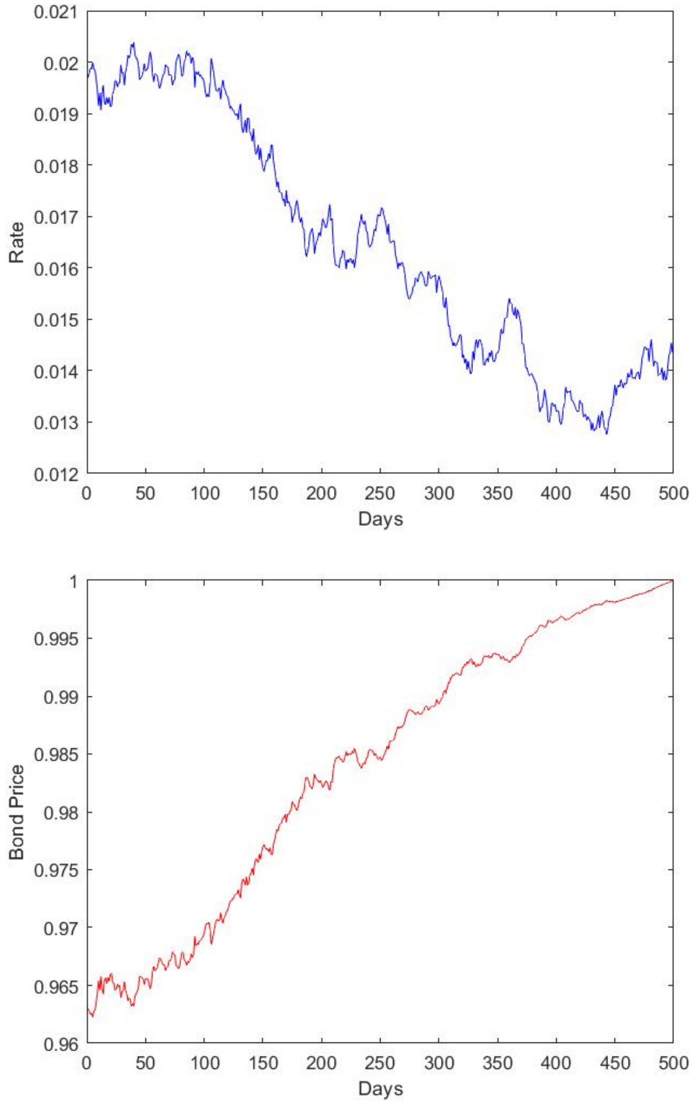

Figure 2.4: Hull-White path for the interest rate in blue and the corresponding bond price with face value of 1, maturity 2 years in red. In general bond prices increase, when interest rates decrease. More information in later chapters.

2.3 Portfolio strategies

In this section, the portfolio strategies of this thesis will be presented. When we say portfolio strategies, we mean trading techniques, able to maximize profits, while minimizing the risk. Their performance of course, is strongly related to the risk appetite of the investor and the market volatility.

2.3.1 OBPI

The Option Based Portfolio Insurance (OBPI), was firstly introduced in 1976 by Leland and Rubinstein [11]. This strategy is designed in such a way, in order to limit the losses of the investor when the markets declines. It is a static strategy, as the amount invested in the risky and the safe asset remains the same until maturity, without changing in between.

The investor at time $t = 0$ is purchasing the underlying asset, which is usually shares of a stock and at the same time they purchase a European put option written on that stock.

The value of the OBPI portfolio at time t is therefore:

$$V^{OBPI}(t) = \bar{q}\,(S(t) + p(t, S(t), K, T)), \tag{2.12}$$

where $\bar{q}$ is the number of shares of the stock and the number of shares of the put $p(t, S(t), K, T)$ written on that stock with strike K and maturity T. We will assume that $\bar{q} \in [0, 1]$ is also possible (purchasing a fraction of one share).

The price of the put will be given, by using the Put-Call parity along with the closed form solution for the call option (2.4):

$$S(t) - C(t, S(t), K, T) = K\, e^{-r(t-t)} - p(t, S(t), K, T). \tag{2.13}$$

Therefore, if we calculate the price of the European call option, by using the Put-Call Parity we can obtain the price of the put option with strike K and payoff $(K - S(T))_+$ at maturity T.

To simplify, assume that $\bar{q} = 1$. At maturity, as we hold both the stock and the put the value of the portfolio will be:

$$V^{OBPI}(T) = S(T) + (K - S(T))_+. \tag{2.14}$$

Therefore, the investor will get at least the strike K. This outcome does not depend on $S(T)$. If $\bar{q} < 1$, this strategy promises at least $\bar{q}\,K$ at maturity. This relation shows that an increase in the initial risk appetite, will increase the guaranteed amount at maturity.

Below, in Figure 2.5, a simulated path of the put price written on Goldman Sachs' (GS) shares with strike $K = 1.25$ and maturity $T = 2$ years, in order to point out the correlation of the put and the stock in case of a market drop. We assume that the initial GS stock price is 1.

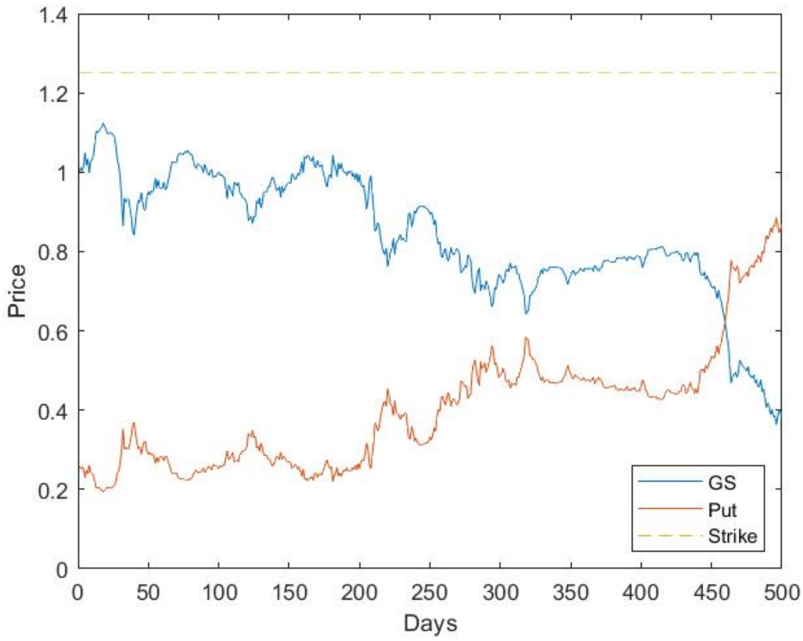

Figure 2.5: Put price written on GS shares with maturity $T = 2$ and strike $K = 1.25$. The 1-year LIBOR was used as the risk-free rate.

2.3.2 CPPI

The Constant Proportion Portfolio Insurance (CPPI) was introduced by Perold (1986) mainly for fixed income, but also for equity instruments later in 1987 by Black and Jones [see 1,13]. A difference from the previous strategy, is that CPPI is a dynamic strategy, as the capital allocated in the risky and the risk-free asset, changes in every time step (for example, daily).

This strategy requires a position in a risk free asset, usually a Treasury bond, in order to guarantee the principal amount at maturity.A leveraged position in a risky asset is taken simultaneously and it is usually shares of a stock. The position in the risky asset is usually called the "performance engine" of the strategy, as the investor enjoys a notable increase in profits when the stock price rises.

The CPPI then, is like taking a long position on a call option, as it has limited losses but unlimited capital gains, because of the long stock. Below, the mechanics behind the strategy are explained step by step [1]:

- The investor chooses a floor $\bar{F}(t)$ which represents the lowest portfolio value they can accept.
- Then, there is a cushion $\bar{c}(t)$ which is computed as the difference between the portfolio value and the floor $\bar{F}(t)$:

$$\bar{c(t)} = V^{CPPI}(t) - \bar{F}(t).$$

- The exposure $\bar{E}(t)$ in the risky asset is then calculated as:

$$\bar{E(t)} = \min\left[m\,\bar{c}(t), V^{CPPI}(t)\right],$$

where m is called multiplier and it is a measure of risk appetite. It is usually between 3 and 5. The bigger m they choose, the bigger is the gain in a notable market increase, but the portfolio will approach the floor faster in a market decline. The remaining funds are usually invested in the risk free asset.

It is important to point out that the floor $\bar{F(t)}$ and the multiplier m depend on the personal choice of the investor and therefore, they are exogenous to the model [1]. The previous steps are executed in every time step (e.g daily), as this is a dynamic strategy. Therefore, the change in value of a portfolio following the CPPI strategy is then given by:

$$dV^{CPPI}(t) = \bar{E(t)}\frac{dS(t)}{S(t)} + (V^{CPPI}(t) - \bar{E(t)})\frac{dP(t,T)}{P(t,T)}, \tag{2.15}$$

where $P(t, T)$ is the price of the risk-free asset, namely the bond, at time t and maturity T, given by equation (2.11). When the portfolio performs well, more capital goes to the stock, otherwise, in the case of a decline (i.e the portfolio value approaches the floor), a bigger percentage is allocated to the bond.

3

Methods

In this chapter, we will introduce the methods used in order to generate the data for the portfolio simulations. First, the calibration of the Heston model will be presented. The calibration for this model is achieved by taking advantage of the closed form solution for the European call option. Later the Hull-White model calibration is illustrated, using historical interest rates data as well as, the initial term structure (yield curve). Lastly, the discrete version of the stochastic differential equations, the pricing formulas and the portfolio value are displayed.

3.1 Calibration

Calibration is another expression for parameter estimation, using historical data, most of the time. In our case, real market data were used for both models. For the Heston model, data from the Nasdaq options chain website were used, while for the Hull-White model, the US Department of the Treasury web page was the main source for acquiring historical rates.

3.1.1 Heston

As stated in chapter 2 and the summary of this chapter, the Heston Model enjoys a closed-form solution for call options. This is a very convenient result because one can use this sophisticated model to price vanilla options fast and also take advantage of this feature of the model in order to estimate the parameters.

From equations (2.4)-(2.7), we can see that the call option price is a function of the five unknown parameters of the Heston model : $\Psi = \left[\bar{V}, V_0, \rho, \eta, a\right]$ which are the long-term average variance, the initial variance, the correlation between the underlying and the volatility, the volatility of variance and the mean reversion speed, respectively.

The plan here is to measure the distance between the model price and the real price of the market and then minimize this distance by running an optimization. One of the most common approaches is by minimizing the Mean Squared Error given by:

$$\bar{H}(\Psi) = \frac{1}{n}\sum_{i=1}^{n}\left[C_i^{\Psi}(K_i, T_i) - C_i^{market}(K_i, T_i)\right]^2, \tag{3.1}$$

where $C_i^{\Psi}(K_i, T_i)$ is the option price using equation (2.4) along with the set of parameters Ψ and $C_i^{market}(K_i, T_i)$ is the real option price from the market.

Additionally, the parameters need to be aligned with the following condition:

$$2a\bar{V} > \eta^2. \tag{3.2}$$

This inequality is also known as the Feller condition (1951) and assures that the process $V(t)$ is strictly positive, as it is impossible for an asset to have negative volatility.

The problem with this approach is that we cannot say if the objective function is convex, giving the possibility of multiple local minima. Therefore, the outcome of the minimization can be dependent on the initial choice of the parameter set Ψ_0. As a result, the valuation of the solution is quite difficult, because we are not able to see if it is a local or global minimum.

In order to estimate the parameters, a constrained nonlinear 5-dimensional local optimization will be performed. MATLAB gives us such possibility by using *fmincon*, an algorithm that finds the minimum of constrained nonlinear multivariable functions. Our objective in this case is (3.1), while our nonlinear constraint for the parameters here is the Feller condition (3.2).

Besides the selection of initial value for the parameters, lower and upper bounds need to be set a priori. The long-term variance and the average variance $\bar{V}, V_0$ will have the same bounds spanning from 0 to 1, as it is not common for the volatility of an asset to go beyond 100 %. It is known that correlation ρ is taking values from -1 to 1. However, we expect mostly negative correlation between the variance and the asset. The volatility of variance η can take large values in general since the volatility as a process is quite volatile. Therefore we let it span from 0 to 10. Lastly, the mean reversion speed a will be bounded automatically, as it has to respect the Feller condition.

The results of this calibration method are presented in the next chapter.

3.1.2 Hull-White

In chapter 2, while introducing the Hull-White model, the time dependent long term mean level was mentioned. This is because of $\theta(t)$, a parameter which changes in every time step (e.g daily) and in order to calibrate it, the initial term structure i.e the yield curve needs to be known.

Park (2004) [12], introduces a way to calibrate the Hull-White one factor model, using the initial term structure of interest rates $R(0,t)$. Next step is calculating the forward rates $F(0,t)$ using $R(0,t)$ in equation (2.10). Then, it is assumed that $F(0,t)$ is a polynomial.

Using MATLAB, we can approximate the forward curve, with a least squares approximation package. The result is a n-degree polynomial describing the forward rates as a function of time:

$$F(0,t) = p_0 + p_1 t + p_2 t^2 + ... + p_n t^n. \tag{3.3}$$

The degree of the polynomial, depends of course on the current market conditions.

As soon as we have the fitted curve, to express $\theta(t)$ we use (also seen in the previous chapter) :

$$\theta(t) = F_t(0,t) + a\,F(0,t) + \frac{\sigma^2}{2\,a}(1 - e^{-2\,a\,t}).$$

In addition, Hull (2009) [9] assumes that the term :

$$\frac{\sigma^2}{2\,a}(1 - e^{-2\,a\,t})$$

is very small and it can be left out in most cases, when the parameter $\theta(t)$ is calculated. Therefore, equation (2.9) becomes:

$$\theta(t) = F_t(0,t) + a\,F(0,t), \tag{3.4}$$

where $F_t(0,t)$ is the partial derivative with respect to time t of the polynomial (3.3) and a is the mean reversion speed.

It is obvious that we need to estimate a, in order to have an expression for $\theta(t)$.This parameter can be estimated by minimizing the following expression:

$$HW(a) = \sum_{i=1}^{n} \Big[r(t_i) - r(t_{i-1}) - F_t(0,t_{i-1})\Delta t - a(F(0,t_{i-1}) - r(t_{i-1}))\Delta t \Big]^2, \tag{3.5}$$

where $r(t_i)$ is the historical 12-months interest rate data from the US Department of the Treasury web page. These data, were collected daily so $\Delta t = 1/250$ (250 trading days per year). It is assumed that we already have an expression for $F(0,t)$, so its values can be used in the minimization (3.5). Having $F(0,t)$, $\hat{a}$ and the expression (3.4), it is possible to estimate θ as a function of t.

Lastly, to estimate the volatility σ of the model, the one-step prediction equation is used [6]:

$$r(\tilde{t_i}) = (F_t(0,t_{i-1}) + \hat{a}F(0,t_{i-1}))\Delta t + (1 - \hat{a}\Delta t)\, r(t_{i-1}),$$

where $\hat{a}$ is the estimated value given by the minimization of (3.5).

Next, the standard deviation s of the errors:

$$\hat{e_r}(i) = r(\tilde{t_i}) - r(t_i)$$

is calculated. Finally, the volatility for the Hull-White one factor model is given by:

$$\hat{\sigma} = \frac{s}{\sqrt{\Delta t}}. \tag{3.6}$$

Having all the parameters estimated, we can use the model to generate future paths for the interest rates. The calibration results will be presented in the next chapter.

3.2 Simulation

In this section, we will introduce how the calibrated models generate future paths for the variables. For the stock and the put option, a discrete version of the Heston model will be used. To simulate paths for the interest rates, the discrete version of the Hull-White model is used. These interest rates help in pricing the risk-free asset, which in our case is a Zero Coupon Bond. Employing the portfolio processes along with the simulated stock, put and bond paths, one can obtain paths for both strategies.

3.2.1 Heston

To simulate the Heston stochastic volatility model, we will need a discrete form of the SDE's (2.1) and (2.2). To achieve this goal, the Milstein scheme discretization for the processes $V(t)$ and $\ln S(t)$ will be employed.

Starting with a value V_0, which is estimated by (3.1) and given a value for $V(t_i)$, we get $V(t_{i+1})$ as:

$$V(t_{i+1}) = V(t_i) + a(\bar{V} - V(t_i))dt + \sigma\sqrt{V(t_i)dt}\,W_V + \frac{1}{4}\sigma^2 dt(W_V^2 - 1),$$

which can be written as [14]:

$$V(t_{i+1}) = \Big(\sqrt{V(t_i)} + \frac{1}{2}\sigma\sqrt{dt}\,W_V\Big)^2 + a(\bar{V} - V(t_i))dt - \frac{1}{4}\sigma^2 dt, \tag{3.7}$$

where W_V is a random number from the standard normal distribution.

Knowing the volatility at time t_i and assuming that $S_0 = 1$ the discretization for the stock price is then [14]:

$$S(t_{i+1}) = S(t_i)\exp\Big(\Big(r - \frac{1}{2}V(t_i)\Big)dt + \sqrt{V(t_i)dt}\,W_S\Big), \tag{3.8}$$

where W_S is a random number from the standard normal distribution and r is the risk free rate. The proof of the formulas can be found in the paper: Rouah F D. Euler and Milstein discretization [14].

In the introduction, we saw how the volatility is correlated with the stock price by letting:

$$dW(t)_S\,dW(t)_V = \rho\,dt.$$

To achieve correlated Brownian motions here, we need to generate two independent standard normal variables Z_i and Z_j such that:

$$W_V = Z_i$$

and

$$W_S = \rho Z_i + Z_j \sqrt{1-\rho^2}.$$

Finally, the discrete formula of the call price at time t_i, is given by:

$$C(t_i, S(t_i), K, t_n) = S(t_i)\Delta_1 - e^{-r(t_n - t_i)}\, K\Delta_2 \tag{3.9}$$

,where the integrals Δ_1 and Δ_2 ((2.5) and (2.6) respectively) are evaluated numerically in MATLAB for each time step. The term t_n symbolizes maturity.

The Put Option price is easily derived by the Put-Call parity:

$$S(t_i) - C(t_i, S(t_i), K, t_i) = K\, e^{-r(t-t_i)} - p(t_i, S(t_i), K, t_i),$$

so that

$$p(t_i, S(t_i), K, t_n) = K\, e^{-r(t_n - t_i)} - S(t_i) + C(t_i, S(t_i), K, t_n). \tag{3.10}$$

3.2.2 Hull-White

The discrete version of the model in order to simulate future paths for interest rates is the following:

$$r(t_i) = (F_t(0, t_{i-1}) + a(F(0, t_{i-1}))dt + (1 - a\, dt)\, r(t_{i-1}) + \sigma\sqrt{dt} Z_i \tag{3.11}$$

Observe that:

$$F_t(0, t_{i-1}) + a(F(0, t_{i-1}) = \theta(t_{i-1}).$$

In this case, r_0 will be the last data point from the data set used for calibration, denoting today's 12-months rate and the beginning of the future path. It is the same for every simulation.

Now the discrete form of equation (2.11) for the bond price is:

$$P(t_i, t_n) = A(t_i, t_n)\, e^{-B(t_i, t_n) r(t_i)}, \tag{3.12}$$

where

$$B(t_i, t_n) = \frac{1}{a}\Big[1 - e^{-a(t_n - t_i)}\Big],$$

and

$$A(t_i, t_n) = \frac{P(0, t_n)}{P(0, t_i)}\, exp\Big[B(t_i, t_n)\, F(0, t_i) - \frac{\sigma^2}{4a^3}\Big(e^{-at_n} - e^{-at_i}\Big)^2 \Big(e^{2at_i} - 1\Big)\Big].$$

3.2.3 Portfolio Processes

OBPI is the static strategy, so the amount invested in the risky and the risk free asset remains the same in every time step. The initial value of the portfolio will be: $V^{OBPI}(t_1) = 1$. The value of the portfolio in discrete form is given by:

$$V^{OBPI}(t_i) = \bar{q}\,(S(t_i) + p(t_i, S(t_i), K, t_n)) + \bar{h}P(t_i, t_n),$$

where $\bar{q}$ is the amount invested in the risky asset and $\bar{h}$ symbolizes the remaining capital invested in the bond, given by:

$$\bar{h} = \frac{V^{OBPI}(t_1) - \bar{q}\,(S(t_1) + p(t_1, S(t_1), K, t_n))}{P(t_1, t_n)}.$$

As $\bar{q}$ depends on the personal choices of the investor and remains constant, the same happens for $\bar{h}$. The stock, the put and the bond price at time t are given by (3.8), (3.10) and (3.12), respectively.

At maturity, the investor secures that the portfolio value will be at least:

$$floor = \bar{q}\,K + \bar{h}$$

,which is the floor of the strategy. Observe how the portfolio manager earns at least a percentage of the strike K from the risky asset plus a percentage of the face value of the Zero Coupon Bond, which in this thesis is taken to be 1. The floor depends on the initial capital allocation.

CPPI is the dynamic strategy, so the allocation between the stock and the bond will change in every time step. Also starting with $V^{CPPi}(t_1) = 1$, the dynamics of CPPI are given by:

$$V^{CPPI}(t_i) = V^{CPPI}(t_{i-1}) + \bar{E}(t_{i-1})\frac{S(t_i) - S(t_{i-1})}{S(t_{i-1})}$$

$$+(V^{CPPI}(t_{i-1}) - \bar{E}(t_{i-1}))\frac{P(t_i, t_n) - P(t_{i-1}, t_n)}{P(t_{i-1}, t_n)},$$

where

$$\bar{E(t_i)} = \min\left[m\,\bar{c}(t_i), V^{CPPI}(t_i)\right]$$

and

$$\bar{c(t_i)} = V^{CPPI}(t_i) - \bar{F}.$$

$\bar{F}$ is the lower value of the portfolio and is chosen by the investor, such as the multiplier m. The exposure $\bar{E}$ in the risky asset changes in every time step. As we want to make comparisons between the two strategies, the initial exposure in the risky asset has to be the same for both strategies. So, we set $\bar{E(t_1)} = \bar{q}$.

4

Results

In this chapter, the data and the results of the calibration will be presented ,first. We then feed the estimated parameters in the models and employ them for the future path generation. When the assets are simulated, the portfolio processes can be simulated as well, letting us make comparisons between them in a falling or increasing market, as well as their overall performance at maturity. The risky assets are the Goldman Sachs stock and a put option written on that stock, while the risk free asset will be a Zero Coupon Bond paying 1 at maturity.

4.1 Results from the calibration

To calibrate the Heston model, 15 Exchange traded call options written on the Goldman Sachs (GS) shares were used:

Spot	Strike	Maturity	Mid	Rate
195.5	172.5	0.1260	24.45	0.0241
195.5	185	0.1260	13.65	0.0241
195.5	192.5	0.1260	8.70	0.0241
195.5	195	0.1260	7.20	0.0241
195.5	200	0.1260	4.90	0.0241
195.5	180	0.2794	20.25	0.0245
195.5	185	0.2794	16.65	0.0245
195.5	190	0.2794	13.20	0.0245
195.5	195	0.2794	10.45	0.0245
195.5	200	0.2794	7.85	0.0245
195.5	175	0.6054	27.90	0.0237
195.5	180	0.6054	24.30	0.0237
195.5	185	0.6054	21.20	0.0237
195.5	190	0.6054	18.15	0.0237
195.5	195	0.6054	15.35	0.0237

Table 4.1: Call Option prices (Mid price) with three different maturities and five different strikes for each maturity were used (in the money, at the money and out of the money calls). The risk free rate is aligned with each maturity. The spot price is the GS stock closing price observed when the data were collected (June 10th, 2019).

Using the data from the table along with the minimization formula (3.1), the parameters for the Heston model minimizing the distance between the market price and model price are the following:

Param.	Value
$\bar{V}$	0.0692
V_0	0.0755
ρ	-0.7170
η	1.1703
a	9.8938

Table 4.2: Calibrated values for long term average variance, initial variance, correlation, volatility of variance and mean reversion speed, respectively. Observe how the parameters respect the Feller condition (3.2), which yields a strictly positive variance process.

In order to evaluate our calibration, a repricing of these options were performed using the calibrated parameters and the closed form solution for calls of the Heston model (2.4). The results are summarized in the following figure:

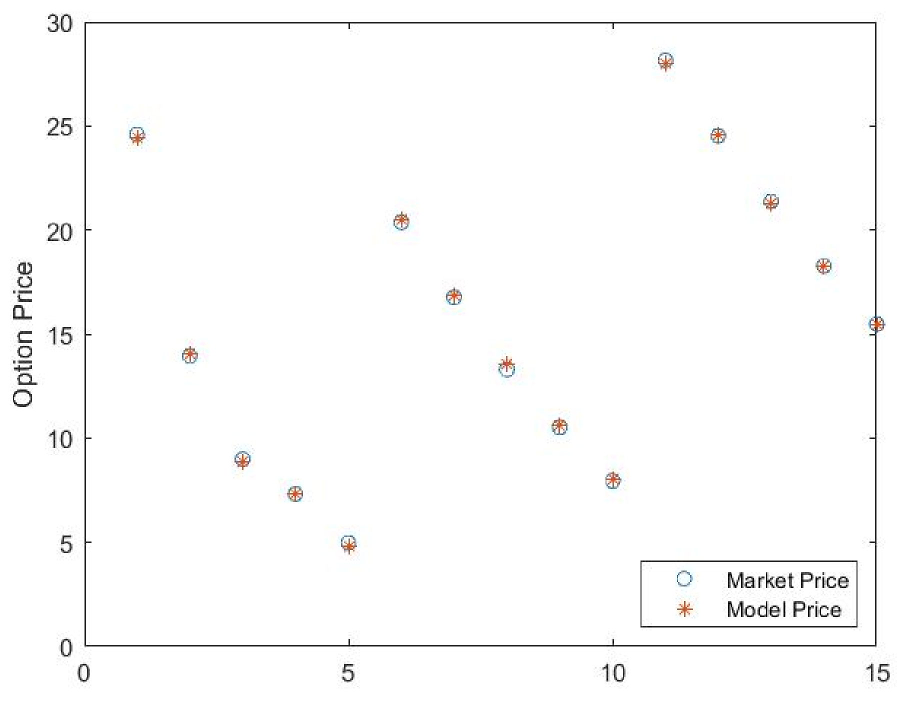

Figure 4.1: The repricing of our 15 Options using the estimated parameters. The blue circle denotes the real market price, while the red star denoted the model price. The model did a good job in capturing the price. The parameters will later be used for simulations.

For the Hull-White model, the yield curve is needed in order to compute forward rates and eventually obtain a polynomial expression for $F(0,t)$. Yield curve data can be seen below:

Rate	Maturity
0.0212	0.0833
0.0215	0.1667
0.0206	0.2500
0.0209	0.5000
0.0197	1
0.0183	2
0.0179	3
0.0183	5
0.0194	7
0.0208	10
0.0238	20
0.0261	30

Table 4.3: Different yields across different contract lengths spanning from 1 month, to 30 years. The plot of the yield curve can be found in Figure 2.3.

Using the data from the previous table, it is possible to calculate the forward rates. A plot of the forward rates was seen in Figure 2.3. We approximate the forward curve with a fourth degree polynomial:

$$F(0,t) = p_0 + p_1\, t + p_2\, t^2 + p_3\, t^3 + p_4\, t^4, \tag{4.1}$$

where

$$\begin{aligned} p_0 &= 0.021, \\ p_1 &= -0.0025, \\ p_2 &= 5.5430 \cdot 10^{-4}, \\ p_3 &= -3.1347 \cdot 10^{-5}, \\ p_4 &= 5.3518 \cdot 10^{-7}. \end{aligned}$$

A visualization is illustrated here:

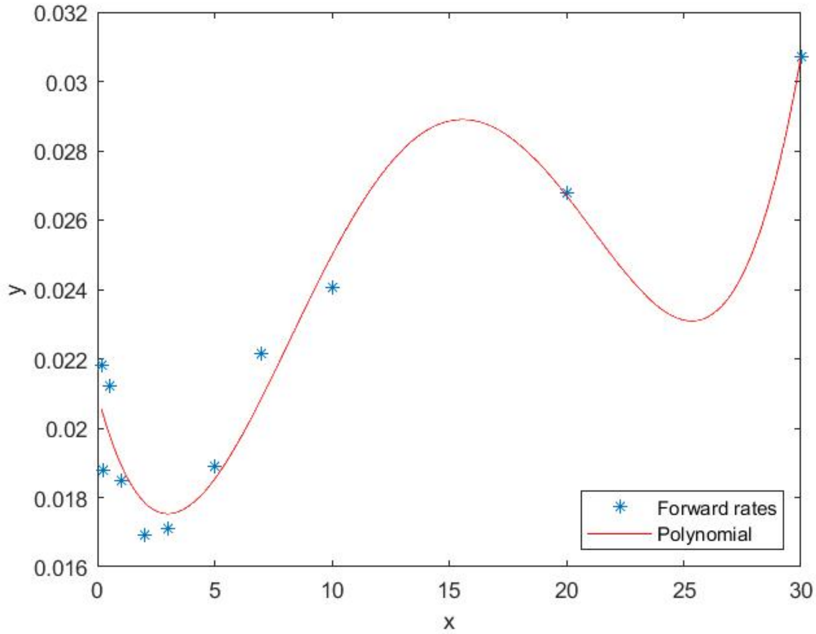

Figure 4.2: The polynomial (4.1) approximating the forward rates which were derived using formula (2.10).

To calibrate the parameters a and σ of the Hull-White model, we used the 12-months historical rates from the US Department of the Treasury. Data span from January 2018 to July 2019. A snippet can be seen below:

12-Months Rate
0.0183
0.0181
0.0182
0.0180
.
.
.
0.0190
0.0194
0.0195
0.0197

Table 4.4: Data from January 2018 to July 2019 from the US Department of the Treasury web page (389 values).

The parameters obtained by the previous data are the following:

Param.	Value
$\hat{a}$	0.2894
$\hat{\sigma}$	0.0036

Table 4.5: Parameter estimation for the Hull-White model with the procedures described in the previous chapter such as equations (3.5) and (3.6) along with the historical rates.

Finally, using the results from Table 4.5 and the equation (3.4), the expression for $\theta(t)$ is the following:

$$\theta(t) = p_0 + p_1(t + \hat{a}) + p_2 t(t + 2\hat{a}) + p_3 t^2(t + 3\hat{a}) + p_4 t^3(t + 4\hat{a}),$$

where p_i is estimated from the fitted polynomial and given below equation (4.1).

Putting everything together, using the discrete form of Hull-White (3.14) and the first value of the Table 4.4, we compare the obtained path to the data. We generate 389 values, since this is the number of our historical data points used for calibration:

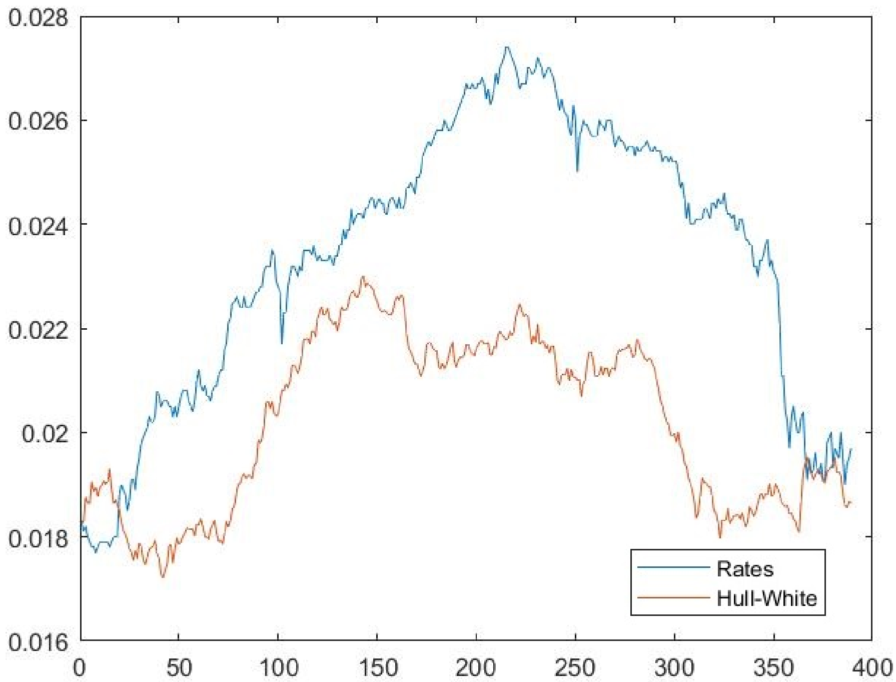

Figure 4.3: The 12-months rates used for calibration with blue compared to the path generated using the calibrated Hull-White model in orange. The model seems to capture the whole trend of the data set, however it lays below the data almost always.

4.2 Assets and Portfolio results

Having our models calibrated, the last thing is to employ the simulated paths obtained from them in the portfolio strategies and draw the final results. Comparisons will be made mostly from their performance in a market decline and in a rising market. In the end, their overall performance at maturity will be discussed.

Before presenting the results, the exogenous parameters of each strategy have to be stated, as well as the selection of the strike for the put option and the maturity of each portfolio process.

Starting with the OBPI static strategy, imagine an investor with an initial capital of 1 at $t = 0$. The investor chooses to invest $\bar{q} = 0.75$ in Goldman Sachs shares with $S_0 = 1$ and a put option written on these shares with strike $K = 1.25$ (ITM) and maturity $T = 2$ years. Such put costs now ≈ 0.2595. Therefore, they invest $\bar{h} \approx 0.0575$ in a bond that costs now 0.963 and pays 1 in 2 years. The floor of this strategy will be then ≈ 0.99. It is important to state again that the parameters $\bar{q}$ and therefore $\bar{h}$ are exogenous to the model, since they reflect personal preferences. It is now time to see how the strategy performs under different market conditions:

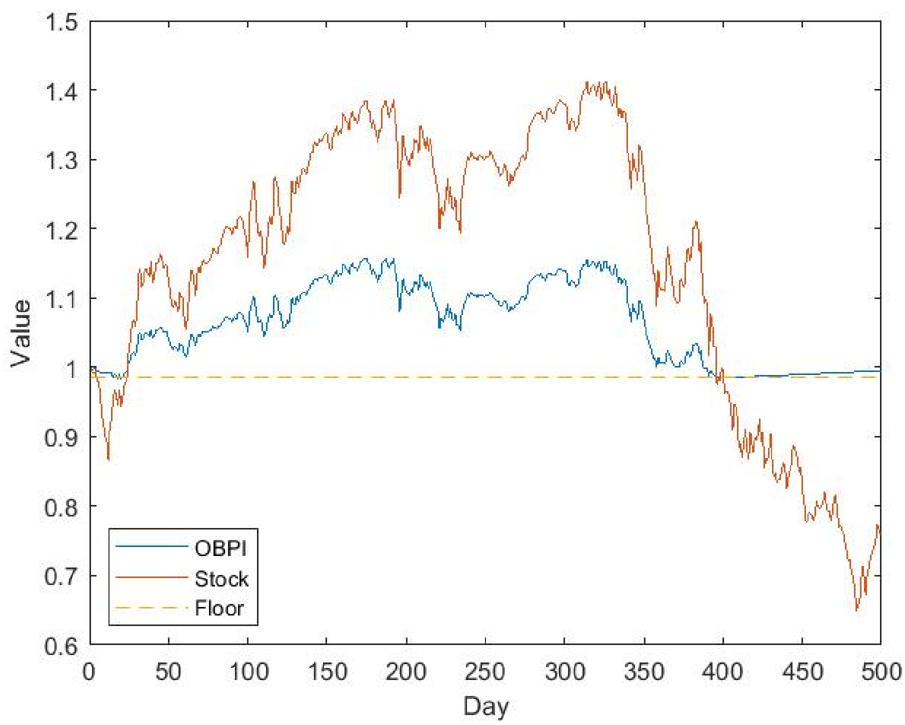

Figure 4.4: The performance of the OBPI strategy with floor 0.99 in a declining market, maturity 2 years (500 trading days). Observe how the strategy remains above the floor, even if the Goldman Sachs share price plummets after the first year.

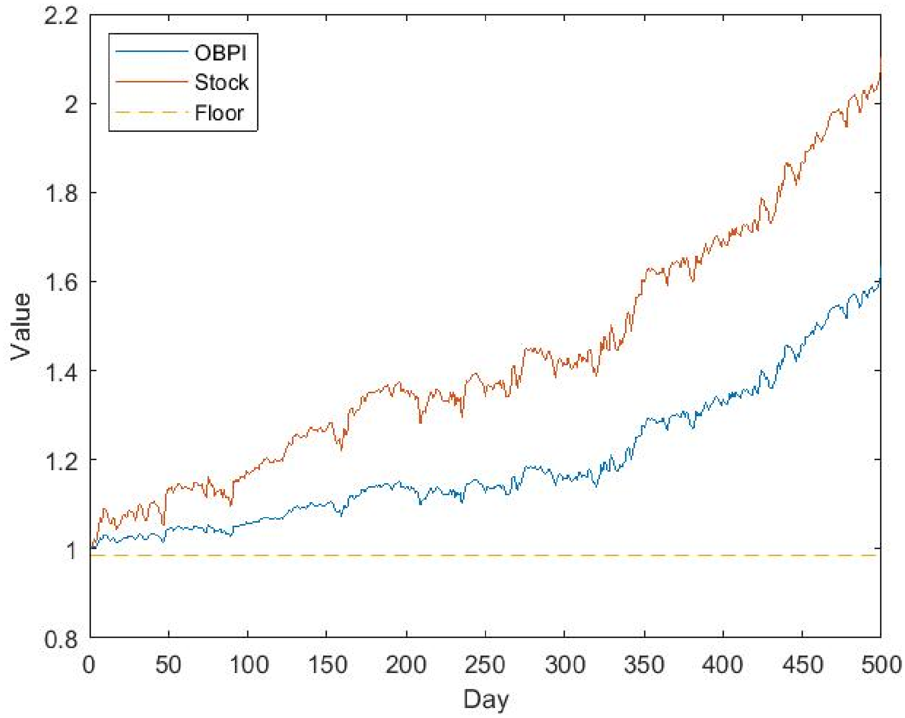

Figure 4.5: The performance of the OBPI strategy compared to a booming market. The portfolio increases slower in value, as the Goldman Sachs share price grows. When the stock passes the strike price $K = 1.25$, the portfolio value increases faster.

To compare the OBPI with the CPPI dynamic strategy, we also consider an investor with capital 1 at time $t = 0$, who chooses the initial exposure to be $\bar{E}(t_0) = 0.75 = \bar{q}$. This means that OBPI and CPPI have the same initial exposure to the risky asset. However, the allocation between the risky and the risk free asset changes in every time step for CPPI. The investor chooses a floor $\bar{F} = 0.75$ and a multiplier $m = 3$. Then, rebalances the portfolio daily, according to the market conditions and the performance of the Goldman Sachs stock. Results in different market scenarios below:

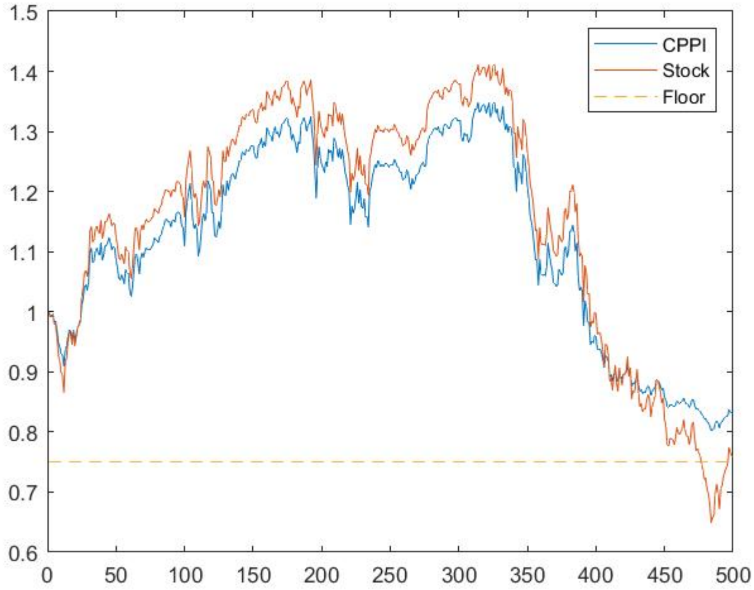

Figure 4.6: The performance of the CPPI strategy with floor 0.75, compared with a declining market. In this strategy, the only risky asset is the stock, so CPPI is more similar with the GS stock performance. However, when the share price drops below the floor, the portfolio value stays above, as the portfolio manager will allocate more capital to the risk free asset.

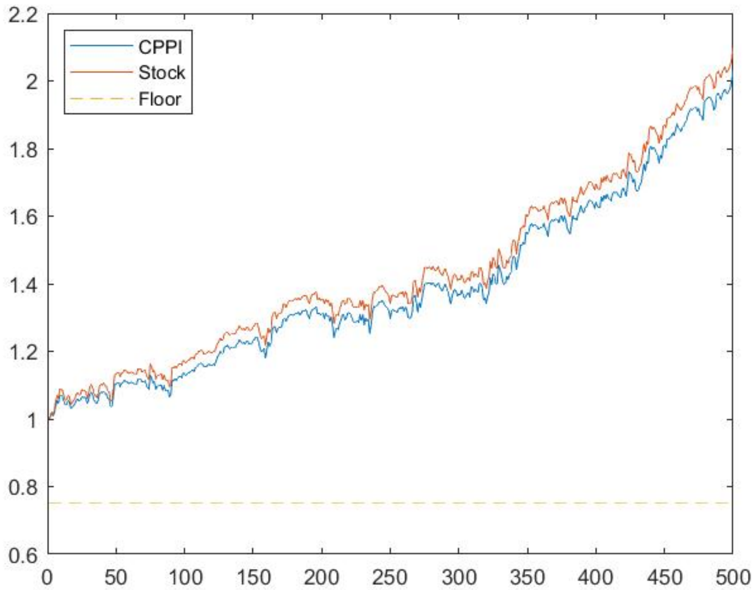

Figure 4.7: The performance of the CPPI strategy compared to an increasing market. Here the portfolio seems to be similar to a long stock strategy, as it grows almost together with the GS share price and the investor enjoys more or less equal returns.

After a number of Monte-Carlo simulations for the OBPI and CPPI paths using the Goldman Sachs stock and the 1-year LIBOR risk free rate, their value at maturity $T = 2$ years (500 trading days) is summarized below:

	Initial	$T = 2$	σ	**Floor**
OBPI	1	1.057	0.13	0.99
CPPI	1	1.052	0.29	0.75

Table 4.6: Mean and standard deviation of the portfolio value using the Goldman Sachs stock for both strategies after 2 years, with an initial investment of 1.

5
Conclusion

The OBPI strategy seems to provide protection from $t = 0$ until the maturity of 2 years. The initial capital is almost guaranteed at maturity, as the floor of the strategy is very high. OBPI preserves the investment in a significantly falling market and gently participates in a bull market. Observe in Figure 4.5 how the portfolio value rises faster when the stock passes the strike $K = 1.25$. This is because the put option is very cheap, while the stock gains value. Holding a put option means that the investor expects a market decline and wants to protect the long position in the stock. To purchase a Put Option requires paying a premium, which is an extra expense. Options are known for their time-decay, which makes them loose their value, especially when maturity approaches. That leaves them with just their intrinsic value in the end.

The CPPI strategy has a lower floor, which in this case was selected by the portfolio manager. This strategy reaches the floor faster when the share price declines, but that also depends on the multiplier. However, as it is a dynamic strategy, when the stock price increases, the strategy will invest more and more in the risky asset and less in the risk free asset, taking advantage of the market performance. From the previous figures, it can be seen that CPPI's value is very close to the stock for a sharply rising market. On the other hand, it also follows the market performance in a decline, but does not fall below the floor, as seen in Figure 4.6.

In general, OBPI is safer in a falling market since it is hedged by the long put, but does not take good advantage of an increasing market compared to the other strategy. The CPPI provides higher returns in a rising market than OBPI but suffers bigger losses when the market declines. From Table 4.6, we see that OBPI has a mean value slightly above CPPI at maturity, for the Goldman Sachs stock. On the other hand, for the CPPI, the standard deviation is more than double, making it a riskier strategy. Riskier strategies means fatter distribution tails and therefore access in more extreme events like higher returns. The higher the risk, the higher the reward. It is obvious that the most important part of both strategies is the stock, which is the main performance engine for both portfolios. Both of the strategies have limited losses but unlimited gains, because of the long position in GS shares. In those cases, higher volatility works in favor of both investors.

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