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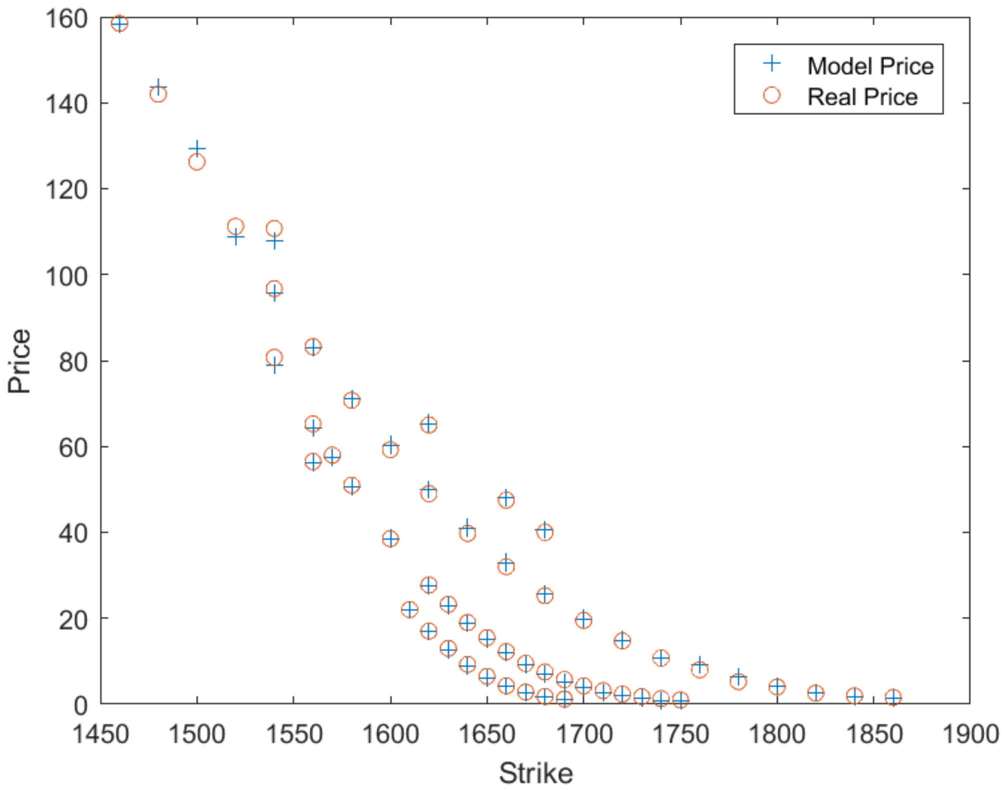

Stochastic volatility enhanced Lévy processes in financial asset pricing

Pricing European call options

Master's thesis in Engineering Mathematics and Computational Science

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CHALMERS UNIVERSITY OF TECHNOLOGY
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Cover:
Calibration result of European call option prices with the OMXS30 index as the underlying. Prices modelled by the mean corrected exponential CGMY process.

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Abstract

This report investigates several stochastic processes used for pricing European call options. The pure jump Lévy processes are the cornerstone in the different models, here presented. These do not have a Brownian motion component, therefore the stochastic volatility is instead introduced as a stochastic time-changing effect. In the paper "Stochastic volatility for Lévy processes" written by Carr, Geman, Madan and Yor, the types of stochastic time-changed mean corrected exponential Lévy processes (type 2 models) used are claimed to be martingales without proof. In the book "Financial modelling with jump processes" written by Cont and Tankov, an attempt to prove the martingale property of these has been given but is insufficient. In this report, a proof of the martingale property is made and presented. Additionally, mean corrected stochastically time-changed exponential Lévy processes (type 1 models) are introduced as proposed by Carr, Geman, Madan and Yor. The models are calibrated against OMXS30 European call options and the calibration performances of the different models are evaluated.

Keywords: Lévy process, Stochastic time-change, European call option, OMXS30, Martingale, Calibration, Option pricing, Fast fourier transform, FFT.

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Contents

1 Introduction **1**

1.1 Background . . . 1

1.2 Purpose . . . 2

1.3 Objective . . . 2

1.4 Scope . . . 2

1.5 Report structure . . . 2

2 Theory **5**

2.1 Lévy processes . . . 5

2.1.1 Basic properties . . . 5

2.1.2 Further extension of Lévy processes and its properties . . . 8

2.2 Option pricing . . . 15

2.2.1 Numerical method (FFT) . . . 18

2.3 Stochastic processes . . . 21

2.3.1 The CGMY process . . . 21

2.3.2 The Meixner process . . . 21

2.3.3 The NIG process . . . 22

2.3.4 Chronometers . . . 22

2.3.4.1 The integrated CIR process . . . 23

2.3.4.2 The integrated Gamma-OU process . . . 23

2.3.5 Stochastically time-changed Lévy processes . . . 24

3 Methods **27**

3.1 Data selection . . . 27

3.2 Numerical optimisation . . . 27

4 Results **29**

4.1 Calibrated models . . . 29

5 Conclusion **33**

5.1 Theoretical framework 33

5.2 Optimisation and data selection 34

5.3 Future work 34

Bibliography **36**

A Appendix **37**

A.1 Option quotes 37

1

Introduction

1.1 Background

The well-known Black and Scholes model, see [3], has for a long time been used in the pricing of financial option derivatives. Unfortunately, the model's accuracy has been questioned and is today known for its drawbacks in financial option pricing. This has led to a research intensive field of financial mathematics to find more accurate models describing option prices. Most models rely on finding another suitable stochastic process to describe the underlying asset of the derivatives. Such improved models often include Lévy jump processes, see [18]. It has been argued that a price process need not to have a diffusion component but has to have a jump component, see [9]. Examples of these kinds of processes fulfilling these conditions are the normal-inverse gaussian (NIG) process and the tempered stable processes, in particular a special case of the tempered stable process called the CGMY process, as proposed by [6]. Note that these processes are pure Lévy jump processes with infinite activity (for the CGMY process when $1 < Y < 2$ where Y is a model parameter, see Section 2.3.1) and henceforth are able cover the effects of any diffusion component, see [6].

Even though these pure jump processes model option prices well, there is room for improvements. In particular, the variation of the option price over different maturities is not well described by pure Lévy jump processes, as explained by [6]. A likely reason for this is that the volatility of the underlying asset usually is both stochastic and inhibits clustering properties (sometimes also called volatility persistence).

Stochastically time-changed exponential Lévy processes have been claimed to possess martingale property, see [5]. However, no proof has been provided for type-2 processes while proving almost martingale property for type-1 processes (see Section 2.3.5 for the definition of process types). An insufficient proof was found in [7], which only considers bounded stopping times. An attempt to a satisfactory proof of martingale property of stochastically time-changed exponential Lévy processes is presented in this report. Some of the models calibrated are of the mentioned type, which essentially possess the martingale property required in arbitrage-free option pricing.

The different processes in this report are calibrated against European call options with the OMXS30 index as the underlying asset. The price data used can be found in Appendix A.1.

1.2 Purpose

The work involve finding suitable stochastic volatility enhanced Lévy processes for option pricing and determining the validity of these. Whether the models are applicable and performs better than the non volatility enhanced versions of the processes, will be assessed.

1.3 Objective

It will be investigated whether there are pre-existing stochastic volatility enhanced Lévy process models to better describe option prices. The models will be compared with their corresponding Lévy process models without the stochastic volatility enhancing properties. The calibration results will be presented.

1.4 Scope

All models are calibrated against European call options based on the OMXS30 index as the underlying asset. The time to maturity will be at least one calendar month for all options used for the calibration.

1.5 Report structure

The outline of the report is the following

- Chapter 2 - Theory; Presenting Lévy processes, stochastic time-changed Lévy processes and option pricing. Numerical method for the calculation of option prices with fast Fourier transform (FFT) is presented.
- Chapter 3 - Methods; Describing the data selection procedure. Presenting the numerical optimisation procedure used for calibrating the models against the selected options data.
- Chapter 4 - Result; The resulting calibrated model parameters are presented.

- Chapter 5 - Conclusion; Discussing important aspects of the models and optimisation procedure. Suggestions for future investigations are presented.

2

Theory

2.1 Lévy processes

In this chapter the basic properties of Lévy processes, required for the construction of option price models, are presented.

2.1.1 Basic properties

Recall the definition of a Lévy process.

Definition 2.1 (Lévy process, [7]). A stochastic càdlàg[1] process $(X_t, t \geq 0)$ in $\mathbb{R}^d$ is called a Lévy process if it has the following properties

1. $X_0 = 0$ a.s.

2. Independent increments: given $n \in \mathbb{N}$ and any partition of time $0 \leq t_0 < \cdots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

3. Stationary increments: given $t \geq 0, h > 0$, the law of $X_{t+h} - X_t$ does not depend on t.

4. Stochastic continuity: $\forall \varepsilon > 0, \forall t \geq 0, \lim_{h \to 0^+} \mathbb{P}(|X_{t+h} - X_t| \geq \varepsilon) = 0$.

Remark 2.2. Condition 4 about stochastic continuity should not be confused with path-wise continuity of the process. The stochastic continuity condition given in the definition of Lévy processes does not allow the process to suddenly jump at a given deterministic time. Although, if the time itself is stochastic, jumps are allowed (see Example 2.5).

Remark 2.3. If condition 3 is removed, the process is defined to be an additive process. An additive process is sometimes called an inhomogeneous Lévy process.

[1] Abbreviation for the French sentence; continue à droite, limite à gauche (right continuous with left limits).

Example 2.4. The standard Brownian motion process is a Lévy process. Standard Brownian motion processes with or without drift are the only path-wise continuous Lévy processes.

Example 2.5. The Poisson process is a Lévy process. Note that jumps are evident in the Poisson process by definition. This does not violate the stochastic continuity property of Lévy processes because the time at which the jumps arrive are stochastic (i.e. not deterministic).

Definition 2.6 (Infinite divisibility, [1])**.** The law of a random variable X is said to be infinite divisible if for all $n \in \mathbb{N}$ there exists i.i.d. random variables $Y_1^{(n)}, \ldots, Y_n^{(n)}$ such that the law of their sum equals the law of X. That is if $\forall n \in \mathbb{N}, X \stackrel{\mathrm{d}}{=} Y_1^{(n)} + \cdots + Y_n^{(n)}$.

Remark 2.7. Often, the random variable X itself is called infinite divisible if the conditions in Definition 2.6 hold.

Example 2.8. Let $L = (L_t, t \geq 0)$ be a Lévy process. Then L_t is infinite divisible for each $t > 0$. Indeed, the random variable L_t can be expressed as a telescope sum for each $n \in \mathbb{N}$ by noting that $L_t = (L_t - L_{t/2}) + (L_{t/2} - L_{t/3}) + \cdots + (L_{t/(n-1)} - L_{t/n}) + (L_{t/n} - L_0)$. Thus, the conclusion follows.

It is necessary to be acquainted with the definition of a measure to characterise the random jumps of Lévy processes to be able to move on to an important theorem.

Definition 2.9 (Lévy measure, [1, 16])**.** A Borel measure ν defined on $\mathbb{R}^d$ is called a Lévy measure if

$$\nu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\, \nu(dx) < \infty.$$

Example 2.10. A Lévy measure of the form $\nu(dx) = a \exp{(-bx)} x^{-1} \mathbf{1}_{(x>0)}\, dx$, where $\mathbf{1}_{(x>0)}$ is the indicator function which equals 1 for $x > 0$ and 0 otherwise, describes the intensity of the jumps of size x of a process with marginals following the Gamma(a, b) distribution.

Theorem 2.11 (Lévy-Klintchine formula, [7])**.** *A random variable X on $\mathbb{R}^d$ with infinite divisible law μ_X has the characteristic function*

$$\mathbb{E}\left[e^{iu^\intercal X}\right] = \phi_X(u) = e^{\psi(u)},\ u \in \mathbb{R}^d$$

where

$$\psi(u) = -\frac{1}{2} u^\intercal A u + i\gamma^\intercal u + \int_{\mathbb{R}^d} (e^{iu^\intercal x} - 1 - iu^\intercal x \mathbf{1}_{|x| \leq 1})\, \nu(dx)$$

for which A is a symmetric positive $d \times d$ matrix, $\gamma \in \mathbb{R}^d$ and ν is the Lévy measure of the law μ_X.

Proof. The proof can be found in [13]. □

Remark 2.12. The triplet (γ, A, ν) is called the generating triplet of the law μ_X. In the one dimensional case, they are often written as (γ, σ^2, ν).

Remark 2.13. In some textbooks, see [7, 1, 16], $\psi(u)$ is referred to as the characteristic exponent or Lévy exponent.

Corollary 2.14. *If $X = (X_t, t \geq 0)$ is a Lévy process, the generating triplet for each X_t is given by $(t\gamma, tA, t\nu)$ and for the one dimensional case $(t\gamma, t\sigma^2, t\nu)$.*

Proof. See [13]. □

Remark 2.15. Given corollary 2.14, it is noted that the characteristic function of L_t, $L = (L_t, t \geq 0)$ being a Lévy process, is $\phi_{L_t}(u) = e^{t\psi(u)}$ where $\psi(u)$ is the characteristic exponent of L_1.

Lemma 2.16. *Let $X = (X_t, t \geq 0)$ be a Lévy process with the characteristic exponent $\psi(u)$. Assume further that the moment generating function for X_t exists for each $t \geq 0$. Then*

$$\frac{e^{X_t}}{\mathbb{E}\left[e^{X_t}\right]} = e^{X_t - t\psi(-i)}$$

where $e^{X_t - t\psi(-i)}$ is a martingale with respect to the natural filtration of X.

Proof. By Remark 2.15 it follows that

$$\mathbb{E}\left[e^{X_t}\right] = \mathbb{E}\left[e^{i(-i)X_t}\right] = e^{t\psi(-i)}.$$

Thus,

$$\frac{e^{X_t}}{\mathbb{E}\left[e^{X_t}\right]} = e^{X_t - t\psi(-i)}.$$

Now, denote the natural filtration of X up to time t as $\mathcal{F}_t^X$. Recall the independent and stationary increment conditions of Lévy processes stated in Definition 2.1. Then for $0 \leq s \leq t$,

$$\begin{aligned}
\mathbb{E}\left[e^{X_t - t\psi(-i)} \;\middle|\; \mathcal{F}_s^X\right] &= \mathbb{E}\left[e^{X_t - X_s + X_s - t\psi(-i)} \;\middle|\; \mathcal{F}_s^X\right] \\
&= e^{X_s - t\psi(-i)}\mathbb{E}\left[e^{X_t - X_s} \;\middle|\; \mathcal{F}_s^X\right] \\
&= e^{X_s - t\psi(-i)}\mathbb{E}\left[e^{X_t - X_s}\right] \\
&= e^{X_s - t\psi(-i)}\mathbb{E}\left[e^{X_{t-s}}\right] \\
&= e^{X_s - t\psi(-i)}e^{(t-s)\psi(-i)} \\
&= e^{X_s - s\psi(-i)}.
\end{aligned}$$

Conclusion follows. □

Remark 2.17. The result given in Lemma 2.16 is a standard result usually given as exercises in textbooks. The procedure is sometimes called to mean correct the exponential Lévy process to retrieve a martingale process.

2.1.2 Further extension of Lévy processes and its properties

Before proving the martingale property for stochastically time-changed mean corrected exponential Lévy processes, some important definitions are introduced.

Definition 2.18 (Power set, [10])**.** Let Ω be a sample space. The power set of Ω is defined as $\mathcal{P}(\Omega) = \{A \colon A \subseteq \Omega\}$.

Definition 2.19 (σ-field, [10])**.** Given sample space Ω. The set $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ non-empty, is called a σ-field given that the followings are satisfied

1. $\Omega \in \mathcal{A}$.

2. $A \in \mathcal{A} \implies A^{\mathsf{c}} \in \mathcal{A}$.

3. $A_n \in \mathcal{A}, n \geq 1 \implies \bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$.

Remark 2.20. The intersection of two σ-fields is still a σ-field.

Remark 2.21. The smallest (by intersection) σ-field containing a set $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is denoted by $\sigma(\mathcal{A})$. Given the σ-fields $\mathcal{G}$ and $\mathcal{F}$, a shorthand notation for the generated σ-field of their union is $\sigma(\mathcal{G} \cup \mathcal{F}) = \mathcal{G} \vee \mathcal{F}$. For efficient typing reasons, the smallest σ-field for which a process X is defined on is denoted by $\sigma(X)$ or $\sigma(X_t, t \geq 0)$.

A theorem which will come in handy for proving the martingale property of a stochastically time-changed mean corrected exponential Lévy process is Doob's optional stopping theorem[2]. For it to make any sense, stopping times have to be defined.

Definition 2.22 (Stopping time, [13])**.** A random variable τ is called an $\mathcal{F}$-stopping time whenever $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in T$ where T is some (ordered, $\leq$) index set. Its associated σ-field is denoted by $\mathcal{F}_\tau = \{A \in \mathcal{F}\colon A \cap \{\tau \leq t\} \in \mathcal{F}_t, t \in T\}$.

Remark 2.23. For a càdlàg process X_t on the index set T adapted to $\mathcal{F}_t$, X_τ is $\mathcal{F}_\tau$-measurable for a T-valued optional time τ. Note that T can be either countable or $T = \mathbb{R}_+$, see [13].

Finally, Doob's optional stopping theorem can be presented.

Theorem 2.24 (Doob's optional stopping theorem, [1])**.** *Let $X = (X_t, t \geq 0)$ be a càdlàg martingale process with respect to the filtration $\mathcal{F}_t$ and S and T bounded*[3] *$\mathcal{F}$-stopping times where $S \leq T$ a.s., then X_S and X_T are both integrable with*

$$\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S \quad a.s.$$

Proof. This is a well established result and the proof can be found in almost any textbook about martingales and stochastic processes. The reader can find the proof in [13, 15]. □

Fortunately, the boundedness condition of the stopping times can be relaxed. The property of uniform integrability of a set of random variables is an important and useful concept which is needed to be able to relax the boundedness condition of the stopping times in Doob's optional stopping theorem. The concept of uniform integrability was found to be a sufficient additional property of an almost surely convergent sequence of L^1 random variables to also be L^1 convergent (to the same limit), see [10].

Definition 2.25 (Uniform integrability, [10])**.** A set of random variables $\mathcal{X}$ is called uniformly integrable if and only if

$$\forall \varepsilon > 0, \exists K \geq 0, \forall X \in \mathcal{X},\ \mathbb{E}\left[|X|\mathbf{1}_{|X|\geq K}\right] < \varepsilon$$

where $\mathbf{1}_{|X|\geq K}$ is the indicator function.

It could be difficult to verify uniform integrability of a set directly through the definition, a sufficient condition can therefore come in handy.

[2] Also called the optional sampling theorem in various textbooks.

[3] There exists $M, N > 0$ such that $S \leq M$ and $T \leq N$ a.s.

Theorem 2.26 (Sufficient condition for uniform integrability, [10])**.** *Let $\mathcal{X}$ be a set of random variables. Further assume* $\sup_{X\in\mathcal{X}} \mathbb{E}[|X|^p] < \infty$ *for some* $p > 1$*. Then $\mathcal{X}$ is uniformly integrable.*

Proof. See [10]. □

Remark 2.27. Note that there could exist uniformly integrable sets which does not satisfy the conditions given in Theorem 2.26.

A more general optional stopping theorem can now be presented which includes conditions on uniform integrability.

Theorem 2.28 (Generalised Doob's optional stopping theorem, [13])**.** *Let the càdlàg process X be a submartingale, supermartingale or a martingale with respect to the right continuous filtration $(\mathcal{F}_t)_{t\geq 0}$. Further assume σ and τ to be two $\mathcal{F}_t$-stopping times, where τ is bounded. Then X_τ is integrable and*

$$\begin{aligned} X_{\sigma\wedge\tau} &\leq \mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \quad \textit{a.s.}, \\ X_{\sigma\wedge\tau} &\geq \mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \quad \textit{a.s.} \end{aligned}$$

or

$$X_{\sigma\wedge\tau} = \mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \quad \textit{a.s.},$$

respectively. The statements hold for unbounded τ if and only if X is uniformly integrable.

Proof. The proof for the case when X is a submartingale can be found in [13]. The proof can easily be extended to include the cases of supermartingales and martingales.

Consider the case when X is a supermartingale. Then $-X$ is a submartingale and thus

$$-X_{\sigma\wedge\tau} \leq \mathbb{E}[-X_\tau \mid \mathcal{F}_\sigma] \quad \text{a.s.} \iff X_{\sigma\wedge\tau} \geq \mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \quad \text{a.s.}$$

Now consider the case when X is a martingale. Then it is both a sub- and supermartingale at the same time. Henceforth,

$$\begin{cases} X_{\sigma\wedge\tau} \leq \mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \quad \text{a.s.} \\ X_{\sigma\wedge\tau} \geq \mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \quad \text{a.s.} \end{cases} \implies X_{\sigma\wedge\tau} = \mathbb{E}[X_\tau \mid \mathcal{F}_\sigma] \quad \text{a.s.}$$

Conclusion follows. □

Remark 2.29. If $\sigma \leq \tau$ a.s. then $X_{\sigma\wedge\tau} = X_\sigma$ a.s.

To continue with the presentation of properties of martingales, some additional definitions and theorems are needed.

Definition 2.30 (π-system, [10])**.** Given sample space Ω. The set $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ non-empty, is called a π-system given that the following is satisfied

1. $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$.

Remark 2.31. In words, a π-system is a non-empty set which is closed under finite intersections.

Definition 2.32 (λ-system, [10])**.** Given sample space Ω. The set $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ non-empty, is called a λ-system given that the followings are satisfied

1. $\Omega \in \mathcal{A}$.
2. $A, B \in \mathcal{A}, B \subseteq A \implies A \setminus B \in \mathcal{A}$.
3. $A_n \in \mathcal{A}, n \geq 1, A_1 \subseteq A_2 \subseteq \dots \implies \bigcup_{n=1}^\infty A_n \in \mathcal{A}$.

There is also an equivalent definition, given by replacing condition 2 and condition 3 by

2. $A \in \mathcal{A} \implies A^c \in \mathcal{A}$.
3. $A_n \in \mathcal{A}, n \geq 1, A_i \cap A_j = \emptyset, i \neq j \implies \bigcup_{n=1}^\infty A_n \in \mathcal{A}$.

Remark 2.33. A λ-system is sometimes also called a Dynkin[4] system.

Remark 2.34. In words, a λ-system is a non-empty set containing Ω which is closed under increasing limits and by difference.

Remark 2.35. The smallest λ-system containing a set $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ is denoted by $\mathcal{D}(\mathcal{A})$.

Theorem 2.36 (Monotone class theorem, [12])**.** *Given a sample space Ω. Let $\mathcal{C} \subseteq \mathcal{P}(\Omega)$ be a π-system containing Ω. Then $\mathcal{D}(\mathcal{C}) = \sigma(\mathcal{C})$.*

Proof. See [12]. □

Theorem 2.36 is an important result often used in probability theory. It will be used to prove that a martingale with respect to a filtration $\mathcal{F}_t$ is still a martingale with respect to an enlarged filtration $\mathcal{F}_t \vee \mathcal{G}$ given that $\mathcal{G}$ and $\mathcal{F}_t$ are independent at every $t \in T$, T being an (ordered, $\leq$) index set. Before continuing, the dominated convergence theorem and a lemma is introduced.

[4]After the mathematician Eugene Dynkin, 1924-2014.

Theorem 2.37 (Dominated convergence theorem, [10])**.** *Let $X_n \to X$ a.s. as $n \to \infty$ and further suppose $|X_n| \leq Y$ a.s. for all n where Y is an integrable random variable. Then*

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \mathbb{E}\left[\lim_{n\to\infty} X_n\right] = \mathbb{E}[X].$$

Proof. See [10]. $\square$

Lemma 2.38. *Let X and Y be two random variables on the probability space $(\Omega, \mathcal{F}, P)$, satisfying $\mathbb{E}[X] = \mathbb{E}[Y]$. Then $\mathcal{B} = \{B \in \mathcal{F} \colon \mathbb{E}[X\mathbf{1}_B] = \mathbb{E}[Y\mathbf{1}_B]\}$ is a λ-system.*

Proof. Because $\mathbb{E}[X] = \mathbb{E}[Y]$ is satisfied, it is trivial that $\Omega \in \mathcal{B}$.

Now consider $A, B \in \mathcal{B}$ and that $B \subseteq A$ then it will be shown that $A \setminus B \in \mathcal{B}$. Note that $A \setminus B = A \cap B^{\mathsf{c}}$. It follows that

$$\begin{aligned}
\mathbb{E}[(X-Y)\mathbf{1}_{A\cap B^{\mathsf{c}}}] &= \mathbb{E}[(X-Y)\mathbf{1}_A\mathbf{1}_{B^{\mathsf{c}}}] \\
&= \mathbb{E}[(X-Y)\mathbf{1}_A(1-\mathbf{1}_B)] \\
&= \mathbb{E}[(X-Y)(\mathbf{1}_A-\mathbf{1}_A\mathbf{1}_B)] \\
&= \mathbb{E}[(X-Y)(\mathbf{1}_A-\mathbf{1}_{A\cap B})] \\
&= [\text{By recalling that } B \subseteq A] \\
&= \mathbb{E}[(X-Y)(\mathbf{1}_A-\mathbf{1}_B)] \\
&= [\text{By recalling that } A, B \in \mathcal{B}] \\
&= 0.
\end{aligned}$$

Finally, given a sequence $(A_n)_{n=1}^{\infty}$ where $A_n \in \mathcal{B}, \forall n \in \mathbb{N}$ and where $A_n \subseteq A_{n+1}, \forall n \in \mathbb{N}$ it is to be shown that $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$. Note that $\bigcup_{i=1}^{n} A_i = A_n \in \mathcal{B}$ for all $n \in \mathbb{N}$ because $A_i \subseteq A_{i+1}$ and $A_i \in \mathcal{B}$ for all $i \in \mathbb{N}$. This implies that

$$\mathbb{E}\left[(X-Y)\mathbf{1}_{\bigcup_{i=1}^{n} A_i}\right] = \mathbb{E}[(X-Y)\mathbf{1}_{A_n}] = 0, \quad \forall n \in \mathbb{N}$$

so

$$\lim_{n\to\infty} \mathbb{E}\left[(X-Y)\mathbf{1}_{\bigcup_{i=1}^{n} A_i}\right] = \lim_{n\to\infty} \mathbb{E}[(X-Y)\mathbf{1}_{A_n}] = 0.$$

Further note that $\mathbf{1}_{A_n} \to \mathbf{1}_{\bigcup_{i=1}^{\infty} A_i}$ a.s. as $n \to \infty$ which implies that $(X-Y)\mathbf{1}_{A_n} \to (X-Y)\mathbf{1}_{\bigcup_{i=1}^{\infty} A_i}$ a.s. as $n \to \infty$. Additionally, it is evident that $|(X-Y)\mathbf{1}_{A_n}| \leq |X-Y|$ a.s. $\forall n \in \mathbb{N}$ and that $|X-Y|$ is integrable because X and Y are integrable.

Therefore, it follows by the dominated convergence theorem that

$$0 = \lim_{n\to\infty} \mathbb{E}[(X-Y)\mathbf{1}_{A_n}] = \mathbb{E}\left[\lim_{n\to\infty}(X-Y)\mathbf{1}_{A_n}\right] = \mathbb{E}\left[(X-Y)\mathbf{1}_{\bigcup_{n=1}^{\infty} A_n}\right],$$

thus $\bigcup_{n=1}^{\infty} A_n \in \mathcal{B}$. Conclusion follows. □

Preservation of martingale property with respect to an enlarged filtration by an independent σ-field can now be proven.

Theorem 2.39. *Let M_t be a martingale with respect to the filtration $\mathcal{F}_t$ on the probability space $(\Omega, \mathcal{A}, P)$ and let $\mathcal{G} \subseteq \mathcal{A}$ be a σ-field independent of $\mathcal{F}_t$ for all $t \geq 0$. Then M_t is a martingale with respect to the filtration $\mathcal{F}_t \vee \mathcal{G}$.*

Proof. Define the set $\mathcal{B} = \{F \cap G \colon F \in \mathcal{F}_s, G \in \mathcal{G}\}$ where $t > s$, which is evidently a π-system containing Ω. Note that $\sigma(\mathcal{B}) = \mathcal{F}_s \vee \mathcal{G}$. Because M_t is a martingale with respect to the filtration $\mathcal{F}_t$ it follows that for $F \in \mathcal{F}_s$

$$\mathbb{E}[M_t \mathbf{1}_F] = \mathbb{E}[M_s \mathbf{1}_F] \iff \mathbb{E}[(M_t - M_s)\mathbf{1}_F] = 0$$

and so, for any set in $\mathcal{B}$, represented by $F \cap G$ where $F \in \mathcal{F}_s$ and $G \in \mathcal{G}$, it follows that

$$\mathbb{E}[(M_t - M_s)\mathbf{1}_{F\cap G}] = \mathbb{E}[(M_t - M_s)\mathbf{1}_F \mathbf{1}_G] = \mathbb{E}[(M_t - M_s)\mathbf{1}_F]\mathbb{E}[\mathbf{1}_G] = 0.$$

It was shown in Lemma 2.38 that the set of events (call it $\mathcal{C}$) satisfying $\mathbb{E}[(M_t - M_s)\mathbf{1}_C] = 0$ is a λ-system. Because $\mathcal{B} \subseteq \mathcal{C}$ as shown above, it is also true that $\mathcal{D}(\mathcal{B}) \subseteq \mathcal{C}$. By Theorem 2.36 it follows that $\sigma(\mathcal{B}) = \mathcal{D}(\mathcal{B})$ and so $\mathcal{F}_s \vee \mathcal{G} = \mathcal{D}(\mathcal{B})$. Therefore, it can be concluded that the martingale property is true for $\mathcal{F}_s \vee \mathcal{G}$. Note that the integrability and measurable property is inherited with respect to the enlarged filtration because $\mathcal{F}_t \subseteq \mathcal{F}_t \vee \mathcal{G}$. □

Remark 2.40. The importance of Theorem 2.39 is substantial in several areas. It comes in handy especially as a part in proof techniques. In particular, it allows for certain extensions of filtrations such that the extended filtrations inherits some new, possibly sought for properties, while still preserving the martingale property of a certain process.

Finally, a theorem can now be presented giving the martingale property of a composite process if certain conditions are met. The theorem is the central result in this report. It gives the foundations for the martingale property of the type-2 models described in Section 2.3.5. This is important since the martingale property of processes is a necessary property when assuming arbitrage-free markets. More about this in Section 2.2.

Theorem 2.41 (Martingale property for composite mean corrected exponential Lévy processes[5])**.** *Let $M = (M_t, t \geq 0)$ be a mean corrected exponential Lévy process with respect to its natural filtration $\mathcal{F}_t^M$ and $v = (v_t, t \geq 0)$ be an increasing, path-wise continuous and $\mathcal{F}_t^v$-adapted process independent of M and where $v_0 = 0$. Further assume that moment generating function for v_t exists for all $t \geq 0$. Then the composite process $M_v = (M_{v_t}, t \geq 0)$, with its natural filtration $\mathcal{F}_t^{M_v}$, is a martingale with respect to the filtration $\mathcal{F}_t = \mathcal{F}_t^{M_v} \vee \mathcal{F}_t^v$.*

Proof. Fix $s < t$ and define $\overline{\mathcal{F}}_u = \mathcal{F}_u^M \vee \sigma(v_{\tilde{s}}, \tilde{s} \geq 0)$, which is the extension of the natural filtration of M with the sigma field generated from the entire trajectory[6] of the independent process v. It follows immediately that v_s and v_t are $\overline{\mathcal{F}}_u$-stopping times. Additionally, it is noted that $v_t \wedge r$ is a bounded $\overline{\mathcal{F}}_u$-stopping time for all $r \geq 0$. Furthermore, by Theorem 2.39 it is clear that M_u is a martingale with respect to the filtration $\overline{\mathcal{F}}_u$. By Theorem 2.24, it follows that the process $M_{r\wedge v_t}$ is a martingale with respect to the filtration $\overline{\mathcal{F}}_{r\wedge v_t}$, where it shall be noted that r is the independent variable here. Also note that the process $\tilde{M} = (M_{r\wedge v_t}, r \geq 0)$ is adapted to the filtration $(\overline{\mathcal{F}}_{r\wedge v_t})_{r=0}^{\infty}$. It follows that the process $M_{r\wedge v_t}$ is uniformly integrable because for some $p > 2$ it follows that

$$
\begin{aligned}
\mathbb{E}[|M_{r\wedge v_t}|^{p/2}] &= \mathbb{E}\left[\exp\left(\frac{p}{2}(X_{r\wedge v_t} - r \wedge v_t \psi_X(-i))\right)\right] \\
&= \mathbb{E}\left[\exp\left(\frac{p}{2}X_{r\wedge v_t} - \frac{r\wedge v_t}{2}\psi_X(-ip)\right)\right. \\
&\qquad \left.\times \exp\left(\frac{r\wedge v_t}{2}\psi_X(-ip) - p\frac{r\wedge v_t}{2}\psi_X(-i)\right)\right] \\
&\leq \mathbb{E}\left[\exp\left(pX_{r\wedge v_t} - r\wedge v_t\psi_X(-ip)\right)\right]^{1/2} \\
&\qquad \times \mathbb{E}\left[\exp\left(r\wedge v_t\psi_X(-ip) - pr\wedge v_t\psi_X(-i)\right)\right]^{1/2} \\
&= \mathbb{E}\left[\exp\left(r\wedge v_t(\psi_X(-ip) - p\psi_X(-i))\right)\right]^{1/2} \\
&\leq \begin{cases} \mathbb{E}\left[\exp\left(v_t(\psi_X(-ip) - p\psi_X(-i))\right)\right]^{1/2}, & \psi_X(-ip) - p\psi_X(-i)) > 0 \\ 1, & \psi_X(-ip) - p\psi_X(-i)) \leq 0 \end{cases}
\end{aligned}
$$

and by using the fact that the moment generating function for v_t exists and by

[5] Please note that a similar result has been shown in [7] but it is not a sufficient proof for more general processes v. Furthermore, in [5] the martingale property has been taken for granted without actually proving it. The theorem provided here is therefore an extension of the theorem provided by [7] for the case when M is a mean corrected exponential Lévy process and a proof of the claim made in [5].

[6] The author of this report would like to credit Peter Tankov for his explanation and hint about the extension of a filtration in this way.

recalling Theorem 2.26. Note that $\mathbb{E}\left[\exp\left(pX_{r\wedge v_t} - r \wedge v_t\psi_X(-ip)\right)\right] = 1$ because $L_r = \exp\left(pX_r - r\psi_X(-ip)\right)$ is a martingale (which can easily be verified following the same procedure as done in the proof of Lemma 2.16), and so by Theorem 2.24, $L_{r\wedge v_t}$ is a martingale too.

Now by recalling Theorem 2.28, it follows immediately that

$$\mathbb{E}[M_{v_t} \mid \overline{\mathcal{F}}_{v_s}] = M_{v_s} \quad \text{a.s.}$$

It therefore follows that

$$\mathbb{E}[M_{v_t} \mid \mathcal{F}^M_{v_s} \vee \mathcal{F}^v_t] = \mathbb{E}[\mathbb{E}[M_{v_t} \mid \overline{F}_{v_s}] \mid \mathcal{F}^M_{v_s} \vee \mathcal{F}^v_t] = M_{v_s} \quad \text{a.s.}$$

henceforth,

$$\begin{aligned}
\mathbb{E}[M_{v_t} \mid \mathcal{F}_s] &= \mathbb{E}[\mathbb{E}[M_{v_t} \mid \mathcal{F}_s \vee \mathcal{F}^v_t] \mid \mathcal{F}_s] \\
&= \mathbb{E}[\mathbb{E}[M_{v_t} \mid \mathcal{F}^M_{v_s} \vee \mathcal{F}^v_t] \mid \mathcal{F}_s] \\
&= \mathbb{E}[M_{v_s} \mid \mathcal{F}_s] = M_{v_s} \quad \text{a.s.}
\end{aligned}$$

and the conclusion follows. □

Remark 2.42. Parts of the proof above has been inspired by [7, pp. 492-493] and [17, p. 58].

2.2 Option pricing

Moving on to some financial applications, a brief introduction of a cornerstone in financial mathematics is presented, the first fundamental theorem of asset pricing. The theorem presents an important result concerning arbitrage-free markets.

Theorem 2.43 (First fundamental theorem of asset pricing, [19])**.** *If a market model has a risk-neutral probability measure, then it does not admit arbitrage.*

Proof. See [19]. □

The first fundamental theorem of asset pricing has been further generalized by introducing concepts such as no free lunch with vanishing risk, sigma-martingales and semi-martingales where also the converse has been shown to be true, see [8]. For pure Lévy processes, the situation of arbitrage-free markets have been studied extensively, see [7].

However, practitioners do not always strictly follow the first fundamental theorem of asset pricing. Instead, they rely on the fact that martingales are models of so called fair games. That is, given the history of outcomes, a winning bet strategy does not exist, neither for the participant nor the counterpart. This fact is related to the fundamental theorem of asset pricing, showing that if an equivalent martingale measure exists (risk-neutral measures are equivalent martingale measures for discounted processes) such that the discounted[7] price process of the assets in the market are martingales, then the market in the physical world is free of arbitrage. For practitioners, there are two main paths of choice to price an asset.

1. A model is defined (the process describing the underlying asset) on the probability space $(\Omega, \mathcal{F}, P)$. An equivalent risk-neutral measure Q is found to exist. Then, by fundamental theorem of asset pricing, the market described by the model on the probability space $(\Omega, \mathcal{F}, P)$ is arbitrage-free, in accordance with observations of real markets. Only one of the equivalent risk-neutral measures correctly describes the market's asset prices. There are several ways of choosing the measure that correctly describes the market's asset prices, see [18, 14].

2. The assets are assumed to be traded fairly, and thus a martingale process is chosen to describe the underlying asset which is calibrated against real asset price data. If the model fits the asset prices well enough, the practitioner use the result for dynamic hedging purposes or to find abnormal prices in the market which can be exploited to buy or sell underpriced or overpriced assets to make a profit.

Usually, low volume traded assets have abnormal prices, as they do not reach price equilibrium fast enough with respect to the speed of changes of the underlying asset. As such, the market has implicitly been assumed to be frictionless, in the sense that there is no cost of trading options (no spread in option prices) as well as that an option can be immediately sold or bought from the market without delay. More about implicit assumptions made on markets where the first fundamental theorem of asset pricing hold can be found in [18]. There have also been some recent extensions of the theorem which allow for small violations of these assumptions.

For well studied processes, the practitioners path of choice described by 1 would be preferred. Selecting the correct equivalent martingale measure ensures the possibility to construct a replicating portfolio of financial derivatives, i.e. financial derivatives can be priced in a risk-neutral manner.

[7]Assuming a market has at least one riskless asset, such as a very safe bond, the price process is discounted by the amount that could have been earned by the riskless asset.

For newer models, the description in 2 has to be relied on. Unfortunately, the description given by 2 does not necessary imply that financial derivatives are priced by the means of risk-neutral pricing, i.e. by martingale pricing methods. This is due to the fact that replicating portfolios describing the derivatives do not necessary exist for this case. Nevertheless, the practitioner could still assume that the derivatives are priced in a similar manner as in case 1. If the calibrated model describes the prices of a wide variety of financial derivatives well, such as European call options (see Equation 2.2.1), then the practitioner can assume having a clear description of the prices of the derivatives not included in the dataset used for calibration. Another argument (perhaps a harsh one to motivate the use of strategy 2) is to simply assume that the constructed and calibrated martingale model (in the risk-neutral world) has an equivalent measure describing the market in the physical world well enough (by means of statistical terms of historical data), but without proving it. As stochastically time-changed mean corrected exponential Lévy processes are a relatively new type of models, strategy 2 will be considered. This is also what has been proposed in [18], see Section 2.3.5.

This report will focus on European call options. These are contracts which give the holder of the contract an option, but not an obligation, to buy the underlying asset at a predetermined future date, called the time of maturity T. The option contract states at what price K, called the strike price, the underlying asset can be bought for at time T. Denoting the underlying asset's price process as $S(t)$ at time t, the pay-off of the contract at time of maturity is $(S(T)-K)_+ = \max(S(T)-K, 0)$. Note that if the price of the underlying asset is lower than K, the holder of the contract will not exercise the option, as it is cheaper to buy the underlying asset from the stock-market. The writer of the option on the other hand gives away the option to buy an underlying asset to the holder, henceforth takes a risk of losing an asset in the future for a lower price. Therefore the writer of the option will demand a premium, i.e. the price of the option contract itself. A fair price of the option contract depends on the pay-off of the contract. It can be shown, assuming the first fundamental theorem of asset pricing holds, that the fair price C of the European call option at time t in an arbitrage-free market is (assuming the riskless interest rate and the dividend rate to be constant r and q, respectively)

$$C(S(t), t, T, K) = e^{-(r-q)(T-t)}\mathbb{E}^Q[(S(T)-K)_+ \mid \mathcal{F}_t] \tag{2.2.1}$$

where Q is the risk-neutral measure attained by the market, and $\mathcal{F}_t$ a filtration of the market information up to time t. For more detailed explanation of the fundamental theorem of asset pricing, see [19].

2.2.1 Numerical method (FFT)

Closed form analytical expressions are always tractable to be able to quickly calculate option prices. Unfortunately, the option pricing used in Equation 2.2.1 does not always yield analytical closed form expressions, it depends on which process has been chosen to model the behaviour of the underlying asset price.

If the characteristic function of the underlying asset's price process is known, a Fast Fourier Transform (FFT) method could be used, see [4]. A brief explanation[8] of the method follows.

Consider the European call option with strike $K = \exp(k)$, maturity T with the underlying asset's price process $(S(t), t \geq 0)$ and denote $s_T = \log(S(T))$. Further denote the risk-neutral density of the random variable s_T as $q_T(s)$. The characteristic function of s_T (in the risk neutral world) is expressed as

$$\phi_{s_T}(u) = \int_{-\infty}^{\infty} \exp(ius) q_T(s)\, ds.$$

The price of the call option C at $t = 0$ is according to Equation 2.2.1 then expressed as

$$C_T(k) = \int_{k}^{\infty} e^{-(r-q)T}(e^s - e^k) q_T(s)\, ds. \tag{2.2.1}$$

It is argued that $C_T(k)$ is not square integrable because $C_T \to S(0)$ as $k \to -\infty$. It is therefore proposed to modify C_T to obtain a square integrable function by defining

$$c_T(k) = e^{\alpha k} C_T(k)$$

for a suitable $\alpha > 0$. According to [18], choosing $\alpha = 0.75$ should suffice to make $c_T(k)$ square integrable for a wide collection of processes, among those, the processes presented in Section 2.3.

Because $c_T(k)$ is square integrable, it is better suited for the Fourier transform. The Fourier transform[9] of c_T is

$$\mathscr{F}[c_T](v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k)\, dk. \tag{2.2.2}$$

By noting that $\overline{\mathscr{F}[c_T](v)} = \mathscr{F}[c_T](-v)$ because c_T is real, it follows from the inverse

[8]Explanation is provided as there are small errors in the original paper. The errors are mainly the expression of the call price as well as the representation of an approximated integral using Simpson's rule.

[9]Note the sign convention of the Fourier transform being used.

Fourier transform that

$$
\begin{aligned}
C_T(k) &= \exp(-\alpha k)\mathscr{F}^{-1}[\mathscr{F}[c_T]](k) \\
&= \frac{\exp(-\alpha k)}{2\pi}\int_{-\infty}^{\infty} e^{-ivk}\mathscr{F}[c_T](v)\,dv \\
&= \frac{\exp(-\alpha k)}{2\pi}\left(\int_0^{\infty} e^{-ivk}\mathscr{F}[c_T](v)\,dv + \overline{\int_0^{\infty} e^{-ivk}\mathscr{F}[c_T](v)\,dv}\right) \\
&= \frac{\exp(-\alpha k)}{\pi}\operatorname{Re}\left(\int_0^{\infty} e^{-ivk}\mathscr{F}[c_T](v)\,dv\right). \qquad (2.2.3)
\end{aligned}
$$

By inserting Equation 2.2.1 into Equation 2.2.2 it follows that

$$
\begin{aligned}
\mathscr{F}[c_T](v) &= \int_{-\infty}^{\infty} e^{ivk}\int_k^{\infty} e^{\alpha k}e^{-(r-q)T}(e^s - e^k)q_T(s)\,ds\,dk \\
&= \int_{-\infty}^{\infty} e^{-(r-q)T}q_T(s)\int_{-\infty}^{s}(e^{s+\alpha k} - e^{(1+\alpha)k})e^{ivk}\,dk\,ds \\
&= \int_{-\infty}^{\infty} e^{-(r-q)T}q_T(s)\left(\frac{e^{(\alpha+1+iv)s}}{\alpha+iv} - \frac{e^{(\alpha+1+iv)s}}{\alpha+1+iv}\right)ds \\
&= \frac{e^{-rT}\phi_{s_T}(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}. \qquad (2.2.4)
\end{aligned}
$$

Henceforth,

$$
C_T(k) = \frac{\exp(-\alpha k)}{\pi}\operatorname{Re}\left(\int_0^{\infty} e^{-ivk}\frac{e^{-(r-q)T}\phi_{s_T}(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}\,dv\right).
$$

There are several ways to calculate the integral given in Equation 2.2.3. An often used, good approximation of the integral is given by using Simpson's rule, or more precisely a composite Simpson's rule, see [11]. Following this procedure, the interval of integration $[0, A]$ is split into an even N number of equal length subintervals with the discrete points, $v_j = (j-1)\eta$ for $j = 1, \ldots, N+1$ where $v_1 = 0$ and $v_{N+1} = A$. Here, η is the distant between two adjacent points v_j and v_{j+1} for $j = 1, \ldots, N$. Then,

$$
\begin{aligned}
\int_0^A g(v)\,dv &\approx \frac{\eta}{3}\sum_{j=1}^{N/2}(g(v_{2j-1}) + 4g(v_{2j}) + g(v_{2j+1})) \\
&= \frac{\eta}{3}\left(\sum_{j=1}^{N} g(v_j)(3 + (-1)^j - \delta_{j-1}) + g(v_{N+1})\right)
\end{aligned}
$$

where δ_{j-1} is the Kronecker delta function. By truncating the integral given in Equation 2.2.3 through approximation of the interval of integration $[0, \infty]$ by $[0, A]$

and paying attention to

$$g(v) = \exp(-ivk)\mathscr{F}[c_T](v)$$

it follows that

$$\begin{aligned}
&\int_0^\infty e^{-ivk}\mathscr{F}[c_T](v)\,dv \\
&\qquad\approx \frac{\eta}{3}\left(\sum_{j=1}^{N} e^{-iv_jk}\mathscr{F}[c_T](v_j)(3+(-1)^j-\delta_{j-1}) + e^{-iv_{N+1}k}\mathscr{F}[c_T](v_{N+1})\right)
\end{aligned}$$

where $v_j = (j-1)\eta$ for $j = 1, \ldots, N+1$ and where $v_1 = 0$ and $v_{N+1} = A$, as stated earlier. Henceforth,

$$\begin{aligned}
C_T(k) \approx \frac{\eta\exp(-\alpha k)}{3\pi} \\
\times \operatorname{Re}\left(\sum_{j=1}^{N} e^{-iv_jk}\mathscr{F}[c_T](v_j)(3+(-1)^j-\delta_{j-1}) + e^{-iv_{N+1}k}\mathscr{F}[c_T](v_{N+1})\right). \quad (2.2.5)
\end{aligned}$$

Consider only the sum

$$s(k) = \sum_{j=1}^{N} e^{-iv_jk}\mathscr{F}[c_T](v_j)(3+(-1)^j-\delta_{j-1}).$$

By introducing the points $k_u = -b + \lambda(u-1)$ for $u = 1, \ldots, N$ where λ is the space between each point and where $b = \lambda(N-1)/2$, a system of sums have the form

$$\begin{aligned}
s(k_u) &= \sum_{j=1}^{N} e^{-iv_jk_u}\mathscr{F}[c_T](v_j)(3+(-1)^j-\delta_{j-1}) \\
&= \sum_{j=1}^{N} e^{-i\eta\lambda(j-1)(u-1)}e^{ibv_j}\mathscr{F}[c_T](v_j)(3+(-1)^j-\delta_{j-1}), \quad u = 1, \ldots, N. \quad (2.2.6)
\end{aligned}$$

Now taking $\eta\lambda = 2\pi/N$, the sum will be in a suitable form for the FFT algorithm to be used to calculate the set of sums $(s(k_1), \ldots, s(k_N))$, and in turn, the values of the set of call prices $(C_T(k_1), \ldots, C_T(k_N))$ using Equation 2.2.5.

Shrinking the size of η will increase the spacing size λ, and vice versa. Because of this, the need for a fine mesh between the points $v_1, \ldots, v_{N+1}$ will lead to rougher mesh for the spacing of $k_1, \ldots, k_N$. Luckily, in the case of k_u representing logarithm of strike prices, suitable values for λ can be used such that the set of logarithm strike prices $k_1, \ldots, k_N$ contains the logarithm of strike prices represented by real options. This while also yielding a sufficient fine mesh with the enforced η value. The analysis of the numerical precision of the method is out of the scope of this report, the recommended values given by [18] will be considered.

This method works well when the time to maturities of the options are not too short. When the time to maturities are short, the option price tends to the intrinsic value (the pay-off), henceforth, the Fourier transform will become highly oscillatory. Highly oscillatory functions are difficult to integrate numerically, see [4]. Nevertheless, options with such short maturities are outside the scope of this report, thus the FFT method described above is sufficient.

2.3 Stochastic processes

In the following, the processes that will be considered in this report will be presented, even though there are other candidates for option pricing as well.

2.3.1 The CGMY process

A Lévy process $X = (X_t, t \geq 0)$ with X_1 having the Lévy triplet (γ, σ^2, ν) where,

$$\begin{aligned}
\gamma &= C\left(\int_0^1 (\exp(-Mx) - \exp(-Gx))x^{-Y}\, dx\right), \\
\sigma &= 0, \\
\nu &= C|x|^{-1-Y}(\exp(Gx)\mathbf{1}_{(x<0)} + \exp(-Mx)\mathbf{1}_{(x>0)})\, dx
\end{aligned}$$

with $C, G, M > 0$ and $Y < 2$, is called a CGMY process. Note that these restrictions of the coefficients C, G, M and Y are sufficient to make ν a valid Lévy measure. It follows that the characteristic exponent is given by

$$\psi(u) = C\Gamma(-Y)((M - iu)^Y - M^Y + (G + iu)^Y - G^Y).$$

The process X is a CGMY(C, G, M, Y) process where X_t has the marginal law CGMY(tC, G, M, Y). For more properties of the CGMY process, see [18, 5].

2.3.2 The Meixner process

A Lévy process $X = (X_t, t \geq 0)$ with X_1 having the Lévy triplet (γ, σ^2, ν) where,

$$\begin{aligned}
\gamma &= \alpha\delta\tan(\beta/2) - 2\delta\int_1^\infty \frac{\sinh(\beta x/\alpha)}{\sinh(\pi x/\alpha)}\, dx, \\
\sigma &= 0, \\
\nu &= \delta x^{-1}\exp(\beta x/\alpha)\,\mathrm{arsinh}(\pi x/\alpha)\, dx
\end{aligned}$$

with $\alpha > 0$, $-\pi < \beta < \pi$ and $\delta > 0$, is called a Meixner process. It follows that the characteristic exponent is given by

$$\psi(u) = 2\delta \log(\cos(\beta/2)/\cosh((\alpha u - i\beta)/2)).$$

The process X is a Meixner(α, β, δ) process where X_t has the marginal law Meixner$(\alpha, \beta, t\delta)$. For more properties of the Meixner process, see [18].

2.3.3 The NIG process

A Lévy process $X = (X_t, t \geq 0)$ with X_1 having the Lévy triplet (γ, σ^2, ν) where,

$$\begin{aligned}
\gamma &= \frac{2\delta\alpha}{\pi} \int_0^1 \sinh(\beta x) \mathrm{K}_1(\alpha x)\, dx, \\
\sigma &= 0, \\
\nu &= \delta\alpha\pi^{-1}|x|^{-1} \exp(\beta x) \mathrm{K}_1(\alpha|x|)\, dx
\end{aligned}$$

with $\alpha > 0$, $-\alpha < \beta < \alpha$ and $\delta > 0$ is called a NIG process where the modified Bessel function of the third kind with index 1, $\mathrm{K}_1(x)$, is given by

$$\mathrm{K}_1(x) = \int_0^\infty \exp(-x\cosh(u))\cosh(u)\, du, \quad \mathrm{Re}(x) > 0.$$

It follows that the characteristic exponent is given by

$$\psi(u) = -\delta(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}).$$

The process X is a NIG(α, β, δ) process where X_t has the marginal law NIG$(\alpha, \beta, t\delta)$. For more properties of the NIG process, see [18, 5].

NIG is an abbreviation for Normal Inverse Gaussian.

2.3.4 Chronometers

As discussed in [5], one could implement a stochastic volatility effect to the Lévy processes by the use of mean-reverting, positive and monotone increasing process as a time-change. These stochastic time-changing processes are also referred to as chronometers, see [2]. Chronometers considered in this report are presented below. For further explanation see [18].

2.3.4.1 The integrated CIR process

Consider the CIR process $y = (y_t, t \geq 0)$ that solves the stochastic differential equation (SDE)

$$dy_t = \kappa(\eta - y_t)dt + \lambda\sqrt{y_t}dW_t$$

where $W = (W_t, t \geq 0)$ is a standard Brownian motion. Note that y is a non-negative process when $\kappa, \eta, \lambda > 0$. The integrated CIR process $Y = (Y_t, t \geq 0)$ is then defined as

$$Y_t = \int_0^t y_s \, ds.$$

According to [18], the characteristic function of Y_t, given y_0, is

$$\mathbb{E}[\exp(iuY_t) \mid y_0] = \phi(u, t; \kappa, \eta, \lambda, y_0) = \frac{\exp(\kappa^2\eta t/\lambda^2)\exp(2y_0 iu/(\kappa + \gamma\coth(\gamma t/2)))}{(\cosh(\gamma t/2) + \kappa\sinh(\gamma t/2)/\gamma)^{2\kappa\eta/\lambda^2}}$$

where

$$\gamma = \sqrt{\kappa^2 - 2\lambda^2 iu}.$$

2.3.4.2 The integrated Gamma-OU process

The Ohrnstein–Uhlenbeck (OU) process defined by the following SDE

$$dy_t = -\lambda y_t dt + dz_{\lambda t}, \quad y_0 > 0$$

where z_t is a Lévy process with no Brownian part (i.e with the Lévy triplet parameter $\sigma^2 = 0$), non-negative drift term and with positive increments. The process $z = (z_t, t \geq 0)$ is sometimes called a Background Driving Lévy Process (BDLP).

As the Gamma(a, b) distribution is self-decomposable there exists an OU process where y_t follows the Gamma(a, b) law for each $t \geq 0$. The process is denoted as the Gamma-OU process, see [18].

The integrated Gamma-OU process $Y = (Y_t, t \geq 0)$ is then defined as

$$Y_t = \int_0^t y_s \, ds.$$

The characteristic function of Y_t given y_0 is given by

$$\begin{aligned}\mathbb{E}[\exp(iuY_t) \mid y_0] &= \phi(u; t, \lambda, a, b, y_0) \\ &= \exp\left(iuy_0\lambda^{-1}(1 - \exp(-\lambda t))\right) \\ &\quad \times \exp\left(\frac{\lambda a}{iu - \lambda b}\left(b\log\left(\frac{b}{b - iu\lambda^{-1}(1 - \exp(-\lambda t))}\right) - iut\right)\right)\end{aligned}$$

with $a, b, \lambda > 0$.

2.3.5 Stochastically time-changed Lévy processes

Substituting the time variable of a process by a stochastic process yields a composite process, since it constitutes of the composition of two processes. The composition of the CGMY-, Meixner- or NIG process with an integrated CIR or integrated Gamma-OU process yields the processes with the desired properties discussed in [5]. Properties such as that the infinite activity pure jump models (CGMY and NIG) can capture both large and small variations while the chronometers should work as an extension to these models to also capture the variation of option prices over a set of different maturities. The composite process is denoted as $X_Y = (X_{Y_t}, t \geq 0)$, where $X = (X_t, t \geq 0)$, called the base process, is one of either a CGMY-, Meixner- or NIG process and where $Y = (Y_t, t \geq 0)$ follows either the integrated CIR or the integrated Gamma-OU process. It is assumed that X and Y are independent processes. The composite processes are denoted as CGMY-CIR, Meixner-CIR, NIG-CIR, CGMY-Gamma-OU, Meixner-Gamma-OU and NIG-Gamma-OU. The characteristic functions of these composite processes are easily obtainable, given the characteristic function of the chronometers ϕ_Y and the characteristic exponent of the base processes ψ_X,

$$\mathbb{E}[\exp(iuX_{Y_t})] = \mathbb{E}\left[\mathbb{E}[\exp(iuX_{Y_t}) \mid Y_t]\right] = \mathbb{E}\left[\exp(Y_t\psi_X(u))\right] = \phi_Y(-i\psi_X(u)).$$

Note the special case for which $u = -i$ corresponds to the moment generating function (which is assumed to exist, a priori).

It is proposed by [18, 5] that the stochastic price process $S(t)$ is given by

$$S(t) = S_0 \exp\left(t(r-q)\right) \frac{\exp(X_{Y_t})}{\mathbb{E}[\exp(X_{Y_t})]} \tag{2.3.1}$$

where S_0 is the price at time $t = 0$, r instantaneous interest rate and q the dividend rate. It should be noted however, that nowhere has it been found that the discounted process $S^*(t) = \exp(-t(r-q))S(t)$, where $S(t)$ being given by Equation 2.3.1, is a martingale with the given base and chronometers defined above. In fact, it has been argued that the process do not need to be a martingale. In [5] it has been shown that the process is almost a martingale process, in the sense that a martingale part of the process is being perturbed. Although, the perturbation is eventually zero (of no effect to the martingale part of the process), as time increases.

The characteristic function of the logarithm of the process, presented in Equation 2.3.1, is given by

$$\begin{aligned}\mathbb{E}[\exp(iu\log(S(t))) \mid S_0, y_0] &= \mathbb{E}\left[\exp(iu(\log(S_0) + t(r-q)\right.\\ &\qquad \left. + X_{Y_t} - \log\left(\mathbb{E}[\exp\left(X_{Y_t}\right)]\right)))\right]\\ &= \exp\left(iu(\log(S_0) + t(r-q))\right)\frac{\phi_Y(-i\psi_X(u))}{\phi_Y(-i\psi_X(-i))^{iu}}.\end{aligned}$$

By using Lemma 2.16 and Theorem 2.41, one could also construct a discounted price process $S^*(t) = \exp(-t(r-q))S(t)$ with the martingale property, given by

$$S(t) = S_0\exp(t(r-q))\exp(X_{Y_t} - Y_t\psi_X(-i)). \tag{2.3.2}$$

The characteristic function of the logarithm of the process, presented in Equation 2.3.2, is given by

$$\begin{aligned}\mathbb{E}[\exp(iu\log(S(t))) \mid S_0, y_0] &= \mathbb{E}[\exp(iu(\log(S_0) + t(r-q) + X_{Y_t} - Y_t\psi_X(-i)))]\\ &= \exp\left(iu(\log(S_0) + t(r-q))\right)\\ &\quad \times \mathbb{E}[\mathbb{E}[\exp(iu(X_{Y_t} - Y_t\psi_X(-i)))) \mid Y_t]]\\ &= \exp\left(iu(\log(S_0) + t(r-q))\right)\\ &\quad \times \mathbb{E}[\exp(iu(-Y_t\psi_X(-i)))\exp(Y_t\psi_X(u))]\\ &= \exp\left(iu(\log(S_0) + t(r-q))\right)\\ &\quad \times \mathbb{E}[\exp(Y_t(-iu\psi_X(-i) + \psi_X(u)))]\\ &= \exp\left(iu(\log(S_0) + t(r-q))\right)\phi_Y(-u\psi_X(-i) - i\psi_X(u)).\end{aligned}$$

In the following, the different models used will be denoted by `BASE-CHRONO-X` where `BASE` is any of the base processes and `CHRONO` is any of the chronometers used in the model. Additionally, `X` is either 1 or 2 for denoting the used model given by either Equation 2.3.1 or Equation 2.3.2, respectively, also referred to as type-1 or type-2 model.

3

Methods

3.1 Data selection

The option prices used in the calibration have to be liquid as assumptions on frictionless markets have been made, which were discussed in Section 2.2. There are different ways to select options used for calibration of models. The OMXS30 options market is small in comparison to other markets, e.g. the S&P500 options market. As such, the restrictions to select option prices cannot be too narrow as this would result in too few quotes to calibrate the models against. The requirements of the selected quotes used for calibration were that the option strikes lie between -10% to $+15\%$ of the spot price, S_0, and that the minimum number of open interest of each option was 100. These requirements can be questioned for being set too low, but the impact of different requirements for data selection on the calibration results is outside the scope of this report. The liquidity of these options is assumed to be sufficient.

3.2 Numerical optimisation

As discussed in Section 2.2.1, in order to extend the set of European call option prices $C_T(k_u)$ over a set of maturities $T_1, \ldots, T_M$, a matrix of call prices $P = (C_{T_m}(k_u))_{u,m=1}^{N,M}$ is defined, with elements corresponding to different maturities and strikes. The columns represent the maturities and the rows the strikes of the option prices. Each column l of P is represented by $(C_{T_l}(k_1), \ldots, C_{T_l}(k_N))$ which is (partly) calculated by using the FFT algorithm of the system of sums given by Equation 2.2.6.

The option prices given by Equation 2.2.3 depend on the Fourier transform $\mathscr{F}[c_T](v)$ which contains the characteristic function of the underlying asset's price process, see Equation 2.2.4. If the parameters of the characteristic function are not known, they need to be found by calibration against real call option prices. The matrix P is constructed such that the maturity is always represented in one of the columns and the strike in one of the rows, for each real call option price with a corresponding strike and maturity. Note that the maturities can have an exact representation in the columns of P, while the strikes will have an approximate representation with an accuracy depending on the step size λ. The theoretical option prices in P which correspond to the real option prices with respect to strikes and maturities, are selected. Denote the selected theoretical options prices by $C_1^s, \ldots, C_L^s$ and their corresponding real option prices by $C_1^*, \ldots, C_L^*$, the set of unknown parameters θ in the underlying asset's price process is then found by the following optimisation problem

$$\min_\theta \sum_{i=1}^{L} \left(C_i^s(\theta) - C_i^*\right)^2, \tag{3.2.1}$$

possibly with some constraints on θ, see Section 2.3.

The optimisation for all models used in this report were conducted with parameter values of the numerical procedure corresponding to $\alpha = 0.75$, $N = 2^{12}$ and $\eta = 0.25$, see Equation 2.2.5 and Equation 2.2.6. These parameter values do not correspond to the parameter values of the characteristic function of the underlying asset's price process, e.g. η here should not be confused with the η in the characteristic function of the CGMY process as defined in Section 2.3.

The calibration performance was assessed by the root mean square error (RMSE)

$$\text{RMSE} = \sqrt{\frac{\sum_{i=1}^{L} \left(C_i^s(\theta) - C_i^*\right)^2}{L}}$$

and by the absolute percentage error (APE)

$$\text{APE} = \frac{\sum_{i=1}^{L} \left|C_i^s(\theta) - C_i^*\right|}{\sum_{i=1}^{L} C_i^*}.$$

The calibration was conducted using `MATLAB`.

4

Results

4.1 Calibrated models

The models were successfully calibrated. The calibrated stochastically time-changed Lévy models were not substantially different from each other. Even though some deviations seem to be present, they are small, see Table 4.1.1. The pure Lévy base models represent the real option prices correctly as well, even though some improvements can be achieved by the stochastically time-changed models.

The worst and best model fit (with respect to RMSE) to the real option prices are graphically illustrated in Figure 4.1.1(a) and Figure 4.1.1(b), respectively. The calibration and fit of the models are similar and the differences between them are small. In addition, the options deeply in the money[1] are not represented well by the models, compared to the at the money[2] options.

The calibration took much less processing time for the pure Lévy processes than for the stochastically time-changed Lévy processes, most probably due to the increased number of parameters to be estimated for the stochastically time-changed processes.

In Table 4.1.1 it can be further noticed that type-1 processes are not generally a better fit to the option quotes than the type-2 models, and vice versa, the type-2 processes are not generally calibrated better to the quotes than the type-1 processes.

During calibration the starting points were chosen arbitrarily. The choice of starting point could affect the accuracy of the optimal point considerably if the global optimiser is not well suited for the optimisation problem defined in Equation 3.2.1. Nevertheless, the choice of a well-suited starting point was not investigated and neither was the global optimiser, as it is beyond the scope of this work. It is assumed that the calibration procedure was conducted well enough for the purpose of this work.

[1] In the money is a term used for options where the current price of the underlying asset price is larger than the strike of that option.

[2] At the money is the term used for options where the current underlying asset price is close to the strike of that option.

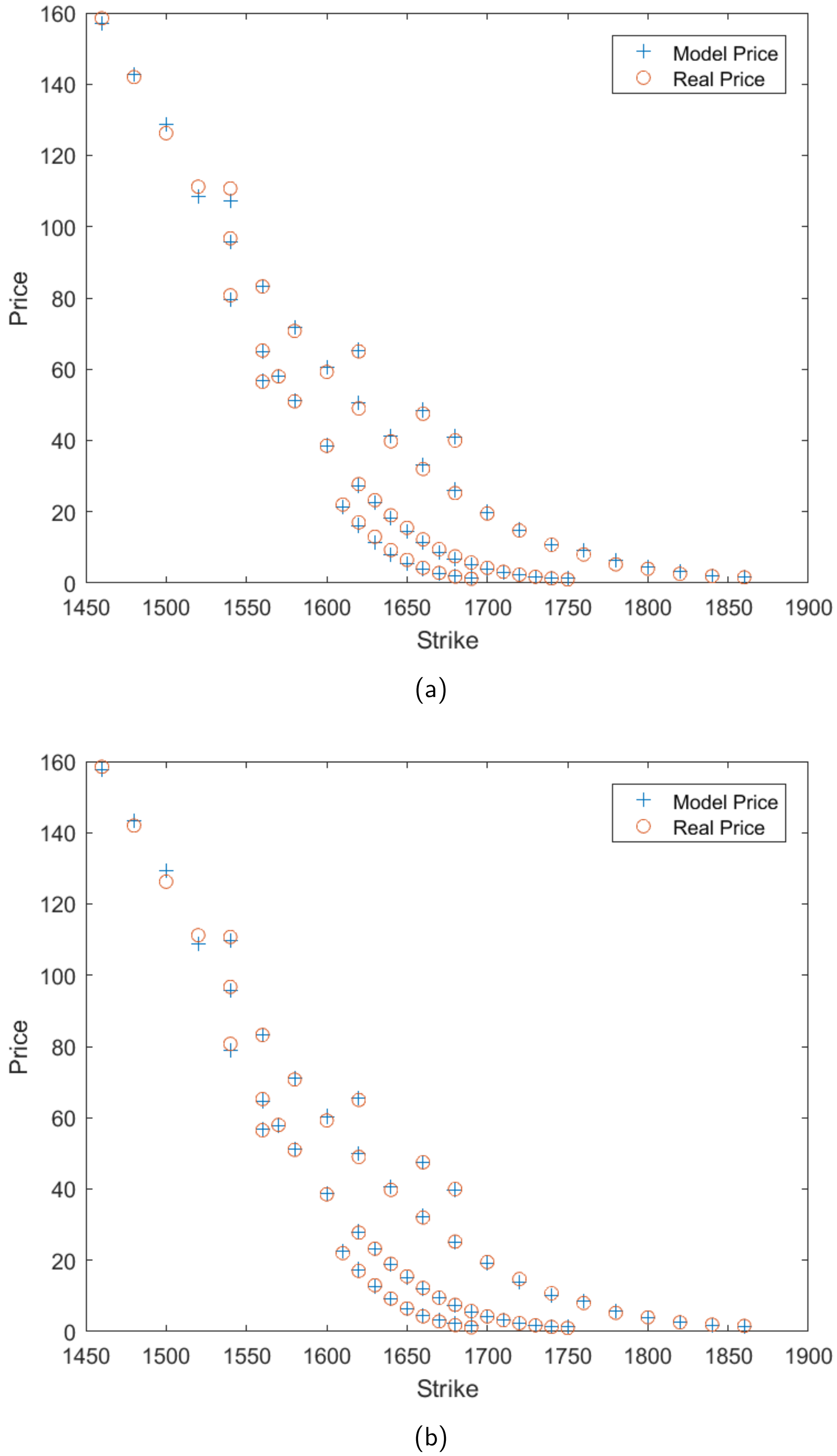

Figure 4.1.1. Calibration of the OMXS30 call options for the models (a) Meixner and (b) Meixner-ΓOU-1.

Table 4.1.1. Result of the calibration of the different models.

Model	Estimated parameters							RMSE	APE
	C	G	M	Y	κ	η	λ		
CGMY-CIR-1	0.2288	5.6382	34.9020	0.7782	0.5243	0.0002	0.8993	0.7612	0.0139
	C	G	M	Y	κ	η	λ		
CGMY-CIR-2	0.1345	3.7480	39.9821	1.0218	20.8166	0.8577	1.0001	0.8512	0.0141
	C	G	M	Y					
CGMY	0.0495	1.9397	62.6217	1.2621				0.9046	0.0174
	C	G	M	Y	λ	a	b		
CGMY-ΓOU-1	0.0752	3.5282	60.1412	1.1725	7.5891 e-05	86.5513	4.1119 e-10	0.9041	0.0174
	C	G	M	Y	λ	a	b		
CGMY-ΓOU-2	1.1918	2.7797	37.3439	1.0910	1957.1290	674.1865	9585.4749	0.8549	0.0143
	α	β	δ		κ	η	λ		
Meixner-CIR-1	0.1037	-1.5246	2.7790		0.6582	0.1540	0.8930	0.7657	0.0142
	α	β	δ		κ	η	λ		
Meixner-CIR-2	0.1035	-1.6802	2.8135		24.1721	0.8369	1.0002	0.8608	0.0150
	α	β	δ						
Meixner	0.1304	-1.7106	1.6676					1.0511	0.0222
	α	β	δ		λ	a	b		
Meixner-ΓOU-1	0.1071	-1.5947	2.6408		1.4943	0.4579	0.7596	0.7490	0.0138
	α	β	δ		λ	a	b		
Meixner-ΓOU-2	0.1478	-1.6967	1.8782		40.9975	763.8758	1083.0070	0.9441	0.0187
	α	β	δ		κ	η	λ		
NIG-CIR-1	22.3016	-14.3449	0.3333		0.5540	0.0477	0.9156	0.7614	0.0140
	α	β	δ		κ	η	λ		
NIG-CIR-2	24.9947	-17.0722	0.3530		21.8060	0.8502	1.0001	0.8531	0.0145

Model	Estimated parameters						RMSE	APE
	α	β	δ					
NIG	21.8639	-15.1313	0.2832				0.9689	0.0198
	α	β	δ	λ	a	b		
NIG-ΓOU-1	21.8634	-15.1308	0.2832	3.3741 e-07	1310.9351	81.6701	0.9689	0.0198
	α	β	δ	λ	a	b		
NIG-ΓOU-2	19.1002	-13.1386	1.0875	382.2985	381.8897	1631.2795	0.8818	0.0157

5

Conclusion

The models considered in this report were successfully calibrated as can be concluded from Table 4.1.1. There were no substantial differences between the calibrated models. The choice of chronometers did not seem to affect the calibration result. The purpose of including stochastic time-change to the models was to better fit them over the domain of strikes and maturities. The market chosen could have been too small to find sufficient number of liquid options so that a larger range of maturities could be used to calibrate the models against. As a result of this, no substantial differences between the models could be detected. A possible solution could be to try out these models on a spectrum of different markets. However, small differences are present and the calibration is in favour for the stochastically time-changed models. On the other hand, the time efficiency in the calibration of the pure base type models was superior to the stochastically time-changed models. This was expected as the number of parameters to be calibrated in the stochastically time-changed models is larger than in the number of parameters in pure base models. The pure base models could therefore be sufficient to describe the OMXS30 vanilla options market.

The Meixner as base process resulted in the best (and worst fit) even though it is a finite activity model. This makes the necessity of the infinite activity property for Lévy processes, as discussed by [5], obsolete for the OMXS30 call options market.

5.1 Theoretical framework

Some remarks on the type-2 models are necessary. Recall Equation 2.3.2 and consider that the chronometer Y_t is a monotone increasing process. As time progress, the chronometer attains a large value and it will be less likely for the base process X to surpass it. Henceforth, the log return will tend to large positive or negative values. Consequential, the log returns are unlikely to switch sign as time passes by, in contrast to empirical evidence of financial assets. Nevertheless, the type-2 models might still be valid price models for short time frames. As have been stated in the report, the type-1 processes are not martingales, but in [5] it has been shown that they are good enough to represent a static arbitrage-free market.

With regard to the choice of chronometers, the Gamma-OU might be a better choice since it is easier to simulate than the CIR process. Simulations could be used to price the options by means of Monte-Carlo methods if desired.

5.2 Optimisation and data selection

In the calibration procedure arbitrary starting points were selected. This might cause significant errors in the result as the objective function is not well behaved, e.g. not convex.

The FFT method used calculates a vast amount of prices that are not included in the calibration procedure. Faster numerical methods for this calibration could exist.

5.3 Future work

Important remaining investigations are the property of replication under a self-financing portfolio strategy described with these stochastically time-changed processes. The replicative property gives a solid foundation to be able to price options in the framework that has been done in this work. Nevertheless, the replicative property is unlikely, since the number of jump sizes are infinite in the models used which makes the market incomplete with infinite number of equivalent risk neutral measures.

It has been assumed that the moment generating function of the processes presented in this report exist. It is recommended to investigate whether this assumption hold.

To find well motivated starting values in the calibration procedure could speed up the calibration as well as minimising possible errors in the calibrated results. A more in-depth investigation of the methods in the choice of global optimiser could also be of interest to conduct.

Other areas of improvement are to calibrate the models against a larger data set, preferably the S&P500 vanilla options, or by also including put options in the data set.

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A

Appendix

A.1 Option quotes

Table A.1.1. The 2017-07-18 OMXS30 European call option closing prices for different maturities and strikes, retrieved from http://www.nasdaqomxnordic.com. The closing price of the underlying was 1607.36 SEK the same day. The riskless interest rate was assumed to be $r = 0.0067$ with the dividend rate of the underlying estimated to $q = 0.036$. Quotes and strikes are given in SEK.

Strike (K)	2017-08-18	2017-09-15	2017-12-15	2018-03-16
1460			158.50	
1480			142	
1500			126.25	
1520			111.25	
1540		80.75	96.75	110.75
1560	56.50	65.25	83.25	
1570		58		
1580		51	70.75	
1600		38.50	59.25	
1610	22			
1620	17	27.75	49	65
1630	13	23.25		
1640	9.25	19	39.75	

Strike (K)	2017-08-18	2017-09-15	2017-12-15	2018-03-16
1650	6.50	15.50		
1660	4.25	12.25	32	47.50
1670	2.85	9.50		
1680	1.80	7.50	25.25	40
1690	1.20	5.75		
1700		4.25	19.50	
1710		3.15		
1720		2.35	14.75	
1730		1.75		
1740		1.30	10.75	
1750		1		
1760			8	
1780			5.25	
1800			4	
1820			2.60	
1840			1.95	
1860			1.60	