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Calibration and Pricing of CBOE Volatility Index Options using Merton's Jump-Diffusion Framework

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Abstract

Market-based Pricing and Calibration of VIX Options under

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This thesis examines the performance of Merton's Jump-Diffusion model (MJD) in a market based valuation of VIX futures option for three different maturities. The first step was to derive the risk neutral dynamics of the MJD model. Second, model parameters are estimated and model prices are computed based on their estimates using the fast Fourier approach. Lastly, the results are compared to its predecessor the Black-Scholes model. The time frame of the options covered months shortly before and past the Lehman Brother collapse which triggered the financial crisis of 2008. The results imply that Merton's Jump-Diffusion model is a significant improvement to the traditional Black-Scholes mode and cannot in fact perform worse than its forerunner since it is embedded as a special case of the MJD model.

This thesis is dedicated to my parents Angela and Robert Gustafsson for their constant love and support up to this day, and my girlfriend Therese who has put up with a student for a boyfriend longer than necessary.

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1 Introduction

The most prominent risk faced by investors are price movements in the marketplace. Consequently, finance practitioners and researchers are concerned with understanding and predicting future market volatility as it mainly impacts the decision making in several areas, such as security valuation, risk management, monetary policymaking, and more. In 1993, the Chicago Board Options Exchange (CBOE) introduced the Market Volatility Index (VIX), which has become a popular way of estimating future volatility. The VIX is intended to provide investors with a snapshot of the market expectation of the volatility of the S&P 500 index option (SPX) over the next 30 days (Implied Volatility). Thus, VIX is used as a proxy for near-term market uncertainty, which investors take into account when determining trading strategies for current market environments.

Continuous-time models relying on Brownian motion play a significant role in modeling and pricing derivatives. The best-known member of this category is the so-called *Black-Scholes model*. Presented for the first time in 1973 in the paper, "The Pricing of Options and Corporate Liabilities," this option pricing model is still one of the most celebrated inventions in modern financial theory. The Black-Scholes model (also called the Black-Scholes-Merton model) has ever since its introduction had a tremendous influence on the way practitioners value and hedge financial derivatives and has largely contributed to the explosive growth of financial innovation over the past 40 years. Nevertheless, the Black-Scholes model relies on several unrealistic assumptions, which misrepresents reality. As a result, succeeding studies and research papers have modified the traditional Black-Scholes model by rejecting the hypothesis that asset returns are normally distributed. This discovery has emphasized the need to revise its traditional elements. After all, practitioners rely on models for risk management and valuation purposes. For example, using an inaccurate model could lead to an inappropriate hedging strategy.

One of the first models beyond the Black-Scholes model was only three years away. It came to be known as the Merton Jump-Diffusion model (MJD) and was a significant improvement to the Black-Scholes model in the sense that it could better capture the negative skewness and excess kurtosis of the return distribution, which have been observed among asset prices since the 1950s [8]. The research paper "Empirical Performance of Alternative Option Pricing Models" by Gurdip Bakshi, Charles Cao and Zhiwu Chen (1997) states that adding a jump component improves model performance for the pricing of short-term options [7].

Despite practitioners' and the research community's eagerness and the endless quest of trying to find the "perfect" model, it is just as important to find a proper and efficient method which can capture its parameters. This paper will treat the ill-posed problem of finding the model parameters of Merton's Jump-Diffusion model such that observed market quotes of the VIX are replicated as closely as possible. The model prices generated by the Merton Jump-Diffusion will also be compared to its predecessor, the Black-Scholes model, which calibration is performed analogously to the one of the MJD.

2 Recalling the Black-Scholes Model

2.1 Model Assumptions and Setting

In its original setting, the Black-Scholes-Merton world presumes a 1+1 dimensional capital market consisting of two types assets, a risk-free asset, B (bond) and a risky asset, S (stock) [9]. The risk-free bond is typically a short rate government note corresponding to the length of the option contract (commonly used are 1, 3, or a 6 month Treasury bill) and generates a risk-free rate of return r_t, which grows at a constant continuously compounding rate. Nevertheless, for simplicity, we will further consider r_t to be nonrandom although it could vary with time. Thus, the price of the Bond, B_t, at time t is assumed to satisfy the differential equation

$$dB_t = rB_t dt,$$

which has the unique solution $B_0 = 1$ as

$$B_t = B_0 e^{rt}.$$

In addition to the preceding assumptions, the authors Fischer Black and Myron Scholes outlined the following vast assumptions assumed by this model:

1. There are no fees or transaction costs from buying or selling the option (i.e. "frictionless market").

2. Investors are allowed to borrow any fraction of the stock price, at the risk-free rate r. This holds for either buying or holding the security.

3. The interest rate r is known and constant throughout the lifetime of the option.

4. The underlying stock pays no dividend or income during the lifetime of the option.

5. The model can only value European options (i.e., options that can only be exercised at maturity).

6. The Stock price process follows a geometric Brownian motion, where the price at each future time is log-normally distributed.

The stock price process $\{S_t\}_{t\geq 0}$ is assumed to be dictated by a geometric Brownian motion with mean rate of return μ and volatility σ (both constant) and satisfy the stochastic differential equation (SDE)

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where $\{W_t\}_{t\geq 0}$ denotes a standard Brownian motion (also referred as standard Wiener process) [6]. Nevertheless, in the risk-neutral setting the Black-Scholes satisfies the SDE

$$dS_t = rS_t dt + \sigma S_t dW_t$$

and its solution is given by l

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma W_t}.$$

3 Surpassing the Black-Scholes

3.1 Merton's Jump-Diffusion Model

One of the very first expansions of the traditional Black-Scholes-Merton model was Merton's Jump-Diffusion model (MJD) developed in 1976 by Robert C. Merton himself. In this framework, the dynamics of the returns of S_t is under the physical probability measure $\mathbb{P}$

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW_t + d(\sum_{i=1}^{N_t}(e^{Y_i} - 1)) \tag{3.1}$$

where $S_{t^-} = \lim_{u \to t^-} S_u$. However, from this point forward, we use that $S_{t-} \equiv S_t$. The Merton Jump-Diffusion model differs from the conventional Black-Scholes model in the last term, which contains two sources of randomness. The parameter N_t is a Poisson process with intensity λ per unit time, which is intended to capture the abnormal price changes caused by the arrival of influential market information (i.e., jump in the asset price). Second, e^{Y_i} is a is non-negative log-normally distributed random variable (i.e., $Y_i \sim N(\mu, \delta^2)$), which measures the magnitude of each jump[8]. The parameters $\mu \in \mathbb{R}^+$ and $\delta > 0$ describes the mean log-return jump-size and standard deviation of log-return jump respectively. Additionally, the jumps in the asset price are assumed to occur independently and identically.

The percentage change in the stock price achieved by jumps is described as follows:

$$\frac{dS_t}{S_t} = \frac{e^{Y_i}S_t - S_t}{S_t} = e^{Y_i} - 1.$$

Observe that whenever $Y_i = 0$ we obtain

$$\frac{dS}{S_t} = \mu dt + \sigma dW_t,$$

the traditional Black-Scholes SDE. Hence, the Black-Scholes model is a special case of the Merton Jump-Diffusion model.

3.2 The Risk-Neutral Pricing Dynamics of MJD

Risk-neutral pricing is a technique used in mathematical finance to value financial derivatives. Similar to most games of chance where the price of a given game is based on its expected payoffs, the value of a financial asset is determined by its discounted future expected payoffs. In games of chance, the probabilities of various outcomes are known and expressed in terms of the real-world probability measure $\mathbb{P}$. Nevertheless, the price of a financial asset is dependent on each investor's level of risk aversion as rational investors demand a higher return for bearing risk (i.e., investors require a risk premium for taking on risk). As a consequence of the latter, the rate at which investors discount their expected payoffs would differ across all investors and so the price of the derivate.

Given the laboriousness and complexity of quantifying each investor's risk aversion, practitioners would rather use a probability measure $\mathbb{Q}$ in which investors are neutral to risk and only expect the risk-free return. The Fundamental Theorem of Asset Pricing affirms the existence of a risk-neutral measure (also called *Equivalent Martingale Measure (EMM)*) if and only if the market is arbitrage free. However, in the cases when the markets are incomplete, this measure is not unique. With the assistance of Radon-Nikodým derivative and Girsanov's theorem, one can define a risk-neutral measure $\mathbb{Q}$ equivalent to the real-world $\mathbb{P}$.

In the Merton Jump-Diffusion framework, the Radon-Nikodým derivative for $t \leq T$ is given by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_T = \exp\bigg(-\frac{\theta^2}{2}T - \theta W_T + \sum_{i=0}^{N_T}(\gamma Y_i + \upsilon)\lambda\kappa' T\bigg). \tag{3.2}$$

With the expected proportional jump size being $\kappa \equiv \mathbb{E}_{\mathbb{P}}[e^Y - 1]$, the moment generating function of all jump sizes is given by

$$\kappa' \equiv e^{\upsilon} M_{\mathbb{P},Y}(\gamma) - 1 = e^{\upsilon}\mathbb{E}_{\mathbb{P}}[e^{\gamma Y}] - 1. \tag{3.3}$$

One can easily verify that (3.2) satisfies the property of a Radon-Nikodým derivative and that it is a martingale in the real world measure $\mathbb{P}$. However, the choices of the parameters υ and γ will dictate which EMM will be generated and as a consequence determine the distribution of the jump size and the jump intensity respectively. Thus, υ and γ can be chosen such that it produces an EMM for (3.1). The following lemma will assist in forming the risk-neutral dynamics for the Merton-Jump Diffusion model.

Lemma 3.1. *Let $\mathbb{P}$ and $\mathbb{Q}$ be equivalent measures. Further, allow $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ to be the probability measure space such that $\mathcal{F}_t$ is the natural filtration generated by a standard Brownian motion W_t, a compounding Poisson process $\sum_{i=0}^{N_t} Y_i$ with the intensity $\lambda > 0$, and a Radon-Nikodým derivative of the form (3.2) where $\lambda, \upsilon \in \mathbb{R}$ and κ' is given by (3.3).*

Then the Brownian motion W_t has drift $-\theta$ under the risk-neutral measure $\mathbb{Q}$ and the compound Poisson process $\sum_{i=0}^{N_t} Y_i$ under the measure $\mathbb{Q}$ has a new intensity $\hat{\lambda} = \lambda(1 + \kappa')$ and a new distribution for the jump-sizes. The moment generating function of the jump-size distribution is given by

$$M_{\mathbb{Q},Y}(u) = \frac{M_{\mathbb{P},Y}(\gamma + u)}{M_{\mathbb{P},Y}(\gamma)}$$

[3].

Recall that under the risk-neutral measure the discounted price process is a martingale. First, assume that $\gamma = 0$ and $\upsilon = 0$. Next, setting $\kappa' = 0$ in Lemma 3.1 will under the measure $\mathbb{Q}$ provide the Poisson process with the intensity $\hat{\lambda} = \lambda$.

The discounted stock process $D_t S_t = e^{-rt} S_t$ can then be expressed as

$$\begin{aligned} d(D_t S_t) &= -rS_t e^{-rt} dt + e^{-rt} dS_t \\ &= -rS_t e^{-rt} dt + e^{-rt} \Big(\mu S_t dt + \sigma S_t dW_t + S_t d(\sum_{i=1}^{N_t} (e^{Y_i} - 1)) \Big) \\ &= \sigma e^{-rt} S_t \Big(\frac{\mu - r}{\sigma} dt + dW_t + \frac{1}{\sigma} d(\sum_{i=1}^{N_t} (e^{Y_i} - 1)) \Big) \\ &= \sigma D_t S_t dW_t^{\mathbb{Q}} \end{aligned} \tag{3.4}$$

where

$$W_t^{\mathbb{Q}} = \int_0^t \theta ds + W_t, \tag{3.5}$$

which yields

$$\theta = \frac{\mu - r + \lambda\kappa}{\sigma} \quad \text{where} \quad \kappa = \mathbb{E}_{\mathbb{Q}}[e^Y - 1]. \tag{3.6}$$

The parameter θ can be interpreted as an analogue to the Sharpe ratio (or Reward-to-Volatility ratio), where the expected jump-magnitude κ have an impact on the excess return.

Recall that the distribution of the jump sizes is characterized by (3.3). Given the choice of γ and υ, the distribution of Y under $\mathbb{Q}$ is equivalent to the distribution under the original measure $\mathbb{P}$, i.e., $Y \sim N(\mu_j, \delta^2)$. Hence, the average jump size can be expressed as

$$\kappa = \mathbb{E}_{\mathbb{Q}}[e^Y - 1] = \int_{\mathbb{R}} (e^y - 1) f(y) dy = e^{\mu_j + \frac{\delta^2}{2}} - 1. \tag{3.7}$$

The new process $dW_t^{\mathbb{Q}} = \theta dt + dW_t$ is a standard Brownian motion under the risk-neutral measure $\mathbb{Q}$ categorized by the Radon-Nikodym derivative, where θ is an adapted process and W_t a standard Brownian motion under $\mathbb{P}$. The parameter θ is also known as the market price of risk, i.e., the additional return an investor requires for bearing risk. The risk-neutral measure $\mathbb{Q}$ as defined in Girsanov's Theorem is equivalent to the physical measure $\mathbb{P}$, which turns the discounted stock price $D_t S_t$ into a martingale. More rigorously, applying Itô's formula on (3.4) results in

$$D_t S_t = S_0 + \int_0^t \sigma D_u S_u dW_u^{\mathbb{Q}}$$

where the second term is an Itô integral, which by definition is a martingale. Substituting dW_t with ${dW_t}^{\mathbb{Q}}$ yield the risk-neutral return dynamics of the MJD model

$$\frac{dS_t}{S_{t^-}} = (r - \lambda\kappa)dt + \sigma {dW_t}^{\mathbb{Q}} + d(\sum_{i=1}^{N_t}(e^{Y_i} - 1)).$$

4 Empirical Discussion

4.1 Data

The data used in the calibration was historical quotes on European call options on the VIX, which was purchased directly from the Chicago Board Options Exchange (CBOE). The date range spans June 16, 2008, to November 19, 2008 (cf. Figure 2). This section focuses on the underlying asset, the VIX volatility index, and its statistical characteristics. The time frame of the valued options spanned 111 days and was evaluated over maturities of one, three and five months. This time frame is considered one of the most distressed and volatile periods in history. Figure 1 display a larger time frame of the VIX, which feature the distressed period following the collapse and bankruptcy of the United States fourth-largest investment bank Lehman Brothers on September 15, 2008.

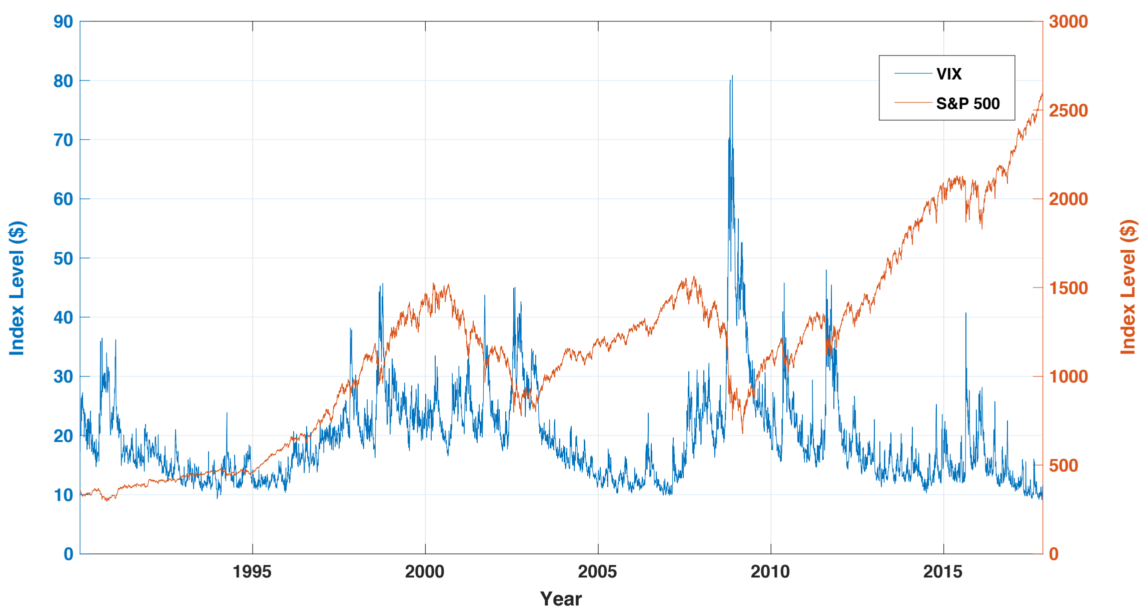

Figure 1: The VIX and the S&P 500 for January 2, 1990, - November 15, 2017. *Data: Bloomberg*

4.2 Empirical Properties

By adding the compounded Poisson jump component to the traditional Black-Scholes model, Merton aimed to capture the empirically observed leptokurtic feature of the return distribution, which is characterized by heavier tails and higher peakedness. Table 2 displays the four empirical moments of log returns for the realizations in Figure 3. Figure 4 and 6 provides us with an pictorial representation of the characteristics listed in table 2. Both Kernel estimates (cf. Figure 5 and 7) seem normally distributed with slight positive skewness. As mentioned earlier, the Merton Jump Diffusion model aims to capture both negative skewness and excess kurtosis, which has been empirically observed in stock log return density $\mathbb{P}[\ln(S_t/S_{t-1})]$.

The negative skewness in aggregate returns is a result of the positive and upward drift that stocks market exhibits over longer time frames. VIX differs from traditional indexes and stocks in the sense that it is supposed to measure the expected volatility in the market. According to CBOE, there has historically existed an inverse relationship between the VIX and the S&P 500 index. Statistics of this relationship based on 3206 trading days are captured in table 1 for the time frame January 1, 2000, to September 28, 2012, [2]. This relationship stresses that the log return distribution of the VIX is positively skewed since it experiences more number of downturns compared to the number of upturns in the long run. Figure 1 illustrates this relationship.

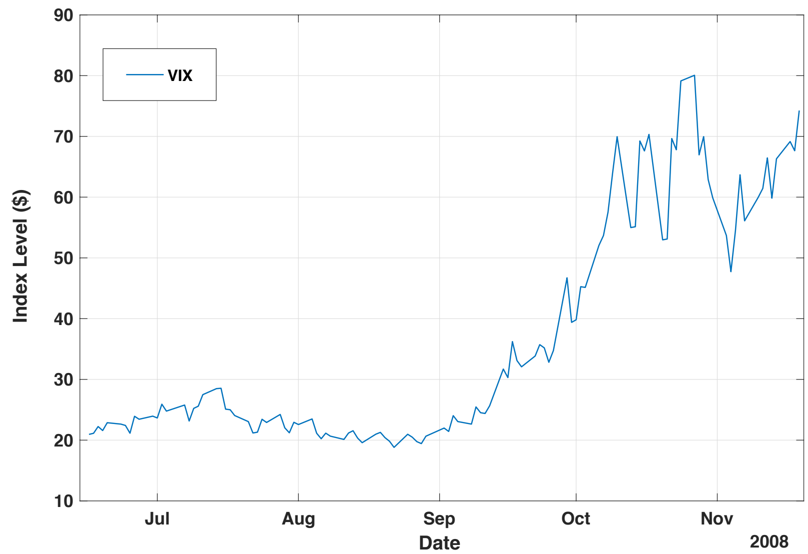

Figure 2: The VIX between June 16, 2008 - November 19, 2008. *Data: Bloomberg*

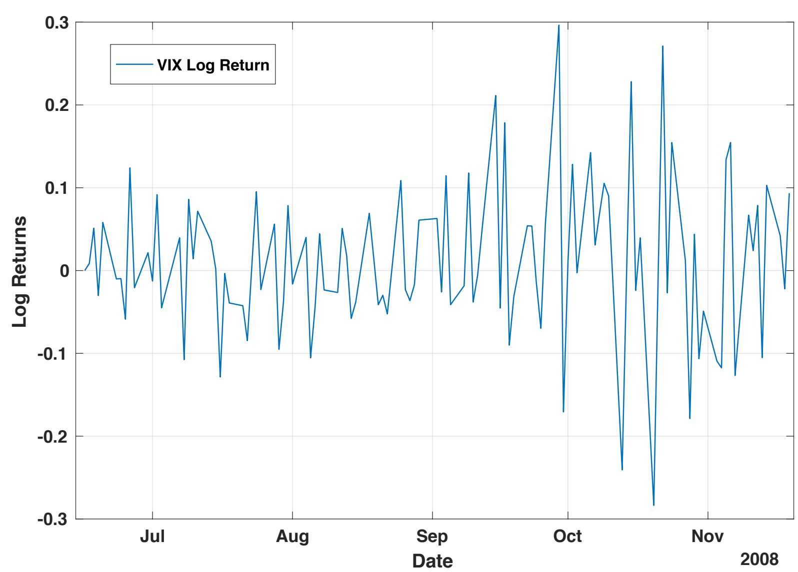

Figure 3: The log returns of VIX spanning June 16, 2008-November 19, 2008.

Table 1: Relationship between the VIX and SPX

(January 1, 2000, - September 28, 2012.)

S&P 500 Up	VIX Index Down	Percent Opposite
1692	1390	82.15%
S&P 500 Down	**VIX Index Up**	**Percent Opposite**
1514	1187	78.40%

Table 2: Descriptive Statistics of the CBOE VIX.

Period	Mean	Standard Deviation	Skewness	Excess Kurtosis
Jun 16 2008 - Nov 19 2008	0.00014433	0.0926517	0.1008	4.3305
Jan 2 1990 - Nov 15 2017	-3.87607e-05	0.0636204	0.6903	7.4532

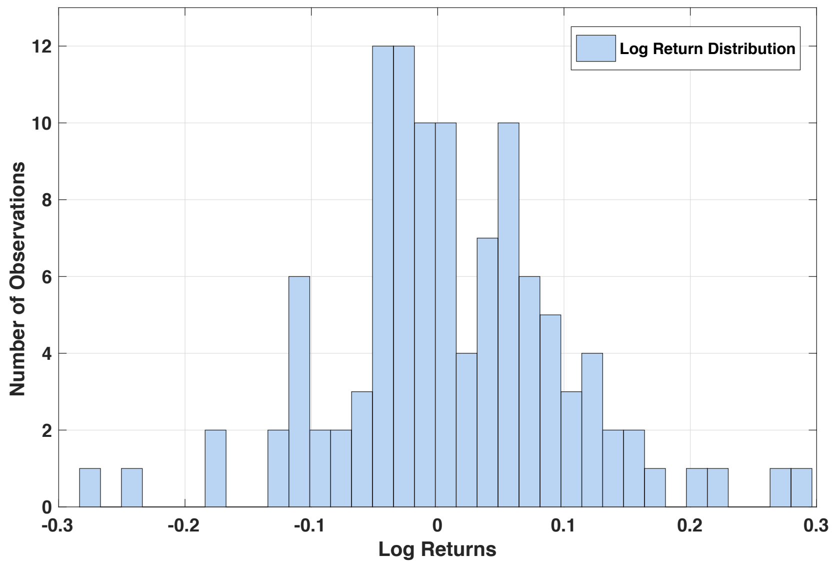

Figure 4: Histogram of VIX log returns for June 16, 2008, - November 19 2008.

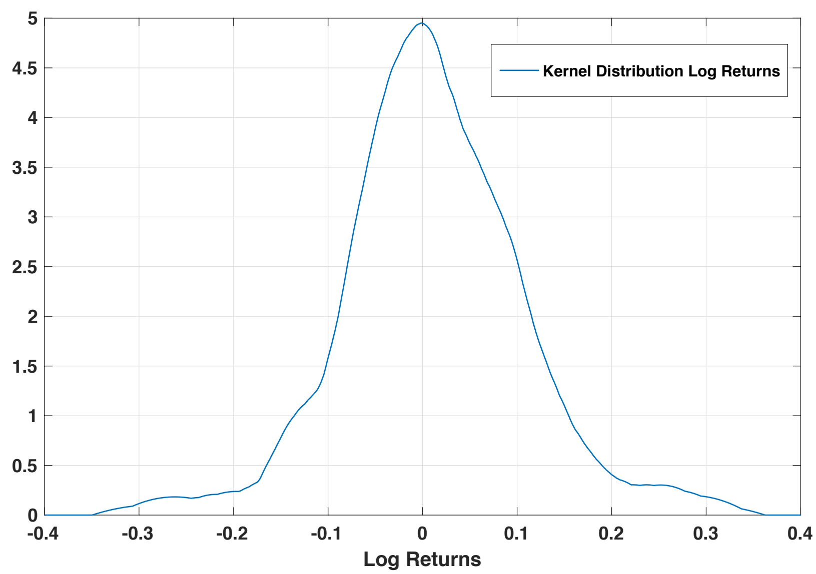

Figure 5: Kernel Estimate of VIX log returns for June 16, 2008, - November 19 2008.

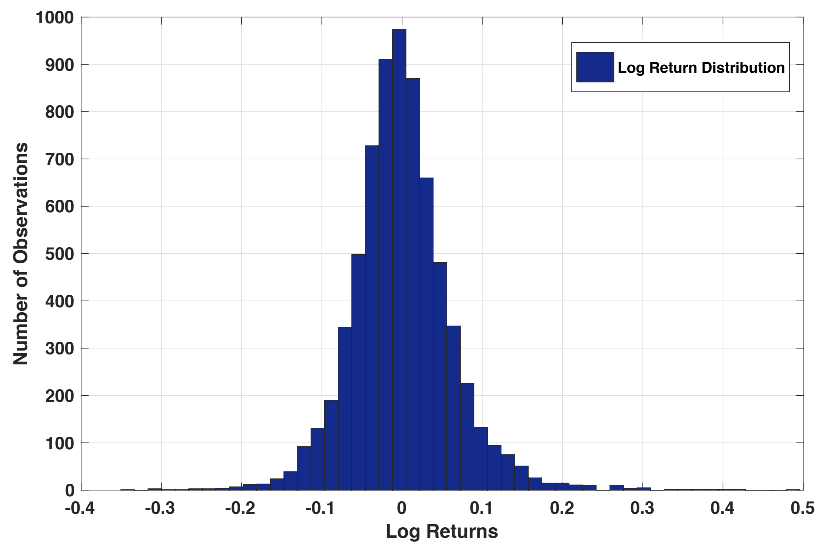

Figure 6: Histogram of VIX log returns for Jan 2, 1990, - Nov 15 2017.

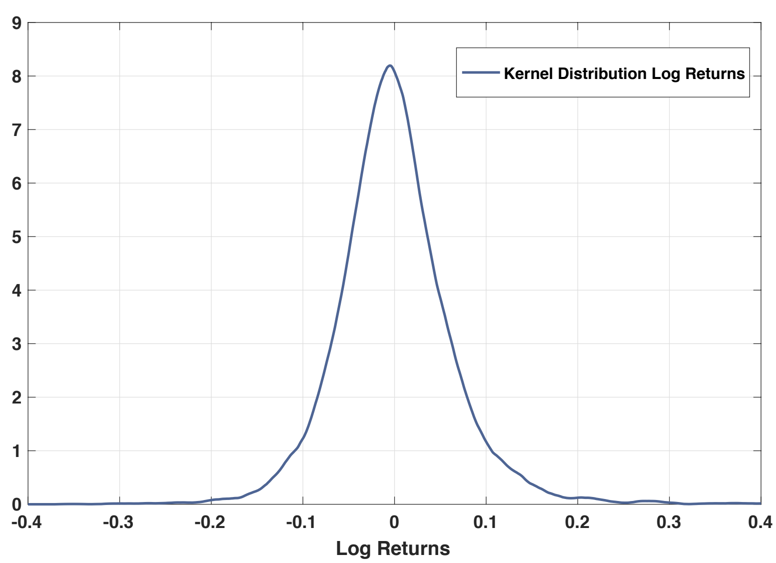

Figure 7: Kernel Estimate of VIX log returns for Jan 2, 1990, - Nov 15 2017.

5 Fourier Based Option Pricing

There are several acceptable approaches to valuing European call options. For example, one can through Monte-Carlo methods simulate a large number of sample paths of the underlying asset. The price of the option is then computed by averaging the sum of the payoffs generated by each sample path. One can also calculate the price by numerically solving partial differential equations through the means of finite difference or finite element methods. However, there exists a more accurate and faster pricing method called the Fast Fourier Transform (FFT). Nevertheless, the method requires that the risk-neutral probability density of the logarithmic stock price is known. Unfortunately, for many pricing processes, the risk-neutral density is unknown. Instead, the Fourier transform of these densities, i.e., their characteristic functions, can in most cases be obtained in closed form. Before going into the details about FFT, a brief, but complete introduction on Fourier transforms will be provided.

5.1 Fourier Transforms

The Fourier transform, $\hat{f}(\omega)$, of the integrable function $f(x) : \mathbb{R} \to \mathbb{C}$ is given by

$$\hat{f}(\omega) = (Ff)(\omega) \int_{-\infty}^{\infty} e^{-i\omega x} f(x) dx \tag{5.1}$$

where $i \in \mathbb{C}$ and $u \in \mathbb{R} \vee \mathbb{C}$.

By inverting (5.1) yield the inverse Fourier transform

$$f(x) = (F^{-1}f)(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} \hat{f}(\omega) d\omega.$$

Also, to ensure the existence of the Fourier transform (5.1), $f(x)$ must be fully integrable, i.e., satisfying

$$\int_{-\infty}^{\infty} \left| f(x) \right| dx < \infty. \tag{5.2}$$

The previous condition is also required for the existence of the inverse transform (5.1). Considering the tools developed in this section, we may move to the next section, which will provide details on the Carr-Madan option pricing approach.

5.2 Carr-Madan Approach (1999)

In the paper, "Option valuation using the Fast Fourier Transform," Peter Carr and Dilip B. Madan (1999) shows how the Fast Fourier transform algorithm can be used to value European options when the characteristic function of the return is known analytically. This method involves generating a Fourier transformation for different values of the underlying asset. This a two step approach, which start with performing a Fourier transformation on the payoff-function with respect to the strike price K. Second, changing the order of the integration enables one to compute the fair price of the option as an inverse Fourier transformation and therefore applying the appropriate characteristic function.

A detailed demonstration of this pricing technique starts with the familiar risk-neutral valuation, which for a European call option with strike price K and maturity T as $C_T(K) = \max[S_T - K, 0]$ satisfies

$$C_T(k) = e^{-rT}\mathbb{E}_{\mathrm{T}}^{\mathbb{Q}}[(e^s - e^k)^+] = e^{-rt}\int_k^{\infty}(e^s - e^k)^+ q(s)ds$$

where $s = \log S_T$, $k = \log K$, and $q(s)$ is the risk-neutral density of S_T. When determining the price of an option, one has to take into consideration the two cases of the option being *In-the-Money* (ITM) or *Out-of-the-Money* (OTM), which for a European call option mean that $k < \log S_0$ or $k > \log S_0$ respectively.

In-the-Money Options

It is obvious that when $k \to -\infty$

$$C_T(k) = e^{-rT}\int_{-\infty}^{\infty} e^s q(s)ds = e^{-rT}\mathbb{E}^{\mathbb{Q}}[e^s] = S_0, \tag{5.3}$$

which is a martingale. Since (5.3) does not converge to zero, the condition (5.2) is unfulfilled and thus $C_T(k) \notin L^1$. To ensure integrability of $C_T(k)$ Carr and Madan introduced a damping factor $e^{\alpha k}$ $(\alpha > 0)$. Consequently, the call price is modified into

$$c_T(k) = e^{\alpha k} C_T(k)$$

and assures that

$$\int_{-\infty}^{\infty} \left| e^{\alpha k} C_T(k) \right| dk < \infty.$$

Assuming that the previous condition is fulfilled, the Fourier transform of a European call at time T is

$$\psi_T(\omega) = \int_{-\infty}^{\infty} e^{i\omega k} c_T(k) dk.$$

Acknowledging the symmetry of the characteristic function, taking the inverse transform will yield the price of the European call as

$$c_T(k) = \frac{e^{-\alpha k}}{\pi} \mathbb{R} \left[\int_0^{\infty} e^{-i\omega k} \psi_T(\omega) d\omega \right]. \tag{5.4}$$

Carr and Madan define the Fourier transform of an ITM European call as

$$\psi_T^{\text{ITM}}(\omega) = \frac{e^{-rT} \phi(\omega - (\alpha + 1)i)}{\alpha^2 + \alpha - \omega^2 + i(2\alpha + 1)\omega} \tag{5.5}$$

where ϕ is the characteristic function $\phi(x) = \mathbb{E}_{\mathrm{T}}^{\mathbb{Q}}[e^{ixs_T}]$ and $s_T = \log S_T$ [1].

Out-of-the-Money Option

To ensure integrability for OTM options Carr and Madan introduced a different damping factor $\sin(\alpha k)$, which provides the Fourier transform

$$\psi_T^{\text{OTM}}(\omega) = \int_{-\infty}^{\infty} e^{i\omega k} \sinh(\alpha k) c_T(k) dk = \frac{\xi(\omega - i\alpha) - \xi(\omega + i\alpha)}{2}$$

where

$$\xi(\omega) = e^{-rT} \left[\frac{1}{1 + i\omega} - \frac{e^{rT}}{i\omega} - \frac{\phi(\omega - i)}{\omega^2 - i\omega} \right] \tag{5.6}$$

which can be derived in a similar fashion as (5.5). By taking the inverse Fourier transform, the value of the European call becomes

$$c_T(k) = \frac{1}{2\pi \sinh(\alpha k)} \int_{-\infty}^{\infty} e^{-i\omega k} \psi_T^{\mathrm{OTM}} d\omega. \tag{5.7}$$

As the intrinsic value of (5.7) is zero the only value embedded in the option will be the time value [1]. The characteristic function for the Merton Jump-Diffusion dynamics is the well-established function

$$\phi^{\mathrm{MJD}}(x) = \exp\left(\left(ix\beta - \frac{x^2\sigma^2}{2} + \lambda(e^{ix\mu_j - x^2\delta^2/2} - 1)\right)T\right) \tag{5.8}$$

where β is the risk neutral drift

$$\beta = r - \frac{\sigma^2}{2} - \lambda\kappa$$

where κ is the same as in (3.6). Hence, valuating the European call essentially becomes a matter of inserting (5.8) into (5.5) or (5.6) and apply the Fast Fourier Transform algorithm.

5.3 Fast Fourier Transform (FFT)

Introduced by Cooley and Tuckey (1965), the Fast Fourier Transform (FFT) is a powerful algorithm to compute sums of the form

$$w(\upsilon) = \sum_{j=1}^{N} e^{-i\frac{2\pi}{N}(j-1)(\upsilon-1)} x(j), \;\; \upsilon = 1, \ldots, N \tag{5.9}$$

in a fast and accurate fashion [4]. Hence, the integral (5.4) can be numerically approximated by the sum

$$C_T(k) \approx \frac{e^{-\alpha k}}{\pi} \mathbb{R}\left[\sum_{j=1}^{N} e^{-i\omega_j k} \Psi(\omega_j)\eta\right], \tag{5.10}$$

where $\omega_j = \eta(j-1)$, $j = 1, \ldots, N$ and $\eta > 0$ denotes the distance between the points in the integration grid. We consider strike prices close to S_0 as those options are usually the most liquid.

Next, we define an array consisting of log-strikes as

$$k_\upsilon = -z + \epsilon(\upsilon - 1) \tag{5.11}$$

where $\upsilon = 1, \ldots, N$ and where $\epsilon > 0$ is considered the spacing parameter for the log-strikes [1]. This spacing will yield N number of log-strikes bounded between $-z$ and z where $z = 0.5\epsilon$. Substituting k for (5.11) in (5.10) yields

$$C_T(k_\upsilon) \approx \frac{e^{-\alpha k_\upsilon}}{\pi} \mathbb{R}\left[\sum_{j=1}^{N} e^{-i\omega_j(-z+\epsilon(\upsilon-1))}\Psi(\omega_j)\eta\right]. \tag{5.12}$$

Recalling that $\omega_j = \omega(j-1)$, (5.12) can be rewritten as

$$C_T(k_\upsilon) \approx \frac{e^{-\alpha k_\upsilon}}{\pi} \mathbb{R}\left[\sum_{j=1}^{N} e^{-i\epsilon\eta(j-1)(\upsilon-1)iz\omega_j}\Psi(\omega_j)\eta\right]. \tag{5.13}$$

Provided that $\epsilon\eta = \frac{2\pi}{N}$, (5.13) is of the form (5.9), the FFT can be applied. Nonetheless, the limitation around the term $\epsilon\eta$ results in a trade-off. A small η will provide a fine grid to integrate over, but will contribute with strike prices further away from the S_0. This issue was solved by using weightings based on Simpson's rule. The approximated call price (5.13) takes on the form

$$C_T(k_\upsilon) \approx \frac{e^{-\alpha k_\upsilon}}{\pi} \mathbb{R}\left[\sum_{j=1}^{N} e^{-i\epsilon\eta(j-1)(\upsilon-1)iz\omega_j}\Psi(\omega_j)\frac{\eta}{3}\big(3 + (-1)^j - \delta_{j-1}\big)\right]$$

where δ_n is the Kroenecker delta function, which

$$\begin{cases} \delta_n = 1 & \text{if } n = 0 \\ \delta_n = 0 & \text{if } n \neq 0. \end{cases}$$

This modification allows us to maintain a fine grid while increasing the size of η [4].

5.4 Model Calibration

Given the options in table 8, 9, and 10, the chosen subset of options was based on strike prices ±10% around the current VIX index level $S_0 = 20.95$ as we found them to be the most liquid ones. The objective of the model calibration is to find model parameters of the Merton Jump-Diffusion model such that the chosen VIX call option quotes are imitated as closely as possible. The calibration was achieved by applying a Root Mean Square Error (RMSE) function of the following form

$$\min_{\sigma,\lambda,\mu_j,\delta} \sqrt{\frac{1}{N}\sum_{n=1}^{N}\big(C_n - \hat{C}_n(\sigma,\lambda,\mu_j,\delta)\big)^2}$$

where C_n is the observed market quotes and $\hat{C}_n$ the MJD model prices. As there might exist multiple local minima, the minimization procedure started with a brute force method, which required an arbitrarily initial guess $\mathbf{x}_0$ for each parameter. It uses a global minimization method, which roughly scans the multidimensional error grid to find a local and more promising area. Subsequently, the yielded result is used as an initial guess for a minimization method based on a simplex algorithm, which scans the suggested local area more thoroughly.

5.5 The Risk-Free Interest Rate

Regarding the risk-free interest rate, the convention for pricing CBOE VIX options is U.S. Treasury bills. The calibration is performed for three maturities and therefore interest rates corresponding to these periods are required. The U.S. Treasury bill rates have been collected from the United States Treasury's website [11]. As the date of these rates did not match the expiration date of the options, a linear interpolation on these rates was necessary. The interpolated rates corresponding to the three maturities are displayed in the table below.

Table 3: U.S. Treasury Rates and Interpolated Rates

Treasury Rates	**4 Weeks (28 days)**	**13 Weeks (91 days)**	**26 Weeks (182 days)**
2008-06-16	1.88%	2.08%	2.42%
Interpolated Rates	**30 days**	**93 days**	**157 days**
2008-06-16	1.8863%	2.0875%	2.3266%

6 Results and Discussion

Judging by the results listed in table 4, the Merton Jump-Diffusion model is far from perfect as it yields prices that are inconsistent with the observed market prices. However, the results suggest that the Merton Jump-Diffusion model improves for longer maturities, while its predecessor, the Black-Scholes model deteriorates at the same maturities. Illustrations of these results are displayed in figure 9. Following the news surrounding the insolvency of the insurance giant American International Group, Inc, and the bankruptcy of the investment bank, Lehman Brothers on September 15, 2008, there was a dramatic increase in market volatility and jumps, which explains why the MJD model dominates the Black-Scholes model during such a market climate. The latter might also explain why the Merton Jump-Diffusion model is closer to replicate the option prices for options with maturity date September 17, 2008. Nevertheless, when markets are smooth, and jumps are absent, the MJD model will price closer to the Black-Scholes model. In other words, the residual of the market price and MJD price should always be equal or greater than the residual of the market price and the Black-Scholes price.

Even though the MJD model seems to have better captured the features of a turbulent market environment, the extended jump parameter has not been enough to capture the real volatility surface as the volatility smile changes with strike price and maturity. Being a simple modification of the original Black-Scholes, the MJD offers a higher degree of freedom (i.e. more parameters) that makes its calibration less challenging than in the BSM case. However, in a survey conducted by Peter Tankov and Ekaterina Voltchkova (2009), Tankov and Voltchkova claim that the inclusion of a jump component to the traditional BSM is insufficient to accommodate for different maturities and strike levels [10]. The previous statement suggests that the real world financial markets are far more complex than the MJD market dynamics.

Summaries of the parameter estimations are displayed in tables 6 and 7. The poor results are likely a consequence of the unrealistic assumption of constant volatility in the Merton Jump-Diffusion model. Also, important to keep in mind is that the MJD and the BSM model are being tested at exceptional circumstances. Either way, in the paper "Implied Calibration and Moments Asymptotics in Stochastic Volatility Jump-Diffusion Models," Stefano Galluccio and Yann Le Cam (2008), p. 9, argues that either a pure jump or stochastic volatility in isolation cannot fit the volatility smile because of the simultaneous presence of both jump and stochastic volatility in the market [5].

Today, there exist models which can cope with several sources of risk such as, risk in the underlying asset, volatility risk, jump risk, and interest rate risk, which all affects the price of equity derivatives. The difficult task is to find ways that are both efficient and accurate regarding time and valuation. Despite the positive development seen in the past 40 years in option theory, the quest in finding the "perfect" model continuous.

Table 4: Calibration Result

Date	**Maturity**	**Trade Vol.**	**Rate (%)**	**Strike**	**Market Price**	**MJD Price**	**BS Price**
2008-06-16	2008-07-16	70	1.8863	19.0	4.10	3.20	2.07
2008-06-16	2008-07-16	638	1.8863	20.0	3.46	2.63	1.08
2008-06-16	2008-07-16	7280	1.8863	22.5	2.35	1.52	0
2008-06-16	2008-09-17	25	2.0875	19.0	5	4.76	2.33
2008-06-16	2008-09-17	100	2.0875	20.0	4.40	4.27	1.35
2008-06-16	2008-09-17	108	2.0875	22.5	3	3.23	0
2008-06-16	2008-11-19	2	2.3266	19.0	5.40	5.84	2.59
2008-06-16	2008-11-19	290	2.3266	20.0	4.66	5.40	1.62
2008-06-16	2008-11-19	10	2.3266	22.0	3.60	4.60	0

Table 5: Error Result

Date	Maturity	Trade Vol.	Market Price	Diff: MJD/Market	Diff: BSM/Market
2008-06-16	2008-07-16	70	4.10	-0.90	-2.03
2008-06-16	2008-07-16	638	3.46	-0.83	-2.38
2008-06-16	2008-07-16	7280	2.35	-0.83	-2.35
2008-06-16	2008-09-17	25	5	-0.24	-2.67
2008-06-16	2008-09-17	100	4.40	-0.13	-3.05
2008-06-16	2008-09-17	108	3	0.23	-3
2008-06-16	2008-11-19	2	5.40	0.44	-2.81
2008-06-16	2008-11-19	290	4.66	0.74	-3.04
2008-06-16	2008-11-19	10	3.60	1	-3.60

Table 6: Parameter Summary Merton's Jump-Diffusion.

$\hat{\sigma}$	$\hat{\lambda}$	$\hat{\mu}$	$\hat{\delta}$	**RMSE**
0.817	4.434	-0.157	0	0.639

Table 7: Parameter Summary Black-Scholes.

$\hat{\sigma}$	**RMSE**
0.004	2.804

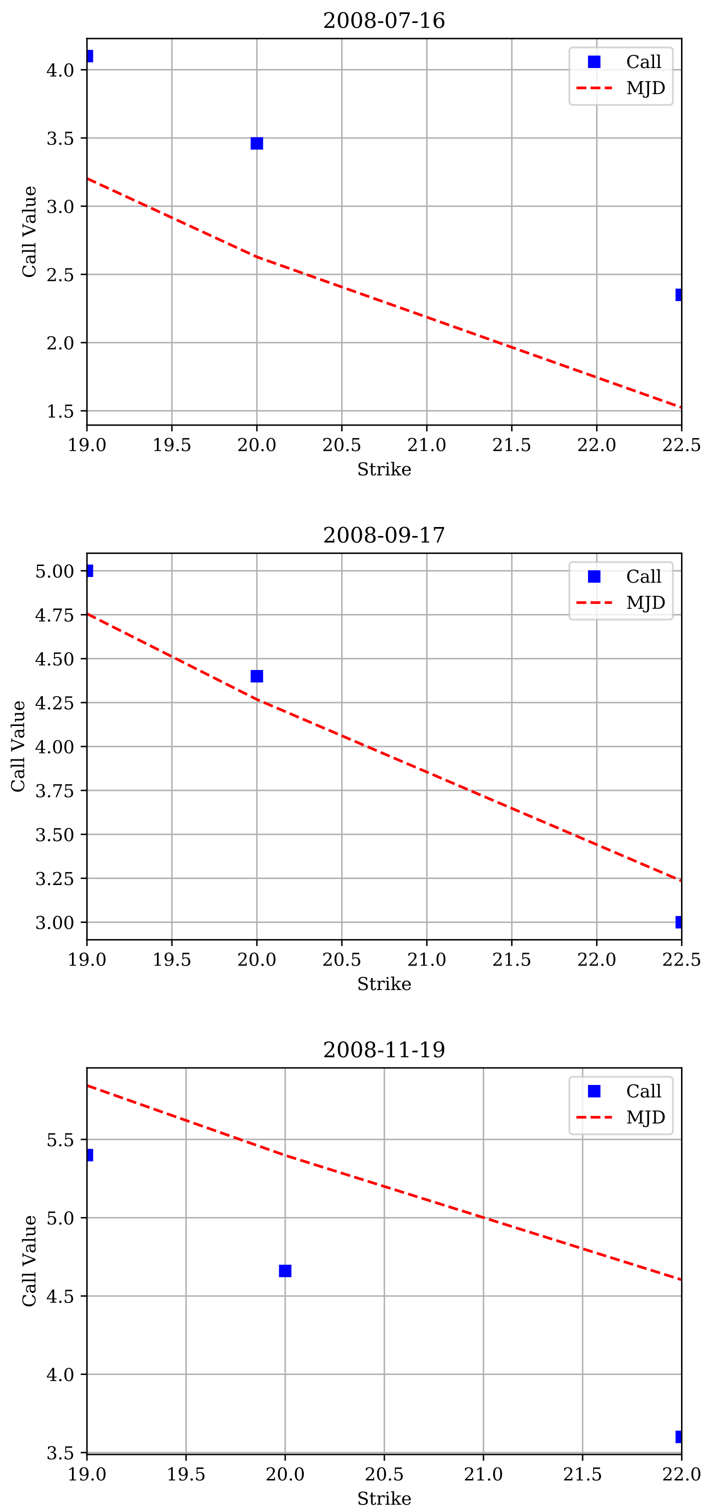

Figure 8: Results of the 3 maturity calibration of the MJD model.

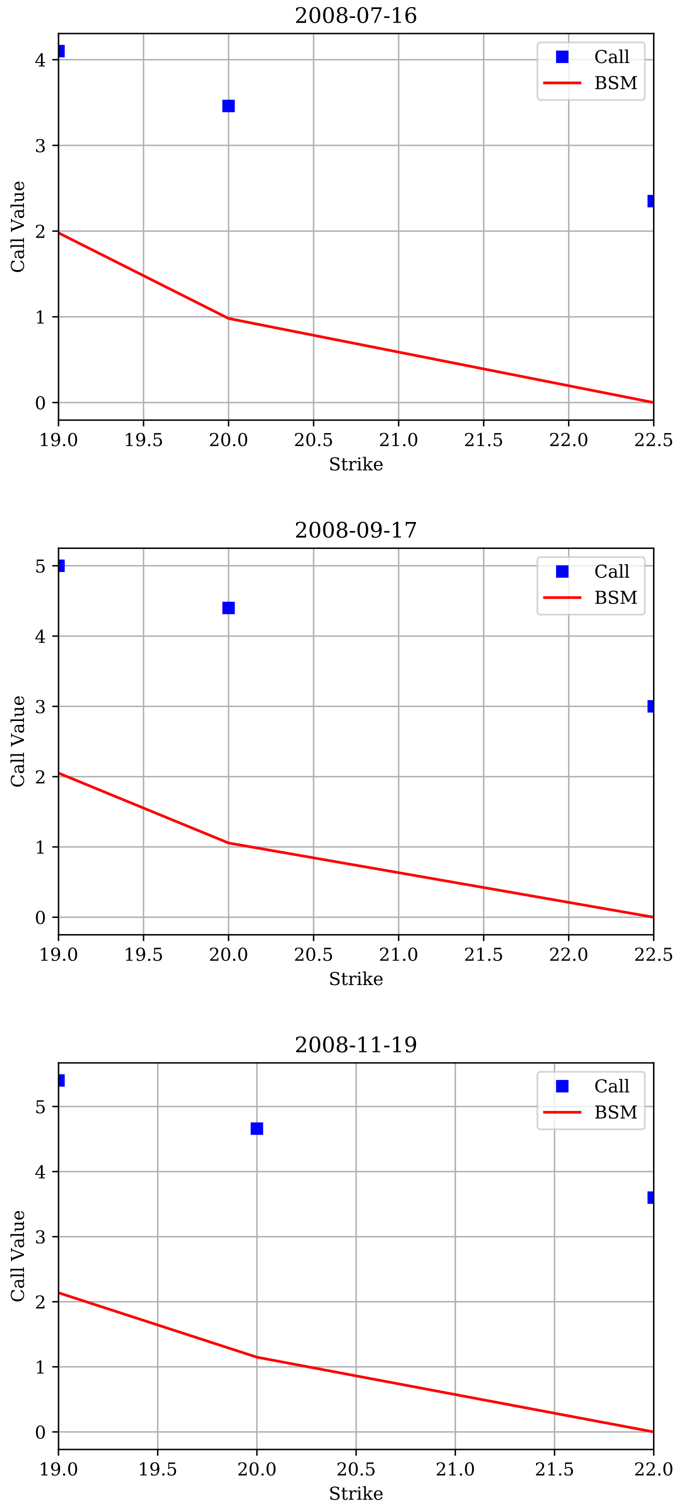

Figure 9: Results of the 3 maturity calibration of the BSM model.

Table 8: VIX Option Quotes for 1 Month Maturity.

Date	**Maturity**	**Strike**	**Call**	**Trade Volume**
2008-06-16	2008-07-16	10	12.60	50
2008-06-16	2008-07-16	15	7.80	154
2008-06-16	2008-07-16	17	0	0
2008-06-16	2008-07-16	18	5.10	80
2008-06-16	2008-07-16	19	4.10	70
2008-06-16	2008-07-16	20	3.46	638
2008-06-16	2008-07-16	22.50	2.35	7280
2008-06-16	2008-07-16	25	1.40	14669
2008-06-16	2008-07-16	27.50	0.85	12603
2008-06-16	2008-07-16	30	0.55	6847
2008-06-16	2008-07-16	32.50	0.35	21311
2008-06-16	2008-07-16	35	0.25	242
2008-06-16	2008-07-16	37.50	0.18	167
2008-06-16	2008-07-16	40	0	0
2008-06-16	2008-07-16	42.50	0	0
2008-06-16	2008-07-16	45	0.05	151
2008-06-16	2008-07-16	50	0	0

Table 9: VIX Option Quotes for 3 Month Maturity.

Date	**Maturity**	**Strike**	**Call**	**Trade Volume**
2008-06-16	2008-09-17	10	0	0
2008-06-16	2008-09-17	15	8.80	2
2008-06-16	2008-09-17	17	0	0
2008-06-16	2008-09-17	18	0	0
2008-06-16	2008-09-17	19	5	25
2008-06-16	2008-09-17	20	4.40	100
2008-06-16	2008-09-17	22.50	3	108
2008-06-16	2008-09-17	25	0	0
2008-0616	2008-09-17	27.50	1.50	210
2008-06-16	2008-09-17	30	1	1120
2008-06-16	2008-09-17	32.50	0	0
2008-06-16	2008-09-17	35	0	0
2008-06-16	2008-09-17	37.50	0	0
2008-06-16	2008-09-17	40	0	0
2008-06-16	2008-09-17	42.50	0	0
2008-06-16	2008-09-17	45	0	0
2008-06-16	2008-09-17	50	0	0

Table 10: VIX Option Quotes for 5 Month Maturity.

Date	Maturity	Strike	Call	Trade Volume
2008-06-16	2008-11-19	10	0	0
2008-06-16	2008-11-19	15	8.80	3
2008-06-16	2008-11-19	17	0	0
2008-06-16	2008-11-19	18	0	0
2008-06-16	2008-11-19	19	5.40	2
2008-06-16	2008-11-19	20	4.66	290
2008-06-16	2008-11-19	22	3.60	10
2008-06-16	2008-11-19	24	2.55	10
2008-06-16	2008-11-19	26	0	0
2008-06-16	2008-11-19	28	0	0
2008-06-16	2008-11-19	30	1.30	613
2008-06-16	2008-11-19	32.50	0.90	3
2008-06-16	2008-11-19	35	0.80	20
2008-06-16	2008-11-19	37.50	0	0
2008-06-16	2008-11-19	40	0	0
2008-06-16	2008-11-19	42.50	0	0
2008-06-16	2008-11-19	45	0	0
2008-06-16	2008-11-19	50	0.10	100

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