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Random measures, stochastic integration and infinitely divisible processes

Master's Thesis in Engineering Mathematics and Computational Science

Olof Elias

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CHALMERS UNIVERSITY OF TECHNOLOGY
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OLOF ELIAS

Department of Mathematics
Chalmers University of Technology
SE-412 96 Göteborg
Sweden
Telephone +46 (0) 31-772 1000

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Abstract

In this thesis we study the spectral representation of infinitely divisible processes. We give a description of Musielak-Orlicz spaces relating to certain infinitely divisible independently scattered random measures. We propose a new method for simulating infinitely divisible integral processes that holds arbitrarily well in probability under general conditions.

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Introduction

THE preliminary aim with this thesis was to study the area known as representations of stochastic processes. This is a thoroughly studied field and classical results such as the Karhunen-Loéve representation or the spectral representation of weakly stationary processes were developed in the 1960s. The most recent results deal with infinitely divisible processes, the main topic of this thesis.

The advantage with representing stochastic processes in this manner is that one can "remove" the randomness of the process by identifying it with a suitable linear space. In the case of weakly stationary processes this space is a Hilbert space, which makes them very suitable for various applications. The linear structure of an infinitely divisible process is far more intricate, but it will nevertheless still constitute a complete metric space.

The thesis is divided into three parts. The first part is an introduction to the basic theory used in this thesis. In the second part we discuss the spectral representation of infinitely divisible processes in detail, and we will emphasize the tight connection between a homogeneous random measure and its control measure. We also give a full description of linear spaces relating to certain random measures. The last part is devoted to approximations of stochastic integral processes using natural assumptions in order to achieve appropriate simulations.

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Preliminaries

In this chapter we introduce the basic theory required for the thesis including some notations that will be used throughout the thesis. They will consist mainly of basic topological notions, more intricate probability theory such as concepts of infinite divisibility and random measures, and functional analysis relating to the theory of Musielak-Orlicz spaces.

2.1 Notation

- Natural numbers $\mathbb{N}$, Integers $\mathbb{Z}$, Rationals $\mathbb{Q}$, Reals $\mathbb{R}$, Complex $\mathbb{C}$.
- The Lebesgue measure is denoted as m.
- The law of a random variable X is denoted as $\mathscr{L}(X)$.
- $\mathbb{R}_+ = [0,\infty), \overline{\mathbb{R}} = [-\infty,\infty], \overline{\mathbb{R}}_+ = [0,\infty]$.
- $a \vee b = \max(a,b), a \wedge b = \min(a,b)$

2.2 Basic probabilistic concepts

In this section we introduce basic concepts and notations, so that the following exposition will be simpler to understand. Throughout the thesis $(\Omega, \mathscr{F}, \mathbb{P})$ will always mean a probability space, where $\mathscr{F}$ is a $\sigma-$algebra and $\mathbb{P}$ is a probability measure. The space $L^p(\Omega, \mathscr{F}, \mathbb{P})$ is the space of all random variables with finite p-th moment, and when it is clear what Ω, $\mathscr{F}$ or $\mathbb{P}$ is they will be omitted in the notation and instead $L^p(\Omega, \mathscr{F}, \mathbb{P})$ will be written as L^p.

In probability theory different notions of convergence exists. Let $\{X_n\} \subset L^p(\Omega, \mathscr{F}, \mathbb{P})$, with $p = 0$ be understood as random variables and we say that X_n converges to X *almost surely* with respect to the probability measure $\mathbb{P}$ if $\mathbb{P}\{\omega \in \Omega : X_n(\omega) \to X\} = 1$, denoted $X_n \to$

$X, \mathbb{P}-$a.s. Moreover a sequence converges in L^p if $\mathbb{E}|X_n - X|^p = 0$ and is denoted $X_n \to_{L^p} X$. There are also weaker notions of convergence, *convergence in probability*, denoted $\to_{\mathbb{P}}$ is defined as $\mathbb{P}\{|X_n - X| > \epsilon\} \to 0, \forall \epsilon > 0$ and *weak convergence* or *convergence in distribution* which is denoted $\xrightarrow{d}$. This means that the distribution functions converge. Indeed, there is a hierarchy between these notions of convergence. The convergence almost surely and in L^p are the strongest and neither imply the other. The weakest is convergence in distribution.

By a *stochastic process*, ξ, on a probability space we mean a family of random variables over some index set T, which in most cases can be understood as a time domain, i.e

$$\xi = \{\xi_t : t \in T\}. \tag{2.1}$$

By a $p-$th order process we mean that the family of random variables are in L^p and by a *separable* process we mean that there exists a countable subset $T_0 = T$ such that $X_{t'} \to_{L^p} X_t$ for all $t \in T$ and $t' \in T_0$, and if $p = 0$ the convergence is understood as convergence in probability. We define the convolution between two measures as

$$\mu * \nu(A) = \int \mathbb{1}_A(x+y)\mu(\,\mathrm{d}x)\nu(\,\mathrm{d}y).$$

A very important concept in more advanced probability theory is the concept of infinite divisibility, abbreviated **ID** . This means that if a random variable ξ is infinitely divisible then

$$\xi \overset{\mathrm{d}}{=} \eta_{1,n} + \eta_{2,n} + \ldots + \eta_{n,n}, \; \forall n \in \mathbb{N}$$

for some finite sequence of *i.i.d.* random variables η. In terms of probability measures this means that a probability measure P is infinitely divisible if it is the n-fold convolution of some probability measure P_n :

$$P = *^n P_n, \; *^n P_n = (*^{n-1} P_n) * P_n$$

so that the Fourier transform, which in probabilistic terms is called the characteristic function, of P satisfies

$$\hat{P}(z) = \int \mathrm{e}^{\mathrm{i}tz} P(\,\mathrm{d}t) = \left(\hat{P}_n(z)\right)^n.$$

Whenever a measure, ν on $\mathbb{R}^d$ satisfies

$$\nu(\{0\}) = 0, \int_{\mathbb{R}^d} |x|^2 \wedge 1 \; \nu(\,\mathrm{d}x) < \infty$$

we call this measure a *Lévy* measure. Now, a remarkable fact, when it comes to infinitely divisible random variables and distributions, is that they can be completely characterized by their characteristic function which is known as the *Lévy-Khintchine representation* and can be found in [14].

Theorem 2.1. *If P is an infinitely divisible probability measure on $\mathbb{R}^d$, then:*

1. $$\hat{P}(z) = \exp\left[\mathrm{i}\langle \mu, z\rangle - \frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle \tau(x)\nu(\,\mathrm{d}x) \right] \tag{2.2}$$

   where A is a symmetric positive semi-definite $d \times d$ matrix, ν is a Lévy measure and τ is a bounded measurable function satisfying

   $$\tau(x) = 1 + o(|x|), |x| \to 0,\ \tau(x) = O(1/|x|), |x| \to \infty.$$

2. (2.2) *is unique.*

3. *Conversely, if A is a symmetric positive semi-definite $d \times d$—matrix, $\mu \in \mathbb{R}^d$, ν is a Lévy-measure and τ satisfy the conditions above then there exists an infinitely divisible probability measure with the characteristic function above.*

If X is an infinitely divisible random variable, i.e $\mathscr{L}(X)$ is infinitely divisible, we write, $X \in \mathbf{ID}(\mu, A, \nu)$.
Remark. The integral with respect to Lévy measure can be written as

$$\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}\langle u,x\rangle} - 1 - \mathrm{i}\langle u, \tau(x)\rangle \, \nu(\,\mathrm{d}x)$$

if we take

$$\tau(x) = [[x]] \stackrel{def}{=} \left(\frac{x_1}{|x_1| \vee 1}, \frac{x_2}{|x_2| \vee 1}, \ldots, \frac{x_d}{|x_d| \vee 1}\right), \tag{2.3}$$

or some other function that satisfy similar conditions. This function will be of use when dealing with **ID** processes.

The class of **ID** random variables is large, to list a few it contains stable random variables, the exponential random variables, gamma distributed random variables and many more, see [12] for a more extensive list.

2.3 Orlicz and Musielak-Orlicz spaces

The Orlicz and Musielak-Orlicz spaces are certain complete metric spaces that arise in different applications. In this thesis they will appear when we discuss the subject of infinitely divisible processes. A function $\Phi : \mathbb{R} \to \mathbb{R}_+$ is a *Young function* if it is even, unbounded and zero at the origin, i.e $\Phi(x) = \Phi(-x)$, $\lim_{x\to\infty} \Phi(x) = \infty$, $\Phi(0) = 0$. In addition, Φ is called a *nice Young function* if $\Phi(x) = 0 \Leftrightarrow x = 0, \lim_{x\to 0} \Phi(x)/x = 0, \lim_{x\to\infty} \Phi(x)/x = \infty$ while $\Phi(\mathbb{R}) \subset \mathbb{R}_+$.

A Young function is said to satisfy the Δ_2 condition, $\Phi \in \Delta_2$ if

$$\Phi(2x) \le K\Phi(x),\ x > x_0 \ge 0,\ K > 0. \tag{2.4}$$

If $x_0 = 0$ then the condition is said to hold globally.

Let $(X, \mathcal{M}, \mu)$ be an arbitrary measure space, and $f : X \to \mathbb{R}$ be a measurable function. If $\mu(X) = \infty$ and Φ is a Young function that satisfy (2.4) then the space

$$L_\Phi = \left\{ f : X \to \mathbb{R} : \int_X \Phi(f(x))\mu(\,\mathrm{d}x) < \infty \right\}, \tag{2.5}$$

is a linear space, according to [10]. On this space we can introduce a norm that makes it a complete normed space if we identify functions as identical $\mu -$a.e,

$$\|f\|_\Phi = \inf \left\{ c > 0 : \int_X \Phi(\frac{f(x)}{c})\mu(\,\mathrm{d}x) \leq 1 \right\}. \tag{2.6}$$

The Musielak-Orlicz spaces are generalized Orlicz spaces introduced by Julian Musielak in his book Orlicz spaces and Modular spaces [8], and relies heavily on the concept of modulars. These are a type of functionals on a vector space that satisfy conditions that are similar to norms.

Definition. Let V be a real or complex vector space. A functional, $\rho : V \to [0,\infty]$ is either a *pseudomodular*, *semimodular*, or a *modular* if for any $x,y \in V$ the following holds

- $\rho(0_V) = 0$, $\rho(\lambda x) = 0,\ \forall \lambda > 0 \Rightarrow x = 0_V,\ \rho(x) = 0 \Rightarrow x = 0_V$, respectively
- $\rho(x) = \rho(-x)$ if V is real and $\rho(\mathrm{e}^{it}x) = \rho(x), \forall t \in \mathbb{R}$ if V is complex
- $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y), \forall x,y \in V, \alpha + \beta = 1$ if $\rho(\alpha x + \beta y) \leq \alpha^s \rho(x) + \beta^s \rho(y)$ then it is said to be a s-convex pseudo-,semi- or modular.

Indeed, any norm is a modular but the converse is not necessarily true as a modular can take infinite values. An example of a p-convex modular is for instance

$$\int_{\mathbb{R}} |f(x)|^p \,\mathrm{d}x, p > 0$$

An F-norm, denoted $\|\cdot\|$ on X is a functional that satisfy

- $\|x\| = 0 \Longleftrightarrow x = 0$,
- $\|x\| = \|-x\|$ or $\|\mathrm{e}^{it}x\| = \|x\|, \forall t \in \mathbb{R}, \forall x \in X$,
- $\|x + y\| \leq \|x\| + \|y\|, \forall x,y \in X$,
- If $\alpha_k \to \alpha$ and $\|x_k - x\| \to 0$ then $\|\alpha_k x_k - \alpha x\| \to 0$.

For every semimodular,ρ, and hence for every modular, we can define a F-norm

$$\|x\|_\rho = \inf \left\{ c > 0 : \rho\left(\frac{x}{c}\right) \leq c \right\} \tag{2.7}$$

that has the following properties.

Lemma 2.1. $\|x\|_\rho$ *satisfy the following:*

- $\rho(\lambda x_1) \leq \rho(\lambda x_2),\ \forall \lambda > 0\ \Rightarrow \|x_1\|_\rho \leq \|x_2\|_\rho$,
- $\|\alpha x\|_\rho$ *is a non-decreasing function of* $\alpha > 0$,
- $\|x\|_\rho < 1 \Rightarrow \rho(x) \leq \|x\|_\rho$.

The difference between a Orlicz space and a Musielak-Orlicz space is subtle. Rather than considering a Young function we consider a function

$$\Psi(x,t) : X \times \mathbb{R}_+ \to \mathbb{R}_+. \tag{2.8}$$

This function satisfy conditions similar to Young functions for fixed x. We let Ψ be an increasing, continuous function of t such that $\Psi(x,0) = 0$, $\Psi(x,t) > 0, t > 0$, $\lim_{t\to\infty} \Psi(x,t) = \infty$ for almost all $x \in X$. For fixed $t \in \mathbb{R}_+$ we let Ψ be measurable with respect to $\mathscr{M}$. Then the functional defined by

$$\rho(f) = \int_X \Psi(x,|f(x)|)\mu(\,\mathrm{d}x) \tag{2.9}$$

is a modular. We introduce the Musielak-Orlicz space, denoted $L_\Psi(X,\mathscr{M},\mu)$, as the functions that satisfy

$$\int_X \Psi(x,|f(x)|)\mu(\,\mathrm{d}x) < \infty \tag{2.10}$$

equipped with the F-norm

$$\|f\|_\Psi = \inf\left\{c > 0 : \int_X \Psi(x,|f(x)|/c) \leq c\right\} \tag{2.11}$$

An important property of Orlicz and Musielak-Orlicz spaces is that they are, under reasonable assumptions, linear complete metric spaces with the F-norms defined above. Another useful property is that the simple functions are dense.

Indeed, we can have different notions of convergence in the Musielak-Orlicz spaces, the two most central are modular and normed convergence. In general the two topologies do not coincide and the following proposition gives a precise condition of when this happens. This property will be of use later.

Proposition 2.1. *Modular and normed convergence are equivalent if and only if*

$$\{x_k\}_{k\in\mathbb{N}} \subset V, \rho(x_k) \to 0 \Rightarrow \rho(2x_k) \to 0 \tag{2.12}$$

2.4 Stable processes

In this section we will introduce several important classes of stochastic processes. These will be characterized either in the sense of their characteristic function or its finite dimensional distributions. The entire section is an excerpt from [13].

First we introduce the simplest concept, symmetric random variables. A random variable,X, is *symmetric* if

$$X \stackrel{\mathrm{d}}{=} -X.$$

Note that this implies that the characteristic function must be real-valued.

The name *stable* comes from the most elementary definition of stable random variables. A random variable is stable if for any $A,B > 0$ there exists a $D \in \mathbb{R}$ such that

$$AX_1 + BX_2 \stackrel{\mathrm{d}}{=} CX + D \tag{2.13}$$

where X_1,X_2 are independent copies of X. If $D = 0$ we call X a strictly stable random variable.

This definition is not suitable for our purpose, and we will use the following equivalent definition.

Definition. To require that a random variable, X, has a *stable distribution* we ask that there exists parameters $0 < \alpha \leq 2$, $\sigma \geq 0$, $|\beta| \leq 1, \mu \in \mathbb{R}$ such that its characteristic function is of the form:

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}tX}] = \begin{cases} \exp\left\{-\sigma^{\alpha}|t|^{\alpha}\left(1 - \mathrm{i}\beta\,\mathrm{sgn}(t)\tan\frac{\pi\alpha}{2}\right) + \mathrm{i}\mu t\right\}, & \alpha \neq 1 \\ \exp\left\{-\sigma|t|\left(1 + \mathrm{i}\beta\frac{2}{\pi}\,\mathrm{sgn}(t)\log|t|\right) + \mathrm{i}\mu t\right\}, & \alpha = 1. \end{cases} \tag{2.14}$$

We write this as $X \in S_{\alpha}(\sigma,\beta,\mu)$

Remark. An alternative definition of stable random variables, which is also useful when it comes to dealing with stable random vectors, is that a random variable is stable if there exists a sequence of *i.i.d.*random variables $\{Y_i\}$, positive numbers $\{d_n\}$ and real numbers $\{a_n\}$ such that

$$\frac{Y_1 + Y_2 + \ldots + Y_n}{d_n} + a_n \stackrel{d}{\rightarrow} X.$$

Remark. The parameters, σ, β and μ are unique. The parameter α is commonly called the *index of stability* whereas μ is the *shift parameter*, β the *skewness parameter* and σ is the *scale parameter*. The reason for the names is due the following result that can be found in Samorodnitsky and Taqqu [13].

Lemma 2.2. *Let $X_i \in S_{\alpha}(\sigma_i, \beta_i, \mu_i), i = 1,2$ then:*

- $X_1 + X_2 \in S_{\alpha}(\sigma, \beta, \mu)$, $\sigma = (\sigma_1^{\alpha} + \sigma_2)^{1/\alpha}, \beta = \frac{\beta_1\sigma_1^{\alpha} + \beta_2\sigma_2^{\alpha}}{\sigma_1^{\alpha} + \sigma_2^{\alpha}}, \mu = \mu_1 + \mu_2$
- $X_i + a \in S_{\alpha}(\sigma_i, \beta_i, \mu_i + a)$
- $aX_i \in S_{\alpha}(|a|\sigma_i, \mathrm{sgn}(a)\beta_i, a\mu_i)$, *if* $\alpha \neq 1$ *and* $aX_i \in S_1(|a|\sigma_i, \mathrm{sgn}(a)\beta_i, a\mu_i - \frac{2}{\pi}\log|a|\sigma\beta)$, *if* $\alpha = 1$
- $X \in S_{\alpha}(\sigma, \beta, \mu)$ *is symmetric about* μ *if and only if* $\beta = 0$

Now what this lemma yields is that we can characterize the symmetric α-stable random variables, denoted $S\alpha S$.

A random variable, X, is $S\alpha S$ if

$$\mathbb{E}e^{itX} = e^{-\sigma^\alpha |t|^\alpha}.$$

For a random vector to be stable the description of the characteristic function is too complicated to be useful but we may define it as in the remark above about the definition of stable random variables.

We call a stochastic process $X = \{X_t : t \in T\}$ stable or $S\alpha S$ stable if for all $t_1, t_2, ..., t_k \in T, a_1, a_2, ..., a_k \in \mathbb{R}$ the linear combination

$$\sum_1^k a_n X_{t_n} \tag{2.15}$$

is stable or $S\alpha S$ stable.

2.5 Infinitely divisible processes

The most inclusive class of stochastic processes is the infinitely divisible processes. It contains all stable as well as all Lévy processes.

Definition. A stochatic process $L = \{L_t : t \in T\}$ is a Lévy process if:

1. The process has independent increments, i.e for any finite strictly increasing sequence $t_1, t_2, ..., t_n$ the random variables $L_{t_0}, L_{t_1} - L_{t_0}, ..., L_{t_n} - L_{t_{n-1}}$ are independent
2. $X_0 = 0$ a.s
3. The process has stationary increments, $L_t - L_s \overset{\mathrm{d}}{=} L_{t+h} - L_{s+h}$, this property is sometimes called temporal homogeneity,
4. The process is continuous is probability, i.e $\mathbb{P}(|L_t - L_s| > \epsilon) \to 0, t \to s$,
5. The sample paths of the process are cádlág a.s.

Indeed a Lévy process is an infinitely divisible process and its characteristic function satisfy

$$\mathbb{E}e^{i\theta L_t} = \mathbb{E}e^{i\theta t L_1}$$

The definition of a infinitely divisible stochastic process is according to Maruyama [7]. A stochastic process $X = \{X_t : t \in T\}$ is an *infinitely divisible process* if $t_1, t_2, ..., t_k \in T$ the random vector

$$(X_{t_1}, X_{t_2}, ..., X_{t_k}) \tag{2.16}$$

is infinitely divisible. In the same article Maruyama proves that every infinitely divisible process admits to a Lévy-Khintchine representation. Below we will, on the other hand, present a simpler construction due to Rosinski [12], but first we need to fix some notations.

For every nonempty set T and $S \subset T$ we define $(\mathbb{R}^S, \mathscr{B}^S)$as

$$(\mathbb{R}^S, \mathscr{B}^S) = \prod_{t \in S} (R_t, \mathscr{B}_t),\ (R_t, \mathscr{B}_t) = (\mathbb{R}, \mathscr{B}_{\mathbb{R}}),$$

and let $p_{U,S} : \mathbb{R}^U \to \mathbb{R}^S, S \subset U \subset T$ denote the canonical projection and for simplicity we write p_S for $p_{T,S}$. By 0_S we mean the origin of $\mathbb{R}^S$ that is $0_S = \prod_{t \in S} 0_t, 0_t = 0 \in \mathbb{R}$, and let $\mathscr{P}_f(T)$ denote the family of finite subsets of T. Moreover we say that a measure ν on $(\mathbb{R}^T, \mathscr{B}^T_{\mathbb{R}})$ does not charge the origin if for all $A \in \mathscr{B}^T$ there exists a countable subset S of T such that

$$\nu(A) = \nu(A \setminus p_S^{-1}(0_S)). \tag{2.17}$$

This construction allows us to properly define what a Lévy measure on the cylindrical σ-algebra $\mathscr{B}^T$ is.

Definition. A measure ν defined on $(\mathbb{R}^T, \mathscr{B}^T)$ is a *Path Lévy measure* if it does not charge the origin and for every $t \in T$

$$\int_{\mathbb{R}^T} (|x_t| \wedge 1)\ \nu(\mathrm{d}x) < \infty. \tag{2.18}$$

This definition ensures the uniqueness of a Lévy path measure.

Theorem 2.2. *Let $X = \{X : t \in T\}$ be an infinitely divisible stochastic process. Then there exists a unique triplet (b, Σ, ν) consisting of*

1. *$b \in R^T$,*
2. *a nonnegative symmetric operator $\Sigma : \mathbb{R}^{(T)} \to \mathbb{R}^T$,*
3. *a Lévy measure ν on $\mathbb{R}^T$,*

such that for any $y \in \mathbb{R}^{(T)}$

$$\mathbb{E}e^{\mathrm{i}\sum_{t \in T} y_t X_t} = \exp\left\{ \mathrm{i}\langle y, b\rangle - \frac{1}{2}\langle y, \Sigma y\rangle + \int_{\mathbb{R}^T} \left(e^{\mathrm{i}\langle y, x\rangle} - 1 - \mathrm{i}\langle y, [[x]]\rangle\right) \nu(\mathrm{d}x)\right\} \tag{2.19}$$

where $[[x]] \in \mathbb{R}^T$, and is defined as

$$[[x]]_t \stackrel{\text{def}}{=} \frac{x_t}{|x_t| \vee 1}, t \in T$$

and

$$\mathbb{R}^{(T)} = \left\{x \in \mathbb{R}^T : x_t = 0 \text{ for all but countably many } t\right\}, \tag{2.20}$$

$$\langle y, x\rangle = \sum_{t \in T} y_t x_t,\ y \in \mathbb{R}^{(T)},\ x \in \mathbb{R}^T \tag{2.21}$$

3

Representation of infinitely divisible processes

In this chapter we discuss the theory of spectral representations of infinitely divisible stochastic processes. The main parts include representations of **ID** stochastic processes, identification of Musilak-Orlicz spaces related to certain **ID** processes and a discussion about the control measure of a homogeneous **ID** random measure.

3.1 Random measures and stochastic integration

This section deals with the concept of stochastic integration and more specifically integrals of Wiener type. We define the random measures as stochastic processes over some properly chosen index-set. Let S be a set and $\mathscr{S}$ be a δ-ring on S with the additional property that there exists a sequence of increasing sets converging to S. A δ-ring is similar to the concept of a σ-algebra with the only difference that $\mathscr{S}$ does not necessarily contain the universal set S.

Let $\Lambda = \{\Lambda(A) : A \in \mathscr{S}\}$ be a stochastic process on $(\Omega, \mathscr{F}, \mathbb{P})$. We call Λ an *independently scattered random measure* if, for every sequence $\{A_n\}$ of disjoint sets in $\mathscr{S}$, the random variables $\Lambda(A_n)$ are independent and if $\bigcup_n A_n \in \mathscr{S}$ then

$$\Lambda(\bigcup_n A_n) = \sum_n \Lambda(A_n), \mathbb{P}\text{—a.s.}$$

assuming that the series converges almost surely. The term random measure is somewhat misleading since for fixed ω this will not necessarily be a proper measure. We can for instance construct a random measure by taking a Wiener process, W, on $[0,1]$ and define the independently scattered random measures as: $\Lambda[a,b] = W(b) - W(a)$. This is not a proper measure because W is not of bounded variation.

Moreover let $\Lambda = \{\Lambda(A) : A \in \mathscr{S}\}$ be an independently scattered infinitely divisible random measure, or **ID** random measure for short. This means that $\Lambda(A) \in \mathbf{ID}, \forall A \in \mathscr{S}$ Throughout this thesis we will use the following centering function (2.3). Since $\Lambda(A), A \in \mathscr{S}$ the law of $\Lambda(A)$, $\mathscr{L}(\Lambda(A))(t)$, is of infinitely divisible kind, which means that by (2.2)

$$\hat{\mathscr{L}}(\Lambda(A))(t) = \mathbb{E}[\mathrm{e}^{\mathrm{i}t\Lambda(A)}] = \exp\left\{\mathrm{i}t\,\nu_0(A) - \frac{1}{2}t^2\nu_1(A) + \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}t\tau(x)\,F_A(\,\mathrm{d}x)\right\}. \quad (3.1)$$

We write $\Lambda \in \mathbf{ID}(\nu_0, \nu_1, F_A)$. This means that, as in the case of Lévy processes, we may decompose Λ as an independent sum of a Gaussian random measure and a Poissonian type random measure, that is

$$\Lambda(A) \overset{\mathrm{d}}{=} \Gamma(A) + \Pi(A),$$

where

$$\hat{\mathscr{L}}(\Gamma(A))(t) = \exp\left\{-\frac{1}{2}t^2\nu_1(A)\right\}, \hat{\mathscr{L}}(\Pi(A))(t) = \exp\left\{\mathrm{i}t\,\nu_0(A) + \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}t\tau(x)F_A(\,\mathrm{d}x)\right\}.$$

The following proposition is due to Rajput and Rosinski and is basically a Levy-Khintchines representation for **ID** random measures:

Proposition 3.1. *Let Λ be an **ID** random measure with characterized by* (3.1)*, then:*

1. *ν_0 is a a signed measure on $\mathscr{S}$, ν_0 is a positive measure on $\mathscr{S}$ and F_A is a Lévy measure on $\mathbb{R}$ for $A \in \mathscr{A}$ and for every $B \in \mathscr{B}_R$ $\mathscr{S} \ni A \to F_A(B)$ is a measure on $\mathscr{S}$*

2. *If ν_0, ν_1 and $F.$ satisfy the conditions above there exists a unique **ID** random measure in the sense of fdd's with characteristic function* (3.1).

3. *The set function defined as*

$$\lambda(A) = |\nu_0|(A) + \nu_1(A) + \int_{\mathbb{R}} 1 \wedge x^2\,F_A(\,\mathrm{d}x), A \in \mathscr{S}$$

   is a measure with the property that $\lambda(A_n) \downarrow 0$ implies $\Lambda(A_n) \to_{\mathbb{P}} 0$ for $\{A_n\}$ and if $\Lambda(A'_n) \to_{\mathbb{P}} 0$ for $A'_n \subset A_n$, then $\lambda(A_n) \to 0$

The measure λ is vital to the construction below. We will from now on call it the *control measure,* which is, by the additional assumption on the δ-ring, a σ-finite measure defined on $\sigma(\mathscr{S})$. Before moving on it should be noted that $\nu_0 << \lambda, \nu_1 << \lambda$, which means that the Radon-Nikodym derivatives with respect to λ exists. Some more vital facts on the corresponding measures will be summarized in the following proposition

Proposition 3.2. *Let Λ be an **ID** random measure with characteristic function* (3.1) *then:*

1. *There exists a unique σ-finite measure on $\sigma(\mathscr{S}) \times \mathscr{B}_{\mathbb{R}}$ such that*

$$F(A \times B) = F_A(B), \forall A \in \mathscr{S}, B \in \mathscr{B}_{\mathbb{R}}.$$

Moreover, there exists a function $\rho : S \times \mathscr{B}_{\mathbb{R}} \to [0,\infty]$ such that $\rho(s,\cdot)$ is a Lévy measure on $\mathscr{B}_{\mathbb{R}}$ and $\rho(\cdot,B)$ is a Borel measurable function for $B \in \mathscr{B}_{\mathbb{R}}$ and for every $(\sigma(\mathscr{S}),\mathscr{B}_{\mathbb{R}})$-measurable function $h : S \times \mathbb{R} \to \mathbb{R}$

$$\int_{S\times\mathbb{R}} h(s,x)\,F(\,\mathrm{d}s,\,\mathrm{d}x) = \int_S \int_{\mathbb{R}} h(s,x)\rho(s,\mathrm{d}x)\lambda(\,\mathrm{d}s)$$

2. *The characteristic function of $\Lambda(A)$, $\hat{\mathscr{L}}(\Lambda(A))(t)$, may be rewritten as*

$$\hat{\mathscr{L}}(\Lambda(A))(t) = \exp\{\int_A K(t,s)\,\lambda(\,\mathrm{d}s)\},$$

where

$$K(t,s) = \mathrm{i}ta(s) - \frac{1}{2}t^2\sigma^2(s) + \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}t\tau(x)\,\rho(s,\mathrm{d}x),$$

and

$$a = \frac{\mathrm{d}\nu_0}{\mathrm{d}\lambda},\ \sigma^2 = \frac{\mathrm{d}\nu_1}{\mathrm{d}\lambda}.$$

For a simple function we define the integration with respect to Λ in the sense of Rosinski and Balram [9]. Let

$$f = \sum_1^N c_j \mathbb{1}_{A_j},\ A_j \cap A_i = \emptyset,$$

and define the integral in the same manner as with the Lebesgue integral:

$$\int f\,\mathrm{d}\Lambda = \sum_1^N c_j \Lambda(A_j).$$

For an arbitrary measurable function $f : (S,\sigma(\mathscr{S})) \to (\mathbb{R},\mathscr{B}_{\mathbb{R}})$, the integral $\int_A f\,\mathrm{d}\Lambda$ is defined as a sequence of simple functions $\{f_n\}$ satisfying $f_n \to f, \lambda -$ a.e and the limit $\lim_n \int_A f_n\,\mathrm{d}\Lambda$ exists in probability for all $A \in \sigma(\mathscr{S})$. We then define the integral as the limit in probability $\int_A f_n\,\mathrm{d}\Lambda$ and call f Λ-measurable.
Remark. Whenever K is independent of s, $K(t,s) = K(t)$, the **ID** random measure is said to be *homogeneous*. This is a property that appears as soon as the **ID** random measure given by the increments of a Lévy process which is due to the temporal homogeneity. Whenever this occurs the characteristic function of Λ simplifies and now reads

$$\mathbb{E}\mathrm{e}^{\mathrm{i}t\Lambda(A)} = \exp\{K(t)\lambda(A)\},\ K(t) = \mathrm{i}t\mu - \frac{1}{2}t^2\sigma^2 + \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}t[[x]]\nu(\,\mathrm{d}x).$$

3.2 Representations of infinite divisible processes

The representation of infinite divisible processes as integrals of some corresponding infinite divisible noice was developed by Jan Rosinski and Balram Rajput, [9]. Let S be a set and $\mathscr{S}$ be a δ-ring on S with the additional property that there exists a sequence of increasing sets converging to S. Moreover let $\Lambda = \{\Lambda(A) : A \in \mathscr{S}\}$ be an independently scattered infinitely divisible random measure, or **ID** random measure for short. Throughout this section we will use the centering function defined in chapter 1 (2.3):

$$\tau(x) = [[x]]. \tag{3.2}$$

Here we will discuss questions relating to the integrability of a function $f : S \to \mathbb{R}$. We introduce three auxiliary functions that will provide sufficient and necessary information on the existence of $\int_A f \, \mathrm{d}\Lambda$. Let $p \in [0,\infty)$:

$$\begin{aligned} U(y,s) &= \quad ya(s) + \textstyle\int_{\mathbb{R}} \tau(xy) - y\tau(x)\, \rho(s, \mathrm{d}x), \\ V_p(y,s) &= \quad \textstyle\int_{\mathbb{R}} \{|yx|^p 1_{\{|yx|>1\}} + |yx|^2 1_{\{|yx|<1\}}\}\, \rho(s, \mathrm{d}x), \\ \Phi_p(y,s) &= \quad |U(y,s)| + y^2\sigma^2(s) + V_p(y,s), \end{aligned} \tag{3.3}$$

and with $p = 0$ we set

$$V_0(y,s) = \int_{\mathbb{R}} 1 \wedge |xy|^2 \rho(s, \mathrm{d}x) = \int_{\mathbb{R}} \tau(xy)^2\, \rho(s, \mathrm{d}x). \tag{3.4}$$

We can then express the characteristic function of $\int f \, \mathrm{d}\Lambda$, $\hat{\mathscr{L}}(\int_S f \, \mathrm{d}\Lambda)(t)$ in terms of $K(t,s)$ which is given in 3.2.

Lemma 3.1. *If f is Λ-measurable then:*

$$\int_S |K(tf(s),s)|\, \lambda(\mathrm{d}s) < \infty, \; \hat{\mathscr{L}}(\int_S f \, \mathrm{d}\Lambda)(t) = \exp\left\{\int_S K(tf(s),s)\, \lambda(\mathrm{d}s)\right\}, \tag{3.5}$$

*and $\int f \, \mathrm{d}\Lambda \in$ **ID**(a_f, σ_f, F_f) where*

$$a_f = \int_S U(f(s),s)\, \lambda(\mathrm{d}s), \sigma_f^2 = \int_S |f(s)|^2 \sigma^2(s)\, \lambda(\mathrm{d}s), \tag{3.6}$$

$$F_f(B) = F(\{s,x\} \in S \times \mathbb{R} : f(s) \in B \setminus \{0\}), \; B \in \mathscr{B}_{\mathbb{R}}. \tag{3.7}$$

In addition we have the following result that gives sufficient and necessary conditions on the Λ-integrability of a function.

Lemma 3.2. *Let L_0 be the space of all Λ-integrable functions. Then any measurable $f : S \to \mathbb{R}$ is Λ-integrable if and only if:*

$$\int_S |U(f(s),s)|\,\lambda(\mathrm{d}s) < \infty, \tag{3.8}$$

$$\int_S |f(s)|^2\sigma^2(s)\,\lambda(\mathrm{d}s) < \infty, \tag{3.9}$$

$$\int_S V_0(f(s),s)\,\lambda(\mathrm{d}s) < \infty. \tag{3.10}$$

What is good news about the functions Φ_p is that they can be used to construct a modular so that we may introduce a suitable Musielak-Orlicz space.

Proposition 3.3. *The functions Φ_p satisfy:*

1. *for all $s \in S$, $\Phi_p(\cdot, s)$ is a continuous non-decreasing function in $\mathbb{R}_+$ with $\Phi_p(0,s) = 0$,*
2. *$\left\{s : \Phi_p(y,s) = 0, y = y(s) \neq 0\right\}$ is a λ nullset,*
3. *$\exists C > 0 : \Phi_p(2u,s) \leq C\Phi_p(u,s)$.*

The last condition implies that all Φ_p satisfy the Δ_2-condition globally, for all s, and that modular convergence is equivalent to normed convergence by 2.1. Following Rosinski Rajput we introduce the Musielak-Orlicz space associated with the **ID** random measure

$$L_{\Phi_p}(S,\lambda) = \left\{f \in L_0(S,\lambda) : \int_S \Phi_p(|f(s)|,s)\,\lambda(\mathrm{d}s) < \infty\right\}, \tag{3.11}$$

with $L_{\Phi_0} = L_0$ equipped with the following norm:

$$\|f\|_{\Phi_p} = \inf\left\{c > 0 : \int_S \Phi_p(|f(s)|/c,s)\,\lambda(\mathrm{d}s) \leq c\right\}. \tag{3.12}$$

Looking back at 3.2 we see that the Musielak-Orlicz space can be written as an intersection of an L^2 space and a Musielak-Orlicz space without Gaussian component, that is:

$$L_{\Phi_p}(S,\lambda) = L^2(S,\sigma^2(s)\lambda(\mathrm{d}s)) \cap L_{\tilde{\Phi}_p}(S,\lambda)$$

where

$$\tilde{\Phi}_p(y,s) = |U(y,s)| + V_p(y,s).$$

A useful theorem, that will be used in the coming chapter, found in [9] describes some properties of the stochastic integral as a mapping from L_{Φ_p} to L^p.

Theorem 3.1. *If Λ is a q-th order **ID** random measure then for $0 \leq p \leq q$ we have:*

$$\left\{f \in L_0(S,\lambda) : \mathbb{E}|\int_S f\,\mathrm{d}\Lambda|^p < \infty\right\} = L_{\Phi_p}(S,\lambda), \tag{3.13}$$

and the linear map,

$$L_{\Phi_p}(S,\lambda) \ni f \to \int_S f \,\mathrm{d}\Lambda \in L^p(\Omega,\mathbb{P}),$$

is continuous.

If $p = 0$ in the previous theorem the topology in L^0 should be be understood as the topology generated by the metric

$$d_0(X,Y) = \inf\{\epsilon > 0 : \mathbb{P}(|X - Y| \geq \epsilon) \leq \epsilon\}, \text{or } \tilde{d}_0(X,Y) = \mathbb{E}|X - Y| \wedge 1. \tag{3.14}$$

Moreover under additional assumptions this is actually an isomorphism between the spaces. The suggested isomorphism condition by [9] is as follows:

Definition. Let $U(y,s),V_p(y,s),\Phi_p(y,s)$ be defined as in (3.3) (3.4) and let Λ be a q-th order **ID** random measure. We say that the isomorphism condition **IC** is fulfilled if there exists a constant $C = C(q,p), 0 \leq p \leq q$ such that for all $u > 0$

$$|U(y,s)| \leq C\left\{y^2\sigma^2(s) + V_p(y,s)\right\}. \tag{3.15}$$

This condition holds trivially if Λ is symmetric, and it can be shown, as in [9], that it also holds true if Λ is centered and L^1. A theorem that will be very useful in the coming chapter is on the behavior of this isomorphism.

Theorem 3.2. *Assume that the* **IC** *is satisfied for some* $0 \leq p \leq q$. *Then the mapping* $f \to \int_S f \,\mathrm{d}\Lambda$ *is an isomorphism from* $L_{\Phi_p}(S,\lambda)$ *into* $L^p(\Omega,\mathscr{F},\mathbb{P})$. *Moreover*

$$\left\{\int_S f \,\mathrm{d}\Lambda : f \in L_{\Phi_p}\right\} = \overline{\mathbf{Span}\{\Lambda(A) : A \in \mathscr{S}\}.}$$

Before we present the main result we must separate three cases, let $q \geq 0$ and let $X = \{X_t : t \in T\}$ be a q-th order, $L^q(\Omega,\mathscr{F},\mathbb{P})$-separable **ID** process:

Assumption 1 X is symmetric and $q \geq 0$,

Assumption 2 X arbitrary and $q \geq 1$,

Assumption 3 X is a centered stable process of index α or centered semi-stable of index (α,r)

For each of these three classes of **ID** processes we associate a corresponding **ID** random measure. This is the main reason for this spectral representation. The **ID** measure inherits the majority of the distributional properties retained by the process. That is, if X is symmetric the random measure,Λ, will be symmetric:

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}t\Lambda(A)}] = \exp\left\{2\int_0^\infty \cos(tx) - 1\, F_A(\,\mathrm{d}x)\right\}, \tag{3.16}$$

and if X is α-stable then Λ will be α-stable and so on. If X is an infinitely divisible sequence, that is $T = \mathbb{N}$ or $\mathbb{Z}$, then the following representation holds regardless of assumptions on X.

Theorem 3.3. *Let $q \geq 0$ and $X = \{X_t : t \in T\}$ be a L^q-separable **ID** process satisfying either of the assumptions and let Λ be the corresponding associated **ID** random measure with control measure λ. Then there exists $f_t \in L_{\Phi_q}, t \in T$ such that*

$$X \stackrel{\mathrm{d}}{=} \left\{ \int f_t \, \mathrm{d}\Lambda : t \in T \right\}. \tag{3.17}$$

Remark. An immediate consequence of the previous theorem is that if T has a notion of distance, say that T is a metric space with metric d, then if the map

$$t \to f_t \in L_{\Phi_p},$$

is continuous then

$$\int f_t \, \mathrm{d}\Lambda,$$

will, at least, be stochastically continuous. This follows immediately from the dominated convergence theorem.
Remark. Now this integral is taken over some uncountable subset of a complete metric space, in the article from Rosinski and Rajput it is taken over the boundary of the closed unit ball in l^2, but as given in remark 4.12 in the same article we can just map this through some Borel isomorphism and take this over say $[0,1]$ and then the **ID** random measure can be taken as a stochastic process with independent increments defined as $X[a,b] = X(b) - X(a)$, see for instance [15].

The equality in the previous theorem can be strengthened to hold almost surely under additional assumptions on the spaces T and S, see [9].

Let us simplify the setting, let $S = \mathbb{R}$ and let $\Lambda = \{\Lambda(A) : A \in \mathscr{B}_{\mathbb{R}}\}$ be an **ID** random measure induced by a Lévy process $L = \{L_t : t \in \mathbb{R}\}$ that is :

$$\Lambda[a,b] = L_b - L_a.$$

Then we have that the characteristic function of $\Lambda(a,b)$ can be written as

$$\begin{aligned} \mathbb{E}\mathrm{e}^{\mathrm{i}t\Lambda(A)} &= \mathbb{E}\mathrm{e}^{\mathrm{i}t(L_b - L_a)} = \\ &= \exp\left\{ \mathrm{i}tam(A) - \frac{1}{2}t^2 m(A) + m(A) \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}t\tau(x)\nu(\,\mathrm{d}x) \right\} \end{aligned}$$

where m is the Lebesgue measure. This means that whenever the **ID** measure is given by a Lévy process the control measure can be taken as the Lebesgue measure. Another consequence is that the kernel $K(t,s)$ in 3.2 will only depend on t, $K(t,s) = K(t)$ so for any $f \in L_{\Phi_p}$ we have that the characteristic function of $\int_{\mathbb{R}} f \, \mathrm{d}\Lambda$ can be written as

$$\mathbb{E}\mathrm{e}^{\mathrm{i}\theta \int f \, \mathrm{d}\Lambda} = \exp\left\{ \int_{\mathbb{R}} \left(\mathrm{i}\theta\mu f(s) - \frac{1}{2}\theta^2\sigma^2 f^2(s) + \int_{\mathbb{R}} e^{\mathrm{i}\theta f(s)x} - 1 - \mathrm{i}\theta f(s)\tau(x)\nu(\,\mathrm{d}x) \right) \mathrm{d}s \right\}. \tag{3.18}$$

Consider an stochastic integral process of this form, that is

$$X_t = \int_{\mathbb{R}} f(t,s)\,\mathrm{d}L(s)$$

then the characteristic function is given, analogously with the argument above, by:

$$\mathbb{E}e^{i\theta X_t} = \exp\left\{\int_{\mathbb{R}}\left(i\mu\theta f(t,s) - \frac{1}{2}\theta^2\sigma^2 f^2(t,s) + \int_{\mathbb{R}} e^{i\theta f(t,s)x} - 1 - i\theta f(t,s)\tau(x)\nu(\,\mathrm{d}x)\right)\mathrm{d}s\right\}.$$

It is to no ones surprise that reverse engineering of a spectral representation of a given infinitely divisible process is a terribly difficult task. It is even more difficult whenever the process is Gaussian because then the regularity of the process is not only dependent on the kernel but rather on the Gaussian **ID** random measure [11].

3.3 The control measure

The reason for this section is to emphasize the tight connection that exists between a homogeneous **ID** random measure and its control measure. Let $T : (S, \sigma(\mathscr{S})) \to (S, \sigma(\mathscr{S}))$ be a measurable map. If

$$\lambda(T(A)) = \lambda(A), \forall A \in \sigma(\mathscr{S}), \tag{3.19}$$

then T is said to be *measure preserving*. This implies that

$$\{\Lambda(T(A))\}_{A\in\mathscr{S}} \stackrel{\mathrm{d}}{=} \{\Lambda(A)\}_{A\in\mathscr{S}} \tag{3.20}$$

since

$$\mathbb{E}e^{it\Lambda(T(A))} = \exp\{\lambda(T(A))K(t)\} = \exp\{\lambda(A)K(t)\} = \mathbb{E}e^{it\Lambda(A)}, \forall A \in \sigma(\mathscr{S}).$$

The converse also holds, that is if T satisfies (3.20) then

$$\exp\{(\lambda(T(A)) - \lambda(A))K(t)\} = 1 \Rightarrow \lambda(T(A)) = \lambda(A), \forall A \in \mathscr{S},$$

and as $\mathscr{S}$ generates $\sigma(\mathscr{S})$ it follows that $\lambda(T(A)) = \lambda(A)$. To phrase this in a more familiar setting; consider the case when $(S,\sigma(\mathscr{S})) = (\mathbb{R},\mathscr{B}_{\mathbb{R}})$, $T(x) = \tau_h(x) = x+h, h \in \mathbb{R}$ and the **ID** random measure is the increments of a Lévy process, L. Now, by the temporal homogeneity of L we have that

$$\mathbb{E}e^{i\theta L_t} = \mathbb{E}e^{i\theta t L_1}$$

so that defining the **ID** random measure as $\Lambda(a,b) = L_b - L_a$ we must necessarily have

$$\Lambda(\tau_h(a,b)) \stackrel{\mathrm{d}}{=} \Lambda(a,b),$$

which by the preceding argument means that τ_h must be a measure preserving map with respect to the control measure. Now consider the integral process

$$X_t = \int_{\mathbb{R}} f(s+t)\,\mathrm{d}L(s),$$

then this is simply a random variable, provided that $\operatorname{supp} f = \mathbb{R}$, because

$$\begin{aligned}\mathbb{E}\mathrm{e}^{\mathrm{i}\theta X_t} &= \exp\left\{\int_{\mathbb{R}} K(\theta f(s+t))\,\lambda(\,\mathrm{d}s)\right\} = \\ &= \exp\left\{\int_{\mathbb{R}} K(\theta f(\tau_t(s)))\,\lambda(\,\mathrm{d}s)\right\} = \exp\left\{\int_{\mathbb{R}} K(\theta f(s))\,\lambda(\,\mathrm{d}s)\right\}\end{aligned}$$

This behavior is not easily observed through simulation as there is no simple way of dealing with unbounded support.

Another important property is that if λ is a finite measure, $\lambda(S) < \infty$, the random variable $\Lambda(S)$ will be well-defined since the characteristic function will be well-defined. If, in addition, we assume that Λ is a second-order random measure, the stochastic integral process

$$X_t = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}st}\,\mathrm{d}\Lambda(s),$$

will actually be a weakly stationary stochastic process.

Another interesting property is that if λ is regular then Λ will be regular, in distribution. This means that if S is equipped with a topology and the following holds

$$\lambda(A) = \sup\{\lambda(C) : C \subset A, C \text{ compact }\} = \inf\{\lambda(O) : O \supset A, O \text{ open }\}, \tag{3.21}$$

and this imply that

$$\Lambda(A) \stackrel{\mathrm{d}}{=} \sup_{\substack{C \subset A \\ C \text{ compact}}} \Lambda(C). \tag{3.22}$$

This will be proven in the coming chapter and is central to our simulation approach.

3.4 Musielak-Orlicz spaces related to certain infinitely divisible random measures

Let S be a set and $\mathscr{S}$ be a δ-ring on S with the property that there exists a sequence of increasing sets converging to S.

Poisson random measure

Let Π be a Poisson random measure, $\Pi \in \mathbf{ID}(\pi(\cdot), 0, \pi(\cdot)\delta_1)$, where π is the control measure. First of all it should be noted that Π is a homogeneous **ID** measure so that $\Phi_p(x,s) = \Phi_p(x)$

and we have

$$\begin{aligned}
K(t) &= e^{it} - 1,\\
U(x) &= x + \int_{\mathbb{R}} [[xy]] - x[[y]]\delta_1(\mathrm{d}y) = x + [[x]] - x = [[x]],\\
V_0(x) &= \int_{\mathbb{R}} [[xy]]^2\delta_1(\mathrm{d}y) = [[x]]^2,\\
V_p(x) &= \int_{\mathbb{R}} |xy|^p \mathbb{1}_{\{|xy|>1\}} + |xy|^2 \mathbb{1}_{\{|xy|<1\}}\delta_1(\mathrm{d}y) = |x|^p \mathbb{1}_{\{|x|>1\}} + |x|^2 \mathbb{1}_{\{|x|<1\}}.
\end{aligned}$$

Observe that $[[x]] = \frac{x}{|x|\vee 1}$ Now we begin with the space of all Π-integrable functions which is defined as

$$L_0^{\Pi}(S,\pi) = \left\{ f : S \to \mathbb{R} : \int_S \Phi_{\Pi}(|f(s)|)\pi(\mathrm{d}s) < \infty \right\},$$

and moreover we have

$$\Phi_{\Pi}(x) = |U(x)| + V_0(x) = |[[x]]| + [[x]]^2 = \frac{|x|}{1 \vee |x|} + \frac{x^2}{1 \vee |x|^2} = 1 \wedge |x| + \frac{x^2}{1 \vee |x|^2}.$$

Clearly the following holds:

$$(1 \wedge |x|) \leq \Phi_0 \leq 2(1 \wedge |x|),$$

so that all Π-integrable functions satisfy

$$L_0^{\Pi}(S,\pi) = \left\{ f : S \to \mathbb{R} : \int_S |f(s)| \wedge 1 \, \pi(\mathrm{d}s) < \infty \right\}. \tag{3.23}$$

The $L_{\Phi_p}^{\Pi}$-space is slightly more involved as the modular, Φ_p, is more involved:

$$\Phi_p(x) = 1 \wedge |x| + |x|^p \mathbb{1}_{\{|x|>1\}} + |x|^2 \mathbb{1}_{\{|x|<1\}}.$$

First we observe that

$$|x|^2 \mathbb{1}_{\{|x|<1\}} \leq 1 \wedge |x|,$$

and by definition we have that

$$L_{\Phi_p}^{\Pi}(S,\pi) = \left\{ f \in L_0^{\Pi}(S,\pi) : \int_S \Phi_p(|f(s)|) \, \pi(\mathrm{d}s) < \infty \right\},$$

so we have for $f \in L_0^{\Pi}(S,\pi)$ that

$$\int_S \Phi_p(|f(s)|) = \int_S 1 \wedge |f(s)| + |f(s)|^p \mathbb{1}_{\{|f(s)|>1\}} + |f(s)|^2 \mathbb{1}_{\{|f(s)|<1\}} \, \pi(\mathrm{d}s) = \tag{3.24}$$

$$= C + C' + \int_{\{|f(s)|>1\}} |f(s)|^p \, \pi(\mathrm{d}s), \tag{3.25}$$

since

$$C = \int_S |f(s)| \wedge 1 \, \pi(\mathrm{d}s) < \infty, \; C' = \int_S |f(s)|^2 \mathbb{1}_{\{|f(s)|<1\}} \, \pi(\mathrm{d}s) \le C < \infty,$$

which yields

$$\int_{|f(s)|>1} |f(s)|^p \, \pi(\mathrm{d}s) \le \int_S \Phi_p(|f(s)|) \le 2C + \int_{|f(s)|>1} |f(s)|^p \, \pi(\mathrm{d}s),$$

and hence

$$L^{\Pi}_{\Phi_p}(S,\pi) = \left\{ f \in L^{\Pi}_0(S,\pi) : \int_{|f(s)|>1} |f(s)|^p \, \pi(\mathrm{d}s) < \infty \right\}. \tag{3.26}$$

Finally for $f \in L_0$ we have that

$$\hat{\mathscr{L}}(\int f \, \mathrm{d}\Pi)(z) = \exp\left\{ \int_S \mathrm{e}^{\mathrm{i}zf(x)} - 1 \, \pi(\mathrm{d}x) \right\}. \tag{3.27}$$

It should be observed that $f \in L^{\Pi}_0$ does not imply $|f| < \infty$ a.e. Let, for instance $f(s) = \infty, s \in A, \pi(A) < \infty$ and $f(s) = 0, s \in A^{\complement}$.

Gamma random measure

The Gamma random measure, Υ, is defined as the homogeneous random measure

$$\Upsilon \in \mathbf{ID}(\upsilon(A)k, 0, \upsilon(A)\eta(\mathrm{d}x)), \; \eta(\mathrm{d}x) = \mathbb{1}_{[0,\infty)} \frac{\mathrm{e}^{-x}}{x} \mathrm{d}x, \; k = \int_0^\infty [[x]] \frac{\mathrm{e}^{-x}}{x} \mathrm{d}x.$$

Moreover we have

$$\begin{aligned}
K(t) &= \log(\frac{1}{1-\mathrm{i}t}) = -\frac{1}{2}\log(1+t^2) + \mathrm{i}\arctan(t),\\
U(x) &= kx + \int_0^\infty ([[xy]] - y[[x]]) \frac{e^{-y}}{y} \mathrm{d}y = |x|(1-\mathrm{e}^{-1/|x|}) + \int_{1/|x|}^\infty \frac{\mathrm{e}^{-y}}{y} \mathrm{d}y\\
V_0(x) &= |x|^2(1-\mathrm{e}^{-1/|x|}(\frac{1}{|x|}+1)) + \int_{1/|x|}^\infty \frac{\mathrm{e}^{-y}}{y} \mathrm{d}y,\\
V_p(x) &= |x|^2\left(1-\mathrm{e}^{-1/|x|}(\frac{1}{|x|}+1)\right) + |x|^p \int_{1/|x|}^\infty y^{p-1}\mathrm{e}^{-y} \mathrm{d}y.
\end{aligned}$$

Straightforward one-variable calculus provides the following estimates when $|x| < 1$, note that $|U(x)| = U(x)$ and we have

$$\Phi^{\Upsilon}_0(x) = U(x) + V_0(x) = |x|(1-\mathrm{e}^{-1/|x|}) + |x|^2\left(1-\mathrm{e}^{-1/|x|}(\frac{1}{|x|}+1)\right) + 2\int_{1/|x|}^\infty \frac{\mathrm{e}^{-y}}{y} \mathrm{d}y.$$

Then we have the following estimates for each term

$$
\begin{aligned}
&(1-e^{-1})|x| \le |x|(1-\mathrm{e}^{-1/|x|}) \le |x|,\\
&0 \le |x|^2\left(1-\mathrm{e}^{-1/|x|}(\frac{1}{|x|}+1)\right) \le |x| \sup_{x\in(0,1)} |f(x)|,\, f(x)=|x|-\mathrm{e}^{-1/|x|}(1+|x|),\\
&0 \le \int_{1/|x|}^{\infty} \frac{\mathrm{e}^{-y}}{y}\,\mathrm{d}y \le \mathrm{e}^{-1}|x|.
\end{aligned}
$$

Hence for $|x|<1$ there exists constants $0<c\le C<\infty$ such that

$$c|x| \le \Phi(x) \le C|x|.$$

Similarly for $|x|>1$ we have the following estimates

$$
\begin{aligned}
&1-\mathrm{e}^{-1} \le |x|(1-e^{-1/|x|}) \le 1,\\
&1-2\mathrm{e}^{-1} \le |x|^2\left(1-\mathrm{e}^{-1/|x|}(\frac{1}{|x|}+1)\right) \le 1/2,\\
&\mathrm{e}^{-1}\log|x| \le \int_{1/|x|}^{\infty} \frac{\mathrm{e}^{-y}}{y}\,\mathrm{d}y \le \log|x|+\mathrm{e}^{-1},
\end{aligned}
$$

which yields

$$c'\log(\mathrm{e}|x|) \le \Phi_0^{\Upsilon}(x) \le C'\log(\mathrm{e}|x|),\ 0<c'\le C'<\infty.$$

Letting

$$\Psi(x)=\begin{cases} |x| & |x|<1\\ \log(\mathrm{e}|x|) & |x|\ge 1\end{cases} \tag{3.28}$$

we finally obtain

$$L_0^{\Upsilon}(S,\upsilon)=\left\{f:S\to\mathbb{R}:\int_S \Psi(|f(s)|)\,\upsilon(\mathrm{d}s)<\infty\right\}. \tag{3.29}$$

To characterize the space $L^{\Upsilon}_{\Phi_p}$ it suffices to estimate the integral

$$\int_{1/|x|} \frac{\mathrm{e}^{-y}}{y^{1-p}}\,\mathrm{d}x$$

for $p\in(0,\infty)$. For $|x|<1$ we obtain the same estimates as before. For $|x|>1$ we find that

$$c''|x|^p \le \Phi(x) \le C''|x|^p,$$

and we have

$$L^{\Upsilon}_{\Phi_p}(S,\upsilon)=\left\{f\in L_0^{\Upsilon}:\int_{|f(s)|>1} |f(s)|^p\,\upsilon(\mathrm{d}s)<\infty\right\}. \tag{3.30}$$

Moreover, for an arbitrary f that is Υ-integrable we have:

$$\hat{\mathscr{L}}(\int f \,\mathrm{d}\Upsilon)(z) = \exp\left\{\int_S -\frac{1}{2}\log(1+f(s)^2) + \mathrm{i}\arctan(f(s))\, m(\,\mathrm{d}s)\right\}. \tag{3.31}$$

Remark. It should be noted here that neither the Poisson random measure nor the Gamma random measure satisfy the isomorphism condition, but for $p = 1$ both $L^{\Pi}_{\Phi_1}$, $L^{\Upsilon}_{\Phi_1}$ are in fact $L^1(S,m)$. Moreover for $p > 1$ we actually have that

$$L^{\cdot}_{\Phi_p}(S,\upsilon) = L^p(S,\upsilon) \cap L^1(S,\upsilon),$$

since if $f \in L^{\cdot}_{\Phi_p}(S,\upsilon)$ then

$$\int_{|f|<1} |f| + \int_{|f|>1} |f|^p < \infty,$$

which implies

$$\|f\|_p < \infty, \|f\|_1 < \infty,$$

since $|f| < 1 \Rightarrow |f|^p \leq |f|$ and $|f| > 1 \Rightarrow |f|^p > |f|$. Hence $L^{\cdot}_{\Phi_p} \subseteq L^p(S,\upsilon) \cap L^1(S,\upsilon)$. Conversely, if $\|f\|_p$ and $\|f\|_1$ are both finite the other inclusion follows immediately, as the modular is finite. This is in fact good news as these actually are Banach spaces with norm $\|f\| = \|f\|_1 + \|f\|_p$ and we have the following inclusion $L^p \cap L^1 \subset L^q, 1 < q < p$. See for instance [4] for more details concerning the basic theory of L^p-spaces. Finally the metrics are actually equivalent since modular convergence is equivalent to convergence in F-norm, by 3.3 and 2.1.

$S\alpha S$ random measure

Now consider a $S\alpha S$ random measure with $\alpha \in (0,1) \cup (1,2)$, Σ, that is:

$$\mathbb{E}e^{it\Sigma(E)} = \exp\{-|t|^{\alpha}\sigma(E)\} \Longleftrightarrow \Sigma \in \mathbf{ID}(0,0,\sigma(\cdot)\theta(\,\mathrm{d}x)), \theta(\,\mathrm{d}x) = c|x|^{-\alpha-1}\,\mathrm{d}x, \tag{3.32}$$

which is once again a homogeneous random measure and as before we begin with characterizing the space of all Σ-integrable functions. We have

$$\begin{aligned}
K(t) &= -|t|^{\alpha},\\
U(x) &= c\int_{\mathbb{R}} [[xy]] - x[[y]]|y|^{-\alpha-1}\,\mathrm{d}y,\\
V_0(x) &= c\int_{\mathbb{R}} [[xy]]^2|y|^{-\alpha-1}\,\mathrm{d}y,\\
V_p(x) &= c\int_{\mathbb{R}} |xy|^p \mathbb{1}_{\{|xy|>1\}} + |xy|^2\mathbb{1}_{\{|xy|<1\}}|y|^{-\alpha-1}\,\mathrm{d}y.
\end{aligned}$$

Indeed $U(x) = 0$ as the integrand is odd, moreover it is an easy exercise in integration to see that

$$c\int_{\mathbb{R}} [[xy]]^2|y|^{-\alpha-1}\,\mathrm{d}y = \frac{4c}{\alpha(2-\alpha)}|x|^\alpha,$$

and hence

$$\Phi_\Sigma(x) = V(x) = \frac{4c}{\alpha(2-\alpha)}|x|^\alpha,$$

which simply means that

$$L_0^\Sigma(S,\sigma) = L^\alpha(S,\sigma). \tag{3.33}$$

The spaces $L^\Sigma_{\Phi_p}$ are actually simpler than expected because $p < \alpha$ must hold for $V_p(x)$ to be finite and it is once again an easy exercise in integration to see that Φ_p actually reduces to $C'\Phi_0, C' > 0$, which means that

$$L^\Sigma_{\Phi_p} = L^\Sigma_0, p \leq \alpha.$$

Moreover we have for $f \in L^\Sigma_0$:

$$\hat{\mathscr{L}}(\int f\,\mathrm{d}\Sigma)(z) = \exp\left\{-|z|^\alpha \int |f(x)|^\alpha\,\sigma(\,\mathrm{d}x)\right\}, \tag{3.34}$$

so that the random variable, $\int f\,\mathrm{d}\Sigma$, is itself $S\alpha S$ with scale parameter, s,

$$s^\alpha = \int |f(x)|^\alpha\,\sigma(\,\mathrm{d}x).$$

This result exists in a more general form in [9] concerning spectral representations of stable and semi-stable random integrals. For a more complete structure of the stable integrals we refer to [13].

Gaussian random measure

A special case of the stable random measures is the Gaussian random measure defined as:

$$\mathbb{E}\mathrm{e}^{\mathrm{i}t\Gamma(A)} = \exp\left\{-\frac{1}{2}t^2\gamma(A)\right\}.$$

It follows immediately that

$$\Phi_p(y) = y^2,$$

so that

$$L_0^\Gamma = \left\{f:\mathbb{R}\to\mathbb{R}: \int_{\mathbb{R}} |f(s)|^2\gamma(\,\mathrm{d}s) < \infty\right\} = L^2(S,\gamma). \tag{3.35}$$

Moreover we obtain for $f \in L_0^\Gamma$,

$$\hat{\mathscr{L}}(\int f\,\mathrm{d}\Gamma)(z) = \exp\left\{-|z|^2 \int |f(x)|^2\,\gamma(\,\mathrm{d}x)\right\}, \tag{3.36}$$

which means that $\int f \,\mathrm{d}\Gamma$ is a Gaussian random variable with variance

$$\sigma^2 = \int |f(x)|^2 \gamma(\mathrm{d}x).$$

So in the case of Gaussian random measures the linear structure of the Γ-integrable functions is in fact that of a Hilbert space, which means that we have notions of orthogonality. Moreover, the behavior of the orthogonality of the subspace $\left\{\int f \,\mathrm{d}\Gamma : f \in L^2(S,\gamma)\right\} \subset L^2(\Omega,\mathscr{F},\mathbb{P})$ is inherited from the orthogonality structure of $L^2(S,\gamma)$. This is really due to an isometry or unitary map between the two. To prove this fairly simple statement let, ϕ and ψ be simple functions

$$\phi = \sum_1^N a_i \mathbb{1}_{A_i},\ \psi = \sum_1^M b_i \mathbb{1}_{B_i},$$

and without loss of generality we may rewrite either, ψ or ϕ in terms of the partition A_i or B_i respectively. Hence we may without loss of generality consider two simple functions of equal number of terms and partition,

$$\psi = \sum_1^N b_i' \mathbb{1}_{A_i}.$$

We get for simple functions

$$\langle \phi, \psi \rangle = \mathbb{E} \int \phi \,\mathrm{d}\Gamma \overline{\int \psi \,\mathrm{d}\Gamma} = \sum_{i,j=1}^N a_i \overline{b_j} \mathbb{E}\Gamma(A_i)\Gamma(A_j) = \sum_{i=1}^N a_i \overline{b_i} \gamma(A_i) = \int \phi \overline{\psi}\, \gamma(\mathrm{d}x).$$

This means is that if we have an orthonormal basis $\{\phi_n\}$ then it will be represented as uncorrelated and hence independent random variables in $L^2(\Omega,\mathscr{F},\mathbb{P})$. From this it follows that the problem of finding a suitable approximation of a stochastic integral

$$\int g \,\mathrm{d}\Gamma,$$

reduces to finding a orthonormal basis, $\{\phi_n\}$, for the space $L^2(S,\gamma)$, and approximating the integrals

$$\int \phi_n \,\mathrm{d}\Gamma.$$

This is possible in the Gaussian case, since we know that the random variables $\int \phi_n \,\mathrm{d}\Lambda$ are in fact Gaussian by (3.36) with variance $\sigma^2 = \int \phi_n(x)\gamma(\mathrm{d}x)$. Now for an arbitrary Γ-integrable function f set

$$f^{(N)} = \sum_{n=1}^N \langle f, \phi_n \rangle_\Gamma \phi_n,\ \langle f, \phi_n \rangle_\Gamma = \int f(x)\overline{\phi_n(x)}\gamma(\mathrm{d}x),$$

we have (with obvious notation)

$$\| \int f \, \mathrm{d}\Gamma - \int f^{(N)} \, \mathrm{d}\Gamma \|_p \leq C \| f - f^{(N)} \|_\Gamma,$$

by theorem 3.1. So any Gaussian stochastic integral process can be approximated by a sum of the form

$$\sum_1^N a_n(t)\xi_n, a_n(t) = \langle f_t, \phi_n \rangle, \xi \in \mathcal{N}(0,1).$$

This conclusion is, on the other hand, *not* true for the stable integral-processes or the Poisson integral-processes. In the next chapter we will discuss a general method for simulating other integral processes under sufficiently inclusive conditions on the space and control measure.

4

Simulation of processes using representations

ONE of the most important parts of the theory of stochastic processes is to be able to simulate sample paths. This is due to the fact that understanding a stochastic process requires a full description of the finite dimensional distributions, fdd's. The spectral representations allow us to control the error of the simulation in a precise way. Take for instance the stationary processes, we know that these are in a one to one correspondence with a separable L^2 space with a finite measure. This implies that we can approximate the stochastic process arbitrarily well in the L^2 sense. For the infinite divisible processes this is *not* true, but we still know that the simple functions are dense in the Musielak Orlicz space and converge in probability. What we prove in this chapter is a confirmation of the idea that

$$\sum_{j=1}^{N} f(t,\eta_j)\Delta L_j,\ \Delta L_j = L(s_j) - L(s_{j-1}), \eta_j \in (s_{j-1},s_j)$$

is a suitable approximation of

$$\int_S f(t,s)\,\mathrm{d}L(s),$$

where L is a Lévy process. This is proven for general **ID** random measures under some conditions.

We will not discuss how to properly simulate Lévy processes, or simulating from infinitely divisible distributions. We refeer to [2], among others for further discussions.

4.1 Infinite divisible processes

Simulation of infinitely divisible processes has been done to some extent by Wolfgang Karcher et al. in [6], but they restrict their attention to functions with compact support on $\mathbb{R}^d$ and assume that the collection of functions $\{f_t\}_{t\in T}$ are in $L^p, p > 1$. In general there are error bounds in terms of L^r estimates and in the almost surely sense, see [3].

Our approach is far more general. We aim with this section to propose a method of simulating infinitely divisible integral processes driven by a **ID** random measure and to verify that the approximation holds in a suitable sense. Let

$$X_t \overset{\text{def}}{=} \int_S f_t(s)\,\mathrm{d}\Lambda, t \in T, \tag{4.1}$$

where $\Lambda = \{\Lambda(A) : A \in \mathscr{S}\}$ is a **ID** random measure with control measure λ, where $\mathscr{S}$ is a δ-ring of a set S, and $\Lambda(A) \in \mathbf{ID}(\nu_0, \nu_1, F_A)$. We will assume that the set S actually is a topological space so that notions of compact and open sets make sense. We will moreover assume that either of the following conditions hold.

Condition 1. *The space L_{Φ_p} is a normed space or satisfy at least, $\|\lambda f\|_{\Phi_p} \leq |\lambda|\|f\|_{\Phi_p}, \forall |\lambda| \leq 1$*

This condition might seem restrictive, but it is actually satisfied for Poisson and Gamma random measure whenever $p > 1$ and for $S\alpha S$-stable measures if $\alpha \in (0,1)\cup(1,2]$.

Condition 2. *The **IC** condition is satisfied*

4.1.1 Homogeneous case

Let $\Lambda = \{\Lambda(A) : A \in \mathscr{R}\}$ be homogeneous, that is

$$\Lambda(\cdot) \in \mathbf{ID}(\lambda(\cdot)\mu, \lambda(\cdot)\sigma, \lambda(\cdot)F) \Leftrightarrow \mathbb{E}\mathrm{e}^{\mathrm{i}t\Lambda(A)} = \exp\{\lambda(A)K(t)\}, A \in \mathscr{S}$$

where

$$K(t) = i\mu t - \frac{1}{2}t^2\sigma^2 + \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}t[[x]]F(\mathrm{d}x).$$

To separate the cases might seem strange at first but the reason becomes clear if one considers an integral process,

$$\int_{\mathbb{R}} f(t,s)\,\mathrm{d}L(s),$$

that is driven by a Lévy process, $L = \{L(s) : s \in \mathbb{R}\}$. Then due to the temporal homogeneity of Lévy processes one may take the control measure as the Lebesgue measure.

In the homogeneous case it should be noted that the Musielak-Orlicz space, L_{Φ_p}, essentially reduces to a Orlicz space with the only difference being that the norm is still of Musielak-Orlicz kind. Let us recall the related functions and modular that constitute the

Musielak Orlicz space,

$$V_0(x) = \int_{\mathbb{R}} [[xy]]^2 F(\mathrm{d}y), \; V_p(x) = \int_{\mathbb{R}} |xy|^p \mathbb{1}_{\{|xy|>1\}} + |xy|^2 \mathbb{1}_{\{|xy|<1\}} F(\mathrm{d}y),$$
$$U(x) = xa + \int_{\mathbb{R}} [[xy]] - x[[y]] F(\mathrm{d}y), \, \Phi_p(x) = |U(x)| + x^2\sigma^2 + V_p(x),$$

$$L_0 = \left\{ f : S \to \mathbb{R} : \int_S \Phi_0(|f(s)|)\,\lambda(\mathrm{d}s) < \infty \right\}, \, L_{\Phi_p} = \left\{ f \in L_0 : \int_S \Phi_p(|f(s)|)\,\lambda(\mathrm{d}s) < \infty \right\},$$
$$\|f\|_{\Phi_p} = \inf\left\{ c > 0 : \int_S \Phi_p(f|(s)|/c) \le c \right\}.$$

What will be the key observation in the proposed simulation method is that the simple functions are dense in L_{Φ_p} and that the sets will be of finite measure.

Lemma 4.1. *Let*

$$\varphi(s) = \sum_{j=1}^{N} c_j \mathbb{1}_{A_j}(s), \, A_j \cap A_i = \emptyset,$$

then $\varphi(s)$ is Λ-integrable if and only if $\lambda(A_j) < \infty, j = 1,2,...N$. Moreover the simple functions are dense in L_{Φ_p}.

Proof. The proof is simply to compute the modular of the function,

$$\int_S \Phi_p(|\varphi(s)|)\,\lambda(\mathrm{d}s) = \sum_{j=1}^{N} \Phi_p(|c_j|)\lambda(A_j).$$

Now as Φ_p is a continuous Young function we know that $\Phi_p(|c_j|) < \infty, \forall j$ and we are done. The proof that the simple functions are dense in L_{Φ_p} can be found in either [8] or [10]. $\square$

Remark. We may add the conclusion that the set of all continuous functions with compact support, denoted $C_0(\mathbb{R}^n)$, is dense in L_{Φ_p}. The argument is the same as in the theory of L^p spaces. Consider first R. Let $A = [a,b], -\infty < a < b < \infty$ be λ-finite, then it suffices to find a continuous function, f, that is 1 on A and vanishes outside $[a-\epsilon, b+\epsilon], \epsilon > 0$ and is linear on $[a-\epsilon, a], [b, b+\epsilon]$ then we have

$$\rho(f - \mathbb{1}_A) \le \Phi_p(1)\lambda([a-\epsilon, a] \cup [b, b+\epsilon]),$$

and by the continuity of measure we may choose ϵ so small that $\lambda([a-\epsilon, a] \cup [b, b+\epsilon]) < \epsilon/\Phi_p(1)$. For $\mathbb{R}^n$ one lets A be a rectangle and the same result holds.
This is the first step towards our goal, because now we know that for every fixed $t \in T$ we can find a simple function φ_t such that

$$\|f_t - \varphi_t\|_{\Phi_p} < \epsilon.$$

What is lacking however is that we do not know how well this approximation holds in probabilistic terms. Now if the control measure, λ, is regular that is

$$\lambda(A) = \inf\{\lambda(O) : O \supseteq A, O \text{ open}\} = \sup\{\lambda(K) : K \subseteq A, K \text{ compact}\}, \tag{4.2}$$

we may approximate the distribution of $\Lambda(A)$ arbitrarily well by a compact set in the Musielak-Orlicz norm. The assumption that λ is regular is motivated once again by considering an integral process driven by a Lévy process as the control measure is regular.

Lemma 4.2. *Let Λ be a **ID** random measure and let λ be its control measure. Set*

$$\phi_A(t) = \mathbb{E}e^{it\Lambda(A)} = \exp\{\lambda(A)K(t)\}, A \in \mathscr{S}, \tag{4.3}$$

and assume that λ is regular. Then for $\epsilon > 0$ there exists a compact set K or open set O such that

$$|\phi_A(t) - \phi_K(t)| < \epsilon, \text{ and } \|\mathbb{1}_A - \mathbb{1}_K\|_{\Phi_p} < \epsilon,$$

or

$$|\phi_A(t) - \phi_O(t)| < \epsilon, \text{and } \|\mathbb{1}_A - \mathbb{1}_O\|_{\Phi_p} < \epsilon.$$

holds.

Proof. By (4.2) there exists a sequence $\{K_n\}_{n\in\mathbb{N}}, K_n \subseteq K_m, n \leq m$ of compact sets such that $\lambda(K_n) \uparrow \lambda(A)$ or equivalently $\lambda(A \setminus K_n) \downarrow 0$. This means that there exists an integer $N \in \mathbb{N}$ such that $\lambda(A \setminus K_N) < \epsilon$ for some $\epsilon > 0$. Now set $K = K_N$ and we have

$$\phi_A(t) = \mathbb{E}e^{\mathrm{i}t\Lambda(A)} = \exp\{\lambda(A)K(t)\} = e^{\lambda(A\setminus K)K(t)}e^{\lambda(K)K(t)} = e^{\lambda(A\setminus K)K(t)}\phi_K(t),$$

which evidently yields

$$|\phi_A(t) - \phi_K(t)| = |\phi_K(t)||1 - e^{\lambda(A\setminus K)K(t)}| \leq |1 - e^{\lambda(A\setminus K)K(t)}| < \epsilon,$$

since ϕ_K is a characteristic function and hence $|\phi_K(t)| \leq 1$. The proof for open sets is identical but instead of taking a sequence of increasing compact set one takes a sequence of decreasing open sets. As for the statement about the Musielak Orlicz norm we know that norm convergence and modular convergence are equivalent by 2.1 and 3.3,

$$\int_S \Phi_p(\mathbb{1}_A - \mathbb{1}_K)\,\lambda(\,\mathrm{d}s) = \Phi_p(1)\lambda(A \setminus K) < \Phi_p(1)\epsilon,$$

which completes the proof. $\square$

Now as simple functions consists of disjoint sets these results imply is that any simple function is arbitrarily well approximated, in distribution and Musielak Orlicz sense, by a different simple function that consist of compact disjoint sets. Before we begin with the last part denote

$$d_p(X,Y) = (\mathbb{E}|X - Y|^p)^{1/p}, X,Y \in L^p(\Omega,\mathscr{F},\mathbb{P}),$$
$$d_{\Phi_p}(f,g) = \|f - g\|_{\Phi_p}, f,g \in L_{\Phi_p}(\lambda,S).$$

Before we prove our main result we must prove a certain property on the continuity of the mappings from the space L_{Φ_p} to any normed space.

Lemma 4.3. *The following are equivalent for a linear mapping*

$$T : L_{\Phi_p} \to Y,$$

where $(Y, \|\cdot\|)$ *is a normed space and* L_{Φ_p} *satisfy Condition 1.*

1. *T is continuous,*
2. *T is continuous at* 0*,*
3. *T is bounded.*

Proof. Indeed the following implications hold: (1) ⇒ (2) and (3) ⇒ (1). So to complete our argument we only need to prove that (2) ⇒ (3) holds. Assume T continuous at 0. Then by definition of continuity we have:

$$\forall \epsilon > 0, \exists \delta > 0, \ : \forall x \in L_{\Phi_p} \|x\|_{\Phi_p} < \delta \Rightarrow \|Tx\| < \epsilon,$$

so that for $\epsilon = 1$ there exists $\delta > 0$ such that for $x \in L_{\Phi_p}$, $\|x\|_{\Phi_p} < \delta$ implies $\|Tx\| < \epsilon$. Let

$$x' = \frac{\delta}{2\|x\|_{\Phi_p}} x,$$

for some arbitrary $x \in L_{\Phi_p}$. For simpler notation let $a = \frac{\delta}{2\|x\|_{\Phi_p}}$, and let $\lfloor a \rfloor = \sup\{n \in \mathbb{Z} : n \leq a\}$. Then we have by 2.1 and Condition 1 that

$$\|x'\|_{\Phi_p} = \|ax\|_{\Phi_p} = \|\lfloor a \rfloor x - (a - \lfloor a \rfloor)x\|_{\Phi_p} \leq \lfloor a \rfloor \|x\|_{\Phi_p} + (a - \lfloor a \rfloor)\|x\|_{\Phi_p} < \delta$$

and hence we obtain

$$\|Tx'\| = \|T \frac{\delta}{2\|x\|_{\Phi_p}} x\| < 1 \Leftrightarrow \|Tx\| < \frac{2}{\delta}\|x\|_{\Phi_p}$$

□

This result is only a slight modification of the same result concerning linear continuous maps between normed spaces and can be found in any book on functional analysis. It should be noted that the condition $\|\lambda f\|_{\Phi_p} \leq |\lambda| \|f\|_{\Phi_p}$ is essential. We are now ready to prove our main result.

Proposition 4.1. *Let* $\{f_t\}_{t \in T} \subset L_{\Phi_p}(S,\lambda)$ *and*

$$X_t = \int_S f_t \, \mathrm{d}\Lambda.$$

Assume that λ *is regular and assume that either condition 1 or condition 2 holds.*

Then there exists a family of simple functions

$$\{\varphi_t(s)\}_{t \in T}, \varphi_t = \sum_{j=1}^{N(t)} x_j(t) \mathbb{1}_{A_j}$$

such that

$$d_p(X_t, \sum_{j=1}^{N(t)} x_j(t)\Lambda(A_j) < \epsilon$$

where A_j can be taken compact.

Proof. If condition 1 holds, then by 3.1 we know that

$$f \to \int_S f \, \mathrm{d}\Lambda, \; f \in L_{\Phi_p}$$

is a linear continuous map. This is equivalent to it being bounded that is

$$d_p(I(f), I(g)) \leq C d_{\Phi_p}(f,g), \, I(h) = \int_S h(s) \, \mathrm{d}\Lambda, \, C > 0$$

by the previous lemma 4.3. For $0 < p < 1$ the same holds due to the homogeneity of d_p. For $p = 0$ we need that the **IC** holds. Then for every fixed t we have that there exists a simple Λ-integrable function depending on t, φ_t such that

$$d_{\Phi_p}(f_t, \varphi_t) < \epsilon.$$

By 4.2 we know that for φ_t there exists a corresponding simple function, $\tilde{\varphi}_t$ consisting of disjoint compact sets such that

$$d_{\Phi_p}(\varphi_t, \tilde{\varphi}_t) < \epsilon.$$

Then by the continuity and linearity of $\int_S \cdot \, \mathrm{d}\Lambda$ we have

$$\begin{aligned} d_p(X_t, I(\tilde{\varphi}_t)) &\leq d_p(I(\varphi_t), I(\tilde{\varphi}_t) + d_p(X_t, I(\varphi_t)) \\ &\leq C(d_{\Phi_p}(\varphi_t, \tilde{\varphi}_t) + d_{\Phi_p}(\varphi_t, f_t)) < 2C\epsilon \end{aligned}$$

and we are done. If the **IC** holds then the result follows immediately by 3.2 □

The advantage with condition 1 is that it provides a tool for measuring the error of our approximation. This will be discussed in the coming section.

Now lets restrict ourselves to a more familiar setting. Let $S = \mathbb{R}$ and $T \subseteq \mathbb{R}$. What this theorem tells us is that given a integral process driven by a homogeneous **ID** random measure all we need to know is the structure of our Musielak Orlicz space in order for the approximation to make sense. A concrete example is to consider an integral process driven by an $S\alpha S$ Levy motion $L = \{L_t : t \in \mathbb{R}\} \in S\alpha S$, that is our independently scattered **ID** random measure is defined as $\Lambda[a,b] = L_b - L_a$ and is distributed accordingly to (3.32) with control measure as the Lebesgue measure. This means that the compact sets are just closed and bounded intervals and we may take the intervals to be of same size so that

$$f(t,s) \approx \sum_{j=-N}^{N} x_j(t) \mathbb{1}_{[j, j+\frac{1}{N}]}(s)$$

where the approximation is in L^{α}-sense and the integral is approximated by

$$\int_{\mathbb{R}} f(t,s)\,\mathrm{d}\Lambda \approx \sum_{j=-N}^{N} x_j(t)\xi_j, \xi_j \in \mathbf{ID}(0,0,\frac{1}{N}\theta(\,\mathrm{d}x)) \Longleftrightarrow \xi_j \in S_{\alpha}(\frac{1}{N^{1/\alpha}},0,0).$$

In a more general sense this means that the approximation

$$\int_{\mathbb{R}} f_t(s)\,\mathrm{d}L(s) \approx \sum_{-N}^{N} f_t\left(\frac{k}{n}\right)\left(L\left(\frac{k+1}{n}\right) - L\left(\frac{k}{n}\right)\right)$$

makes sense for an arbitrary Lévy process, assuming of course that f is L-integrable and that either condition 1 or 2 holds.

4.1.2 Inhomogeneous case

Now the inhomogeneous case is, and should be, somewhat more difficult as we must now investigate the regularity of the measures

$$\nu(A) = \int_A K(t,s)\,\lambda(\,\mathrm{d}s), \tag{4.4}$$

$$K(t,s) = \mathrm{i}t\mu(s) - \frac{1}{2}t^2\sigma^2(s) + \int_{\mathbb{R}} \mathrm{e}^{\mathrm{i}tx} - 1 - \mathrm{i}t[[x]]\,\rho(s,\mathrm{d}x), \tag{4.5}$$

$$\mu_p(A) = \int_A \Phi_p(1,s)\,\lambda(\,\mathrm{d}s). \tag{4.6}$$

We restrict ourselves to the case when S is a σ-compact space, which means that S can be written as a countable union of compact sets. Consider for example the sets $K_n = [-n,n]$ on $\mathbb{R}$. A general result on regular measures can then be proven.

Lemma 4.4. *Let X be a σ-compact topological spaces and let $\mathscr{B}_X$ denote the Borel σ-algebra. Let μ be a inner-regular σ-finite measure on X, and let $f \in L^1_{\mathrm{loc}}(\mu)$. Then the measure*

$$\nu(A) = \int_A f\,\mathrm{d}\mu, \tag{4.7}$$

is a σ-finite inner-regular measure.

Proof. The σ-finiteness is clear by the assumption of a σ-compact space and the locally integrability of f :

$$\int_K f\,\mathrm{d}\mu < \infty, \forall K \text{ compact}.$$

Now define $\nu_n(A) = \nu(A \cap A_n), n \in \mathbb{N}$ where $\bigcup_1^{\infty} A_n = X, A_n \cap A_m = \emptyset$, $\nu(A_n) < \infty, \forall n$. Then ν_n is finite for all n and $\nu_n << \mu, \forall n$ which is equivalent to:

$$\forall \epsilon > 0, \exists \delta > 0 : \mu(A) < \delta \Rightarrow \nu_n(A) < \epsilon \tag{4.8}$$

Now recall the definition of inner regularity:

$$\mu(A) = \sup\{\mu(K) : K \subset A,\, K \text{ compact }\},$$

and by the σ-compactness let K be a compact set such that $\mu(A\backslash K) < \delta \Rightarrow \nu_n(A\backslash K) < \epsilon 2^{-n}$. Then we have

$$\nu_n(A) < \epsilon 2^{-n} + \nu_n(K)$$

and hence for

$$\nu(A) = \sum_1^{\infty} \nu_n(A) < \epsilon + \nu(K)$$

taking the supremum on the left hand side we obtain the result. □

What this general result means is that we essentially have arrived at the same result as in the homogeneous case, with the additional assumption on the space S and as long as the functions $K(t,\cdot)$, and $\Phi(1,\cdot)$ are locally integrable. Now indeed the function $K(t,\cdot)$ is locally integrable as it merely consists of the three σ-finite measures, ν_0, ν_1, and F_A, and moreover we know by 3.2 that $\mathbb{1}_A$ is Λ-integrable whenever $A \in \mathscr{S}$, so that $\Phi(1,\cdot)$ is locally integrable as well. Hence the same argument goes through for the inhomogeneous case and we have the same result.

Proposition 4.2. *Let $\{f_t\}_{t\in T} \subset L_{\Phi_p}(S,\lambda)$ and*

$$X_t = \int_S f_t \, \mathrm{d}\Lambda.$$

Assume that λ is regular, S is a σ-compact space and that either Condition 1 or Condition 2 holds.

Then there exists a family of simple functions

$$\{\varphi_t(s)\}_{t\in T},\, \varphi_t = \sum_{j=1}^{N(t)} x_j(t)\mathbb{1}_{A_j}$$

such that

$$d_p(X_t, \sum_{j=1}^{N(t)} x_j(t)\Lambda(A_j) < \epsilon$$

where A_j can be taken compact.

4.2 Convergence analysis

Before we begin with our discussion on the convergence of the approximation we must make some assumptions. We will begin with considering the case $S = \mathbb{R}$. Assume first that $\Phi(x), \Phi(x,s) \uparrow \infty,\ x \to \infty, \forall s \in S$, so that $f \in L_{\Phi_p}$ implies that $|f| < \infty, \lambda$-a.e. Moreover assume that $\Lambda \in L^q(\Omega,\mathscr{F},\mathbb{P})$ for some $q > 0$ and that Condition 1 holds. What propositions 4.1 and 4.2 imply is that the modulars:

$$\rho_1(f_t) = \int_{\mathbb{R}} \Phi(|f_t(s)|)\,\lambda(\,\mathrm{d}s), \tag{4.9}$$

$$\rho_2(f_t) = \int_{\mathbb{R}} \Phi(|f_t(s)|,s)\,\lambda(\,\mathrm{d}s), \tag{4.10}$$

are suitable for measuring the error in the approximations of

$$\int_S f_t(s)\,\mathrm{d}\Lambda.$$

The case of $\Phi(x) = x^\alpha$ has already been discussed to great extent in, for example, [3] and [6].

Let us consider the homogeneous case first. For every $t \in T$ there exists a simple function ϕ_t such that $\|f_t - \phi_t\|_{\Phi_p} < \epsilon$ and we have

$$\|\int f_t\,\mathrm{d}\Lambda - \int \phi_t\,\mathrm{d}\Lambda\|_p \leq C\|f_t - \phi_t\|_{\Phi_p}.$$

Rewrite $f_t(x)$ as

$$f_t(x) = \sum_{i=1}^{N(t)} f_t(x)\mathbb{1}_{A_i} + f_t(x)\mathbb{1}_{A_t^\complement},\, A_t = \bigcup_{i=1}^{N(t)} A_i,$$

then we have

$$\|f_t - \phi_t\|_{\Phi_p} \leq \|\mathbb{1}_{A_t^\complement} f_t\|_{\Phi_p} + \|\sum_{i=1}^{N(t)}(x_i(t) - f_t(x))\mathbb{1}_{A_i}\|_{\Phi_p} \leq \tag{4.11}$$

$$\leq C'\left(\rho_i(\mathbb{1}_{A_t^\complement} f_t) + \rho_i(\sum_{i=1}^{N(t)}(x_i(t) - f_t(x))\mathbb{1}_{A_i})\right). \tag{4.12}$$

Indeed we may assume that A is contained in some large compact set C and that each $A_i = [a_i, b_i]$ which, simply put, means that we must estimate the integrals

$$\int_{|x|>K_t} \Phi(|f_t(x)|)\,\lambda(\,\mathrm{d}x), \tag{4.13}$$

$$\int_{|x|\leq K_t} \Phi(|\sum_{i=1}^{N(t)}(x_i(t) - f_t(x))\mathbb{1}_{[a_i,b_i]}|)\,\lambda(\,\mathrm{d}x), \tag{4.14}$$

for some large $K > 0$. The second integral is fairly simple as we may estimate this as

$$\int_{|x|\leq K_t} \Phi(|\sum_{i=1}^{N(t)}(x_i(t) - f_t(x))\mathbb{1}_{[a_i,b_i]}|)\,\mathrm{d}x \leq \sum_{i=1}^{N(t)} \operatorname*{ess\,sup}_{x\in[a_i,b_i]} \Phi\left(|x_i(t) - f_t(x)|\right)\lambda\{x : |x| \leq K_t\}. \tag{4.15}$$

What this really means is that we are interested in the behavior of Φ close at the origin, which describes the error of f outside some large compact set, since Φ is a continuous non-decreasing function by 3.3. Moreover, if the family $\{f_t\}_{t\in T}$ has compact support, we obtain a similar estimate provided by [6]. Of course this is not necessarily the best approximation, there are several better approximations if we know what the functional is, see [6] for a proper discussion in the case of L^r. One might attempt to use various interpolation methods such as wavelet approximations, provided that the kernel f_t is regular enough.

4.3 Simulation of some processes and a simple error estimate

This section is devoted to discuss the proposed simulation and convergence analysis. We will for simplicity only consider the Gaussian case and we will consider the Brownian motion $B = \{B_t : t \in T\}$, the fractional Brownian motion $B^H = \{B_t^H : t \in T\}$, and the Ornstein-Uhlenbeck process $X = \{X_t : t \in T\}$ (OU-process for short). We define them in terms of their covariance structure, assuming that all are centered and have unit variance:

$$c_B(t,s) = \mathbb{E}B_tB_s = \min(s,t) = \frac{|s|+|t|-|t-s|}{2}, \tag{4.16}$$

$$c_{B^H}(t,s) = \mathbb{E}B_t^HB_s^H = \frac{|s|^{2H}+|t|^{2H}-|t-s|^{2H}}{2}, H \in (0,1) \tag{4.17}$$

$$c_X(t,s) = \mathbb{E}X_tX_s = \mathrm{e}^{-|t-s|} \tag{4.18}$$

As all of the processes are Gaussian so will their **ID** random measure be. Moreover the control measure can be defined in terms of the variance of the **ID** random measure. Let M, M^H and Λ be the **ID** random measures associated with the Brownian motion, the fractional Brownian motion and the OU-process respectively. As the **ID** measures are Gaussian the linear structure of the Musielak-Orlicz space reduces to a Hilbert space, and more importantly we can define the control measures in terms of the variance

$$m(A) = \mathbb{E}|M(A)|^2,\ m^H(A) = \mathbb{E}|M^H(A)|^2,\ \lambda(A) = \mathbb{E}|\Lambda(A)|^2.$$

To obtain the spectral representation of the processes we observe that the OU-process is stationary whereas Brownian motion and fractional Brownian motion are processes with stationary increments. This means that we can consider the kernel functions:

$$f_X(t,s) = \mathrm{e}^{its},\ f_B(t,s) = \frac{\mathrm{e}^{its}-1}{\mathrm{i}s},\ \ f_B^H(t,s) = \frac{\mathrm{e}^{its}-1}{\mathrm{i}s},$$

so that we may apply techniques from the theory of distributions, see for instance [5] for a detailed discussion on the subject.

We obtain the following control measures

$$\lambda(\mathrm{d}s) = \frac{2}{1+s^2}\,\mathrm{d}s,\ m(\mathrm{d}s) = \frac{1}{2\pi}\,\mathrm{d}s,\ m^H(\mathrm{d}s) = \frac{1}{2\pi}\Gamma(1+2H)\sin(H\pi)|s|^{1-2H}\,\mathrm{d}s.$$

Our **ID** random measures are therefore distributed as

$$N(0,m(\,\mathrm{d}s)),\ N(0,m^H(\,\mathrm{d}s)), N(0,\lambda(\,\mathrm{d}s)),$$

respectively. We consider the approximation suggested in the previous section:

$$\begin{aligned}
\int_{\mathbb{R}} \mathrm{e}^{its}\Lambda(\,\mathrm{d}s) &\approx \sum_{j=1}^{N(t)} \mathrm{e}^{its_j}\Lambda(s_j,s_{j+1}),\\
\int_{\mathbb{R}} \frac{\mathrm{e}^{its}-1}{\mathrm{i}s} M(\,\mathrm{d}s) &\approx \sum_{j=1}^{N(t)} \frac{\mathrm{e}^{its_j}-1}{\mathrm{i}s_j} M(s_j,s_{j+1}),\\
\int_{\mathbb{R}} \frac{\mathrm{e}^{its}-1}{\mathrm{i}s} M^H(\,\mathrm{d}s) &\approx \sum_{j=1}^{N(t)} \frac{\mathrm{e}^{its_j}-1}{\mathrm{i}s} M^H(s_j,s_{j+1}).
\end{aligned}$$

Moreover we have

$$\begin{aligned}
&\Lambda(s_j,s_{j+1}) \in N(0,\lambda(s_j,s_{j+1})),\\
&M(s_j,s_{j+1}) \in N(0,m(s_j,s_{j+1})),\\
&M^H(s_j,s_{j+1}) \in N(0,m^H(s_j,s_{j+1})),
\end{aligned}$$

and

$$\begin{aligned}
\lambda(s_j,s_{j+1}) &= 2(\arctan(s_{j+1}) - \arctan(s_j)),\\
m(s_j,s_{j+1}) &= \frac{|(s_j,s_{j+1})|}{2\pi},\\
m^h(s_j,s_{j+1}) &= \frac{1}{2-2H}\left[s_{j+1}^{2-2H} - s_j^{2-2H}\right],
\end{aligned}$$

so that we may write the approximations as

$$\begin{aligned}
&\sum_{j=1}^{N(t)} \mathrm{e}^{its_j}\sigma_j^X \xi_j,\ \sigma_j = \sqrt{\lambda(s_j,s_{j+1})},\ \xi_j \in N(0,1),\\
&\sum_{j=1}^{N(t)} \frac{\mathrm{e}^{its_j}-1}{\mathrm{i}s_j}\sigma_j^B \xi_j,\ \sigma_J^B = \sqrt{m(s_j,s_{j+1})},\ \xi_j \in N(0,1),\\
&\sum_{j=1}^{N(t)} \frac{\mathrm{e}^{its_j}-1}{\mathrm{i}s}\sigma^{BH} \xi_j,\ \sigma_j^{BH} = \sqrt{m^H(s_j,s_{j+1})},\ \xi \in N(0,1).
\end{aligned}$$

This may not the best method for simulating Gaussian processes, especially when simulating Brownian motion, but it provides some insight on how well the approximation scheme works. Now we may apply the suggested error estimate from the previous section. Here we consider the OU-process on the interval $t \in [0,1]$. Let $g,h : \mathbb{N} \to \mathbb{N}$ be increasing functions with $h(n), g(n) \uparrow \infty$, $n \to \infty$ and let

$$\phi_t(x) = \sum_{j=1}^{g(N)} \mathrm{e}^{\mathrm{i}ts_j} \mathbb{1}(x)_{[s_j,s_{j+1}]}, K = g(N), |(s_j,s_{j+1})| \leq C/h(N).$$

We have:

$$\begin{aligned}\rho(|f_t - \phi_t|) &\leq \int_{|x|>g(N)} \left|\mathrm{e}^{\mathrm{i}tx}\right|^2 \frac{2}{1+x^2}\,\mathrm{d}x + \lambda\{|x| < g(N)\} \sum_{j=1}^{g(N)} \sup_{x\in[s_j,s_{j+1}]} \left|\mathrm{e}^{\mathrm{i}tx} - \mathrm{e}^{\mathrm{i}ts_j}\right|^2 \\ &\leq \frac{4}{g(N)} + \pi \sum_{j=1}^{g(N)} \sup_{x\in[s_j,s_{j+1}]} \left|\mathrm{e}^{\mathrm{i}tx} - \mathrm{e}^{\mathrm{i}ts_j}\right|^2,\end{aligned}$$

and Taylor expanding around s_j yields

$$|\mathrm{e}^{\mathrm{i}tx} - \mathrm{e}^{\mathrm{i}ts_j}|^2 = |(x - s_j)\mathrm{i}t\mathrm{e}^{\mathrm{i}ts_j} + R(x,s_j)(x - s_j)^2|^2,$$

where $R(x,s_j)$ is bounded. We obtain

$$\rho(|f_t - \phi_t|) \leq \frac{C}{g(N)} + C' \frac{g(N)}{h^2(N)} + C'' \frac{g(N)}{h^4(N)}.$$

In general, estimates of this type are fairly simple when the control measure is finite. The case of infinite control measure is far more intricate and relies on knowledge about the decay of the kernel at infinity (see [3] and references therein for a more conclusive discussion). A different approach might be to consider polynomial interpolation to obtain a higher degree of convergence, this can be done as we know that the continuous functions with compact support are dense whenever the **ID** random measure is homogeneous.

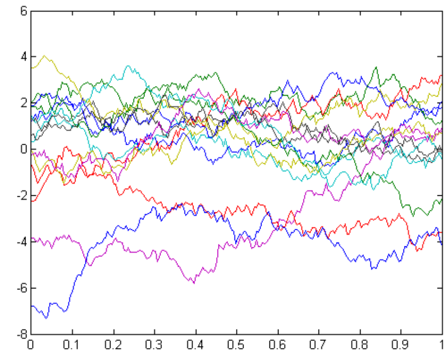

Figure 4.1: Several realizations of Ornstein-Uhlenbeck process.

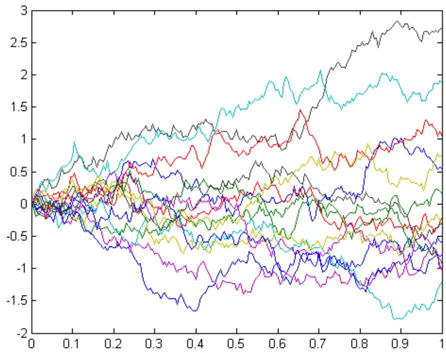

Figure 4.2: Several realizations of Brownian motion.

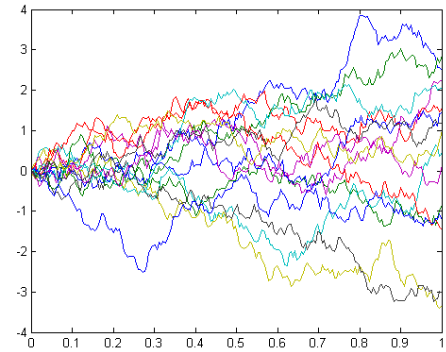

Figure 4.3: Several realizations of fractional Brownian motion.

5

Conclusions and future work

THERE are still a lot of questions to be answered relating to infinitely divisible processes. The most pressing issues are found in the field of inference. For example, how do we, given a stochastic integral process and assuming only knowledge about the noise, estimate the kernel function? Conversely, if we assume knowledge about the kernel how do we estimate the noise? If we know that the process is of a $S\alpha S$-type we know that for every such process there exists a kernel and noise. Linear and non-linear regression for α-stable random variables have been developed to some extent, see [13], but the result are unattractive because the given regression formulae are described in terms of integrals. They are not analytically nor numerically tractable and therefore difficult to use in applications. Another problem is found in the field of simulation. There are seldom any good tools to measure the error of an approximation, and the most pressing issue is that the approximation theory on Musielak-Orlicz spaces have not been developed beyond the case of a finite measure space (see [8] for a very short discussion on this matter). Hopefully the discussion on simulation and convergence provided by us will shed some light on this subject. Some more theoretical questions relating to path properties of **ID** processes are still left unanswered but are being studied intensely by Talagrand and Rosinski among others.

Bibliography

[1] Billingsley, P. (2009). *Convergence of probability measures*, vol. 493. Wiley New York.

[2] Bondesson, L. (1982). On simulation from infinitely divisible distributions. *Advances in Applied Probability*, 855–869.

[3] Cohen, S., Lacaux, C. & Ledoux, M. (2008). A general framework for simulation of fractional fields. *Stochastic processes and their Applications*, 118, 1489–1517.

[4] Folland, G.B. (1999). *Real analysis: modern techniques and their applications*, vol. 361. Wiley New York.

[5] Hörmander, L. (1983). *The Analysis of Linear Partial Differential Operators: Vol: 1: Distribution Theory and Fourier Analysis*. Springer-Verlag.

[6] Karcher, W., Scheffler, H.P. & Spodarev, E. (2013). Simulation of infinitely divisible random fields. *Communications in Statistics-Simulation and Computation*, 42, 215–246.

[7] Maruyama, G. (1970). Infinitely divisible processes. *Theory of Probability & Its Applications*, 15, 1–22.

[8] Musielak, J. (1983). *Orlicz spaces and modular spaces*, vol. 1034. Springer Berlin.

[9] Rajput, B.S. & Rosinski, J. (1989). Spectral representations of infinitely divisible processes. *Probability Theory and Related Fields*, 82, 451–487.

[10] Rao, M.M. & Ren, Z.D. (1991). *Theory of Orlicz spaces*. M. Dekker New York.

[11] Rosinski, J. (1989). On path properties of certain infinitely divisible processes. *Stochastic Processes and their Applications*, 33, 73–87.

[12] Rosinski, J. (2011). Infinitely divisible processes. `http://web.abo.fi/fak/mnf/mate/gradschool/summer_school/tammerfors2011/slides_rosinski.pdf`, [Online; accessed 19-February-2014].

[13] Samorodnitsky, G. & Taqqu, M.S. (1994). *Stable Non-Gaussian Random Processes, Stochastic Models with Infinite Variance*. Chapman & Hall New York London.

[14] Sato, K.i. (1999). *Lévy processes and infinitely divisible distributions*. Cambridge university press.

[15] Sato, K.i. (2006). Additive processes and stochastic integrals. *Illinois Journal of Mathematics*, 50, 825–851.