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MASTER'S THESIS

Option Pricing using Lévy Processes

YONGQIANG BU

Department of Mathematical Statistics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
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Yongqiang Bu

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematical Statistics
Chalmers University of Technology and Göteborg University
SE – 412 96 Göteborg, Sweden
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Abstract

In this Master's thesis we price exotic options using Monte Carlo simulations. The asset price process is modeled as an exponential Lévy process. First we use Lévy processes to fit the log-returns of S&P 500 historical data. By means of both graphical and quantitative tests we find that the NIG process and the Meixner perform better than the Brownian motion. Secondly, we calibrate NIG, Meixner and CGMY Lévy process models using S&P 500 index vanilla options. The calibration results show that non-Gaussian Lévy processes describes the market price better than Brownian motion. At last, we use the calibrated models to price exotic options.

Keywords: Barrier Option; Calibration; Exotic Option; Fast Fourier Transformation; Lévy Process; Monte-Carlo Simulation.

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Contents

Chapter 1

Introduction

The beginning of modern mathematical finance can be attributed to Louis Bachelier who in year 1900 proposed to model the price process $\{S(t)\}_{t\geq 0}$ of an financial asset as

$$S(t) = S(0) + \sigma\, W(t),$$

where $\sigma > 0$ is a parameter and $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion.

The main drawback of the Bachelier model is that it is possible for prices of financial assets to becomes negative. Therefore Samuelson suggested the so called Bachelier-Samuelson model

$$S(t) = S(0)\, \mathrm{e}^{(\mu-\sigma^2/2)\, t+\sigma\, W(t)}, \tag{1.1}$$

where $\mu \in \mathbb{R}$ is another parameter. In this model it is instead the log-price process $\log(S(t))$ that is a (not necessarily standard) Brownian motion (with drift).

In their seminal paper [3] Black and Scholes give a theoretically consistent framework for option pricing based on the model (1.1). This paper changed the world of mathematical finance and initiated an strong growth of derivative markets. The Bachelier-Samuelson model is therefore also called the Black-Scholes model (BS), depending on the context.

The Black-Scholes model assumes log-increments of the stock price are Gaussian. However, there is much empirical evidence for that these log-increments are not Gaussian. This has led researchers to consider a variety of asset price models with non-Gaussian log-increments during the last decade. One of the most important and natural family of such model is that of exponential Lévy processes. In turns out that such processes fit many empirically observed properties of real world data much better than the Black-Scholes model.

In an exponential Lévy process model the price process is given by

$$S(t) = S(0)\, \mathrm{e}^{X(t)} \quad \text{for } \ t \geq 0,$$

where $\{X(t)\}_{t\geq 0}$ is a Lévy process. Some of the most common Lévy processes X that feature in such exponential Lévy process models are normal inverse Gaussian processes (NIG), Meixner processes and CGMY processes. Note that the Black-Scholes model is also an exponential Lévy process model as Brownian motion with drift is a Lévy process.

In this report we first show that NIG and Meixner Lévy process models perform better than the Brownian motion when fitted to log-return of stock prices (Chapter 2). Then we calibrate NIG, Meixner and CGMY Lévy process models by an inverse approach where we fit their predicted theoretical option prices to observed real world S&P 500 index vanilla option prices (Chapter 3). Finally we use the latter calibration results together with Monte Carlo simulations to price European exotic options (Chapter 4).

Chapter 2

The Lévy process framework

In this chapter we give the definitions of the Lévy processes we will use in our work. We also fit the corresponding exponential Lévy process models to S&P 500 historical data.

2.1 Lévy processes

We use the following definition of a Lévy process from the book by Cont and Tankov [8]:

Definition 2.1 *A cádlág*[1] *real valued stochastic process* $\{X(t)\}_{t\geq 0}$ *such that* $X(0) = 0$ *is called a* Lévy process *if it has stationary independent increments and is stochastically continuous.*

An important feature of Lévy process is their intimate link to *infinite divisible distributions* (e.g., Sato [13]): If $\{X(t)\}_{t\geq 0}\}_{t\geq 0}$ is a Lévy process, then every process value $X(t)$ is infinitely divisible. Conversely, to each infinitely divisible distribution there exist a unique in law Lévy process $\{X(t)\}_{t\geq 0}$ such that $X(1)$ has that distribution.

Recall that a probability distribution on the real line is said to be infinitely divisible if for any integer $n \geq 1$ there exists independent identically distributed random variables $Y_1, \ldots, Y_n$ such that $Y_1 + ... + Y_n$ has that distribution.

From the above it follows that a Lévy process $\{X(t)\}_{t\geq 0}\}_{t\geq 0}$ has a unique so called *characteristic exponent* in form of a continuous function $\psi : \mathbb{R} \to \mathbb{R}$ such that the characteristic function of $X(t)$ is given by

$$\mathbf{E}\{\mathrm{e}^{iuX(t)}\} = \mathrm{e}^{t\psi(u)} \quad \text{for } u \in \mathbb{R} \text{ and } t > 0.$$

2.2 Examples of Lévy processes

2.2.1 Brownian motion

Brownian motion with drift is a Lévy process $\{X(t)\}_{t\geq 0}\}_{t\geq 0}$ that has Gaussian increments. Specifically, $X(t)$ is $\mathrm{N}(\mu t, \sigma^2 t)$-distributed where $\sigma > 0$ and $\mu \in \mathbb{R}$ are parameters.

[1] Right continuous with left limits.

2.2.2 Normal inverse Gaussian process (NIG)

The *normal inverse Gaussian process* (NIG) is a Lévy process $\{X(t)\}_{t\geq 0}$ that has normal inverse Gaussian distributed increments. Specifically, $X(t)$ has a NIG$(\alpha, \beta, \delta t, \mu t)$-distribution with parameters $\alpha > 0$, $|\beta| < \alpha$, $\delta > 0$ and $\mu \in \mathbb{R}$.

The NIG$(\alpha, \beta, \delta, \mu)$-distribution has probability density function

$$f_{\text{NIG}}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha\delta}{\pi} \frac{K_1\big(\alpha\sqrt{\delta^2 - (x-\mu)^2}\big)}{\sqrt{\delta^2 + (x-\mu)^2}} \exp\big\{\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)\big\},$$

where

$$K_v(z) = \frac{1}{2}\int_0^\infty u^{v-1} \exp\Big\{-\frac{z}{2}(u + \frac{1}{u})\Big\} du$$

is the modified Bessel function of the third kind, while the characteristic function is given by

$$\phi_{\text{NIG}}(u; \alpha, \beta, \delta, \mu) = \exp\Big(-\delta\big(\sqrt{\alpha^2 - (\beta + iu)^2} - \sqrt{\alpha^2 - \beta^2}\big)\Big)\, \mathrm{e}^{i\mu u}.$$

A NIG$(\alpha, \beta, \delta, \mu)$-distributed random variable has the following stylized features:

Mean	$\dfrac{\beta\delta}{\sqrt{\alpha^2 - \beta^2}} + \mu$
Variance	$\dfrac{\alpha^2\delta}{(\alpha^2 - \beta^2)^{3/2}}$
Skewness	$\dfrac{3\beta}{\alpha\sqrt{\delta}(\alpha^2 - \beta^2)^{1/4}}$
Kurtosis	$3\left(1 + \dfrac{\alpha^2 + 4\beta^2}{\delta\alpha^2\sqrt{\delta}(\alpha^2 - \beta^2)}\right)$

See Barndorff-Nielsen [2] on more information about NIG processes.

2.2.3 Meixner process

The *Meixner process* is a Lévy process $\{X(t)\}_{t\geq 0}$ that has Meixner distributed increments. Specifically, $X(t)$ has a Meixner$(x; a, b, dt, mt)$-distribution with parameters $a > 0$, $|b| < \pi$, $d > 0$ and $m \in \mathbb{R}$.

The Meixner$(x; a, b, dt, mt)$-distribution has probability density function

$$f_{\text{Meixner}}(x; a, b, d, m) = \frac{(2\cos(b/2))^{2d}}{2a\pi\Gamma(2d)} \exp\Big\{\frac{b(x-m)}{a}\Big\} \Big|\Gamma\Big(d + \frac{i(x-m))}{a}\Big)\Big|^2,$$

where Γ denotes the Gamma function, while the characteristic function is given by

$$\phi_{\text{Meixner}}(u; a, b, d, m) = \left(\frac{\cos(b/2)}{\cosh(au - ib)/2}\right)^{2d} \mathrm{e}^{imu}.$$

A Meixner$(x; a, b, dt, mt)$-distributed random variable has the following stylized features:

Mean	$ad\tan(b/2)+m$
Variance	$\frac{1}{2}\frac{a^2 d}{\cos^2(b/2)}$
Skewness	$\sin(b/2)\sqrt{\frac{2}{d}}$
Kurtosis	$3+\frac{2-\cos(b)}{d}$

See Schoutens [14] on more information about Meixner processes.

2.2.4 CGMY process

The *CGMY process* is a Lévy process $\{X(t)\}_{t\geq 0}$ such that $X(1)$ is CGMY(C,G,M,Y)-distributed with parameters $C,G,M>0$ and $Y<2$.

The probability density function of a CGMY(C,G,M,Y)-distribution takes an analytically very complicated form, while the characteristic function is given by

$$\phi_{\text{CGMY}}(u;C,G,M,Y)=\exp\left\{C\Gamma(-Y)((M-iu)^Y-M^Y+(G+iu)^Y-G^Y)\right\}.$$

A CGMY(C,G,M,Y)-distributed random variable has the following stylized features:

Mean	$C(M^{Y-1}-G^{Y-1})\Gamma(1-Y)$
Variance	$C(M^{Y-2}+G^{Y-2})\Gamma(2-Y)$
Skewness	$\frac{C(M^{Y-3}+G^{Y-3})\Gamma(3-Y)}{(C(M^{Y-2}+G^{Y-2})\Gamma(2-Y))^{3/2}}$
Kurtosis	$3+\frac{C(M^{Y-4}+G^{Y-4})\Gamma(4-Y)}{(C(M^{Y-2}+G^{Y-2})\Gamma(2-Y))^{2}}$

See Carr, Geman, Madan and Yor [4] on more information about CGMY processes.

2.3 Modelling S&P 500 index with Lévy processes

Our dataset will be the S&P 500 index adjust closed price from 3rd, June, 2002 to 3rd, June, 2007, 1259 trading days, as listed by Yahoo Finance.

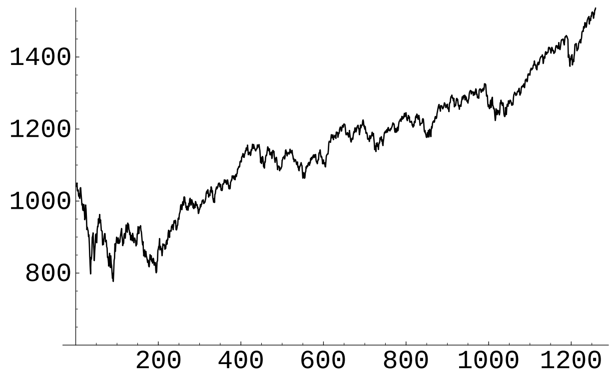

Figure 2.1: S&P 500 index adjusted closed prices

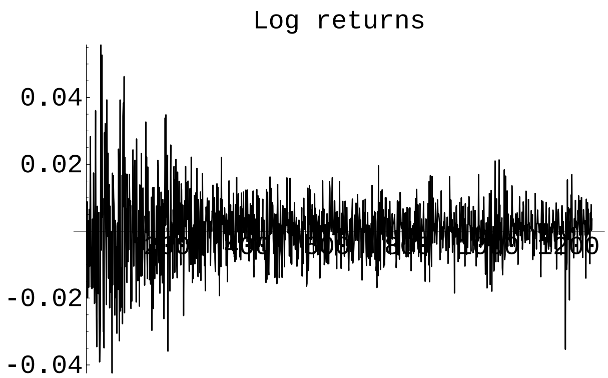

Figure 2.2: Log-return of S&P 500 index adjusted close prices

2.3.1 Stylized facts of financial time series

Now we discuss some stylized facts of financial time series, see Cont [6] for more information.

Skewness and kurtosis

Skewness and kurtosis of Gaussian distributions are 0 and 3, respectively. However, empirical financial time series usually display non-zero skewness and higher kurtosis than 3. In our case the skewness of the daily log-return of S&P 500 data is 0.191731 while the kurtosis is 6.68909. Hence it cannot be completely correct to model this data set with a Black-Scholes model.

Autocorrelations

Here we check the empirical autocorrelations (ACF) for log-return and squared log-returns of our data set. For the definition of ACF, please check with any text book on time series:

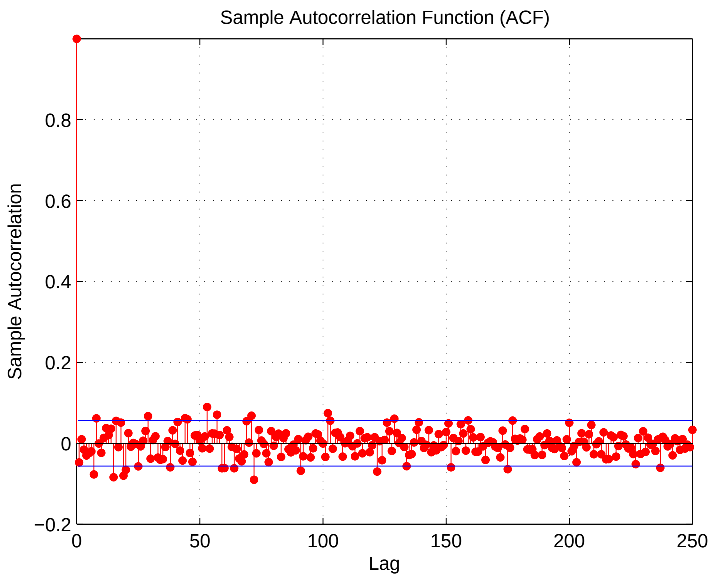

Figure 2.3: Empirical autocorrelations for log-returns

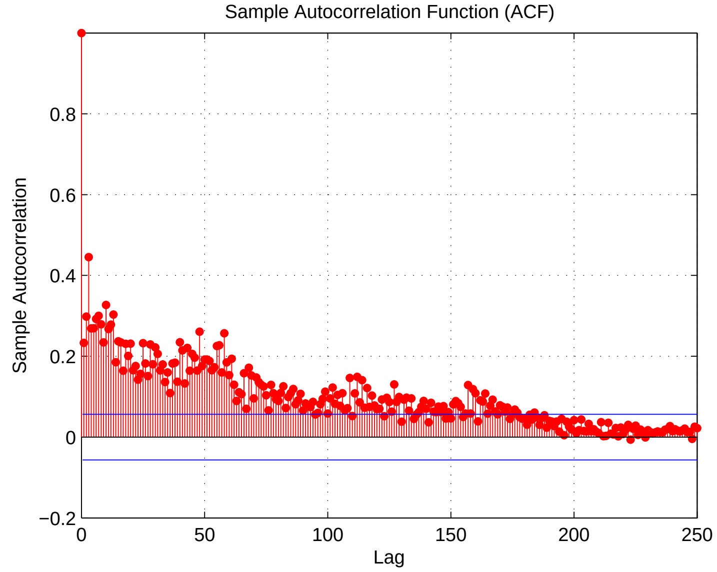

Figure 2.4: Empirical autocorrelations for squared log-returns

From the above two figures we see that the log-returns are uncorrelated, while the squared log-returns instead are correlated. Hence it cannot even be completely correct to model the data with an exponential Lévy process model. However, we will not consider more general models than that anyway.

Volatility clustering

Large changes in financial data tend to be followed by large changes, of either sign, while small changes tend to be followed by small changes, see Cont [7]. This experience is supported by Figure 2.4 above.

2.3.2 Parameter estimation

We will fit the empirical log-return of S&P 500 index to NIG process and Meixner process, as well as to Brownian motion by means of maximum likelihood estimation (MLE).

Due to the high numbers of parameters of the NIG and Meixner distribution and high numbers of data points, it turned out to be close to impossibly time consuming to make a direct MLE fit with the help of standard mathematical software packages, see Jonsson [9]. Therefore we used the methods of moments to get a first parameter estimate to be used as starting point for the MLE fit in order to significantly speed up the fitting procedure. The results were as follows:

Brownian motion	μ	σ		
	0.00031	0.0098		
NIG	a	b	d	m
	78.3512	-5.70771	0.00756726	0.000862369
Meixner	a	b	d	m
	0.0279247	-0.178417	0.244316	0.000919888

The lack of an analytically tractable expression for the density function of CGMY distributions made us refrain from trying to fit CGMY processes.

2.3.3 Test of distributional assumptions

Here we consider two ways to evaluate the corectness of the fitted distributions.

Graphical test of distributional assumption

According to the Glivekno-Cantelli theorem, if the sample $X_1, ..., X_n$ has cummulative distribution function $F(.;\theta)$, then the ordered sample $X_{(1)} \leq ... \leq X_{(n)}$ satisfies

$$\lim_{n\to\infty} \max_{1\leq i\leq n} |(i-0.5)/n - F(X_{(i)};\theta)| = 0,$$

so that a so called *QQ-plot* of

$$\{(X_{(i)}, F^{-1}((i-0.5)/n);\theta)\}_{i=1}^{n}$$

is an approximative 45°, and a systematic deviation therefrom indicates that the $F(.;\theta)$ assumption is not true.

The following three figures depict QQ-plots of our data set fitted to normal distribution, Meixner distribution and NIG distribution, respectively.

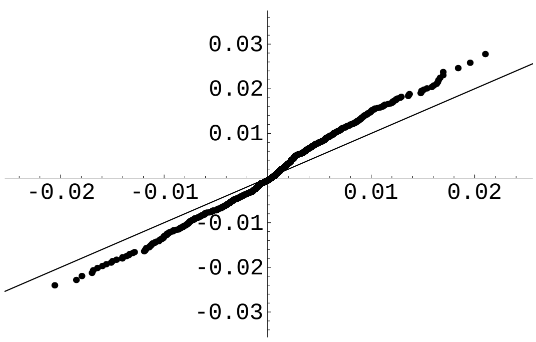

Figure 2.5: Normal QQ-plot

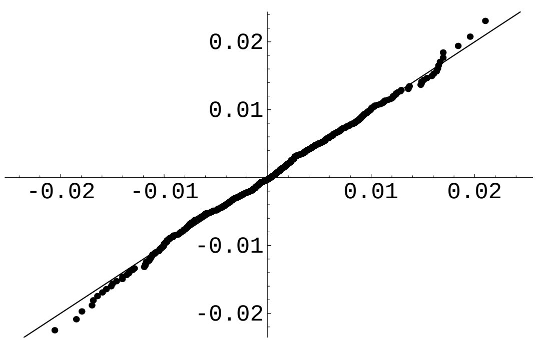

Figure 2.6: Meixner QQ-plot

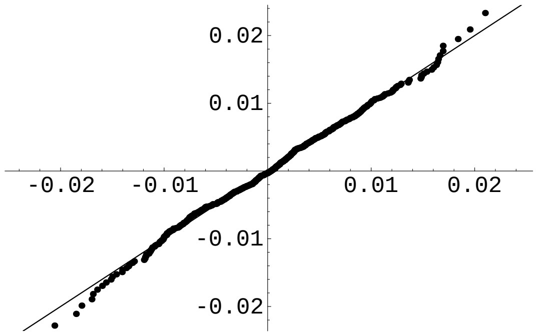

Figure 2.7: NIG QQ-plot

The QQ-plots indicate that the empirical data fits much better to the Meixner and NIG Lévy process models than to the Brownian motion.

Statistical test of distributional assumptions

There are many analytical statistical tests for checking distributional assumptions. Among them the *Kolmogorov-Smirnov distance* (K-S) and *Anderson & Darling statistic* (A-D) [1] are two common choices.

Writing F_{emp} for the empirical distribution of a data set and F_{fit} for the fitted distribution, the K-S distance is given by

$$KS = \max_{x\in\mathbb{R}} |F_{\text{emp}}(x) - F_{\text{fit}}(x)|,$$

while the A-D statistic is given by

$$AD = \max_{x\in\mathbb{R}} \frac{|F_{\text{emp}}(x) - F_{\text{fit}}(x)|}{\sqrt{F_{\text{fit}}(x)(1 - F_{\text{fit}}(x))}}.$$

Note that the A-D statistic pays attention to the fit in the tails by mean of amplifying tail deviations as compared with the K-S statistic. This can be convenient, e.g., in applivations to risk analysis etc.

We obtained the following values of the K-S and A-D statistics for our fitted distributions.

	KS	AD
Normal	0.117641	0.272002
NIG	0.0321954	0.0963174
Meixner	0.0190565	0.0487207

Table 2.1: K-S and A-D test statistics

The smaller value of K-S and A-D means closer of empirical distribution and fitted distribution. Obviously, the statistic for Lévy Process are smaller than the value for Brownian Motion. We find that the NIG process and the Meixner perform better than the Brownian motion.

Chapter 3

Inverse model calibration

Before calibrating our Lévy process models we have to introduce the risk-neutral option pricing model.

3.1 Risk-neutral option pricing

We assume that the price $B(t)$ of a risk-free asset satisfies the ordinary differential equation

$$dB(t) = r\, B(t)\, dt,$$

where $r \geq 0$ is the *interest rate*. Further, we assume that there is a risky asset whose price $S(t)$ is given by

$$S(t) = S(0)\, \mathrm{e}^{X(t)},$$

where $X(t)$ is a Lévy process. In our case this Lévy process will be of the type normal, Meixner, NIG or CGMY. The market does not admit *arbitrage*.

Recall that an arbitrage is a portfolio strategy such that one starts with zero capital and at some later time T is sure not to have lost money and has a positive probability to make money.

By the *first fundamental theorem of asset pricing*, if there exists a *risk-neutral probability measure*, then there is no arbitrage. This risk-neutral probability is a *martingale measure* $\mathbb{Q}$ that is equivalent to the original probability measure $\mathbb{P}$ and such that the underlying asset price is a $\mathbb{Q}$ local martingale.

See Shreve [17] on more information about the above matters.

An *European call option* is the right but not obligation to buy a contingent claim at the *time of maturity* T to a fix *strike price* K. Thus the *payoff function* is given by

$$\max(S(T) - K, 0).$$

The *arbitrage-free value* of the option at time $t < T$ can be defined as

$$\Pi_t = \mathrm{e}^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\max(S(T) - K, 0)], \tag{3.1}$$

where $\mathbb{Q}$ is a risk-neutral measure.

3.1.1 Equivalent martingale measure

We must find an equivalent martingale measure in order to price the derivatives. For this we will use the so called *mean-correcting martingale measure*.

After we have estimated all the parameters of some specific asset price process $S(t)$, then we add a *drift term* $\omega \in \mathbb{R}$ in a way appropriate to make the in this way discounted process a martingale. Specifically, writing $q \in \mathbb{R}$ for the dividend rate, in our exponential Lévy process setting, the condition

$$\mathbb{E}^{\mathbb{Q}}[S(t)] = S(0)\,\mathrm{e}^{t(r-q)}$$

gives that

$$\omega = r - q - \log(\phi(-i)),$$

where ϕ is the characteristic function of $S(1)$.

Here is a list of the mean-correcting risk neutral drift terms for the Lévy processes we consider:

Model	ω
Normal	$r - q - \mu$
CGMY	$r - q - C\Gamma(-Y)((M-1)^Y - M^Y + (G+1)^Y - G^Y)$
NIG	$r - q + \delta(\sqrt{(\alpha^2 - (\beta+1)^2} - \sqrt{(\alpha^2 - \beta^2)})$
Meixner	$r - q - 2\delta(\log(\cos\beta/2)) - \log(\cos((\alpha+\beta)/2))$

3.2 Pricing formulas for European vanilla options

We consider the case of *vanilla options* for which the payoff function only depends on the terminal stock price. We can find an analytical price formula for Brownian motion based price models, but require numerical solutions for other Lévy process based models.

3.2.1 Black-Scholes formula

With the volatility σ, the interest rate r and the dividend rate q in the exponential Brownian motion Black-Scholes asset price model, the asset price $S(t)$ at time t is given by

$$S(t) = S(0)\,\exp\Big\{\sigma W(t) + (r - q - \frac{1}{2}\sigma^2)t\Big\}.$$

As

$$S(T) = S(t)\,\exp\Big\{\sigma(W(T) - W(t)) + (r - q - \frac{1}{2}\sigma^2)(T-t)\Big\},$$

using (3.1), we get the option price

$$\begin{aligned}
\Pi_t &= \mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{-r(T-t)}(S(T) - K)^+\big] \\
&= \mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{-r(T-t)}(x\,\mathrm{e}^{\sigma(W(T)-W(t))+(r-q-\frac{1}{2})(T-t)\sigma^2} - K)^+\big] \\
&= \mathbb{E}^{\mathbb{Q}}\big[\mathrm{e}^{-r\tau}(x\,\mathrm{e}^{-\sigma\sqrt{\tau}Y)+(r-q-\frac{1}{2}\sigma^2)\tau} - K)^+\big],
\end{aligned}$$

where $\tau = T - t$ and

$$Y = -\frac{W(T) - W(t)}{\sqrt{T-t}}$$

is a standard normal random variable. Writing

$$d_1 = d_2 + \sigma\sqrt{\tau} = \frac{1}{\sigma\sqrt{\tau}}\left[\log\Big(\frac{x}{K}\Big) + \Big(r - q + \frac{1}{2}\sigma^2\Big)\tau\right],$$

we thus obtain

$$\begin{aligned}
\Pi_t &= \frac{1}{2\pi}\int_{-\infty}^{d_2} \mathrm{e}^{-r\tau}\left[x\,\exp\Big\{-\sigma\sqrt{\tau}\,y + \Big(r - q - \frac{1}{2}\sigma^2\Big)\tau\Big\} - K\right]\mathrm{e}^{-\frac{1}{2}y^2}dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{d_2} x\,\exp\Big\{-\sigma\sqrt{\tau}\,y - \Big(q + \frac{1}{2}\sigma^2\Big)\tau - \frac{1}{2}y^2\Big\}dy - \frac{1}{2\pi}\int_{-\infty}^{d_2}\mathrm{e}^{-r\tau}K\,\mathrm{e}^{-\frac{1}{2}y^2}dy \\
&= \frac{1}{2\pi}\int_{-\infty}^{d_2} x\,\mathrm{e}^{-q\tau}\exp\Big\{-\frac{1}{2}(y + \sigma\sqrt{\tau})^2\Big\}dy - \mathrm{e}^{-r\tau}K\Phi(d_2) \\
&= x\,\mathrm{e}^{-q\tau}\Phi(d_1) - \mathrm{e}^{-r\tau}K\Phi(d_2) \qquad (3.2)
\end{aligned}$$

If we insert $S(t)$ instead of x in the (3.2), then we get the option pricing at time t.

3.2.2 Option pricing using fast Fourier transformation

For more general Lévy process models than those based on Brownian motion we typically cannot find analytical solutions in the fashion of (3.2). We will therefore now introduce pricing methods based on characteristics function. When using these methods in practice fast Fourier transformation can be employed, see Carr and Madan [5] on more information.

The Fourier transform of an option price

Here we will evaluate an European call option price based on the price asset price process $S(t)$, maturity time T and strike price K. Write $k = \log(K)$ and $s(T) = \log(S(T))$. Let $C_T(k)$ denote the option price and q_T the risk-neutral probability density function of the log price s_T.

The characteristic function of the density q_T is given by

$$\phi_T(u) = \int_{-\infty}^{\infty} \mathrm{e}^{ius} q_T(s)\,ds.$$

The option value which is related to the risk-neutral density q_T is given by

$$C_T(k) = \int_{k}^{\infty} \mathrm{e}^{-rT}\big(\mathrm{e}^{s} - \mathrm{e}^{k}\big)q_T(s)\,ds.$$

Here $C_T(k)$ is not square integrable because when $k \to -\infty$ so that $K \to 0$, we have $C_T \to S(0)$. To obtain a square integrable function, Carr and Madan [5] suggested consideration of the modified price $c_T(k)$ given by

$$c_T(k) = \mathrm{e}^{\alpha k} C_T(k),$$

for a suitable $\alpha > 0$. Here Carr and Madan suggested to choose $\alpha \approx 0.25$, while Schoutens [15] suggests $\alpha \approx 0.75$. The value of α affects the speed of convergence.

The Fourier transform of $c_T(k)$ is given by

$$\psi_T(\upsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{i\upsilon k} c_T(k)\,dk.$$

The inverse corresponding inverse Fourier transform takes the form

$$c_T(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ivk}\psi_T(v)\,dv.$$

We can use these formulas to get the following option price formula for $C_T(k)$:

$$C_T(k) = \frac{\exp(-\alpha k)}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ivk}\psi_T(v)\,dv = \frac{\exp(-\alpha k)}{\pi}\int_{0}^{\infty} \mathrm{e}^{-ivk}\psi_T(v)\,dv, \tag{3.3}$$

where we made use of the fact that the function ψ_T is odd in its imaginary part and even in its real part since $C_T(k)$ is real.

We may express ψ_T in terms of $\phi_T(k)$ as

$$\begin{aligned}
\psi_T(v) &= \int_{-\infty}^{\infty} \mathrm{e}^{ivk}\int_{k}^{\infty} \mathrm{e}^{\alpha k}\mathrm{e}^{-rT}(\mathrm{e}^{s}-\mathrm{e}^{k})q_T(s)\,dsdk \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-rT}q_T(s)\int_{-\infty}^{s}(\mathrm{e}^{s+\alpha k}-\mathrm{e}^{k+\alpha k})\mathrm{e}^{ivk}dkds \\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-rT}q_T(s)\left(\frac{\mathrm{e}^{(\alpha+1+iv)s}}{\alpha+iv}-\frac{\mathrm{e}^{(\alpha+1+iv)s}}{\alpha+1+iv}\right)ds \\
&= \mathrm{e}^{-rT}\int_{-\infty}^{\infty} q_T(s)\frac{\mathrm{e}^{(\alpha+1+iv)s}}{(\alpha+iv)(\alpha+1+iv)}\,ds \\
&= \frac{\mathrm{e}^{-rT}\phi_T(v-(\alpha+1)i)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}.
\end{aligned} \tag{3.4}$$

Using known expressions for the characteristic function of NIG, CGMY and Meixner in (3.4), we can use (3.3) to get the option price.

Fast Fourier transformation

Fast Fourier transformation (FFT) is an efficient algorithm to compute the following sum

$$\omega(k) = \sum_{j=1}^{N} \mathrm{e}^{-i2\pi(j-1)(k-1)/N}x(j), \tag{3.5}$$

where N is usually a power of 2. FFT is a commonly employed discrete approximation technique of Fourier transform used to reduce computational labour.

Here we will reexpress the relation (3.3) approximately using the FFT (3.5) as

$$C_T(k) \approx \frac{\exp(-\alpha k)}{\pi}\sum_{j=1}^{N} \mathrm{e}^{-iv_j k}\psi_T(v_j)\eta \tag{3.6}$$

with the following conventions and parameter values (as suggested by Carr and Madan [5])

$$v_j = \eta(j-1),\quad N = 4096,\quad a = N\eta = 600,\quad b = \frac{N\lambda}{2},\quad k_u = -b + \frac{2b}{N}(u-1),\quad \lambda\eta = \frac{2\pi}{N}.$$

Here a is the upper limit for the integration, while k_u is a vector with N values of k and b sets a bound on the log strike to range between $-b$ and b.

Our formula (3.6) for C_T can now be rewritten as

$$C_T(k) \approx \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-i\lambda\eta(j-1)(u-1)} \mathrm{e}^{ib\upsilon_j} \psi_T(\upsilon_j)\eta.$$

Here we cannot combine a too fine integration grid with a wide enough region for strikes, as if we choose a too small η we get a fine integration grid but few strikes lying in the region.

Carr and Madan suggest to use Simpson's weighting rule to obtain an accurate integration with large η. Then we rewrite our price formula as

$$C_T(k) \approx \frac{\exp(-\alpha k_u)}{\pi} \sum_{j=1}^{N} \mathrm{e}^{-i2\pi(j-1)(u-1)/N} \mathrm{e}^{ib\upsilon_j} \psi_T(\upsilon_j) \frac{\eta}{3} \left(3 + (-i)^j - \delta_{j-1}\right), \tag{3.7}$$

where δ_n is the Kronecker delta function.

We will use Black-Scholes formula for the normal model together with (3.7) for our other Lévy process based models to compute call option prices.

3.3 Model calibration

In this section we use historical option prices to calibrate model parameters. In this way we avoid many problems that are associated with calibrations that are based on the underlying asset prices.

While the *pricing problem* is concerned with computing option values given the model parameters, the *calibration problem* is concerned with computing the model parameters given the option prices. Thus the calibration problem is the *inverse problem* to the pricing problem.

One of the most popular calibration methods are to use *least squares*, they idea of which is very simple: Observed market prices $(C_i)_{i=1}^N$ at $t = 0$ with different strikes (K_i) and maturities (T_i) should be the same as those proposed by the risk-neutral model prices C^θ described in last section with the model parameters θ. Thus we find the best parameter values θ by means of minimizing the sum of quadratic deviations between these prices

$$\theta^* = \arg\min \sum_{i=1}^{N} (C^\theta(T_i, K_i) - C_i)^2. \tag{3.8}$$

We will compute the following statistics suggested by Schoutens [15] to measure the quality of fits:

$$\begin{aligned}
\text{APE} &= \sum_{i=1}^{N} \frac{|C^\theta(T_i, K_i) - C_i)|}{N} \bigg/ \sum_{i=1}^{N} \frac{C_i}{N}, \\
\text{AAE} &= \sum_{i=1}^{N} \frac{|C^\theta(T_i, K_i) - C_i)|}{N}, \\
\text{ARPE} &= \frac{1}{N} \sum_{i=1}^{N} \frac{|C^\theta(T_i, K_i) - C_i)|}{C_i}, \\
\text{RMSE} &= \sqrt{\sum_{i=1}^{N} \frac{(C^\theta(T_i, K_i) - C_i))^2}{N}}.
\end{aligned}$$

3.4 Calibration results

We use S&P 500 historical call option prices on 1st of June 2007 from Yahoo Finance that are listed in Appendix A below. The market prices were chosen from June 2007 to December 2008. The strike is from 1300 to 2000 with the increment of 25 from 1300 to 1700 and the increment 100 from 1700 to 2000. The index closed price is 1536.34.

Some of the options have two different prices with the same maturity and strike. In that case, we choose the price with highest trading volume. We didn't include the option prices that were smaller than 1.

3.4.1 Data selection

There are four different versions of the option prices, namely the close price, the bid, the ask and the mean of the bid and ask.

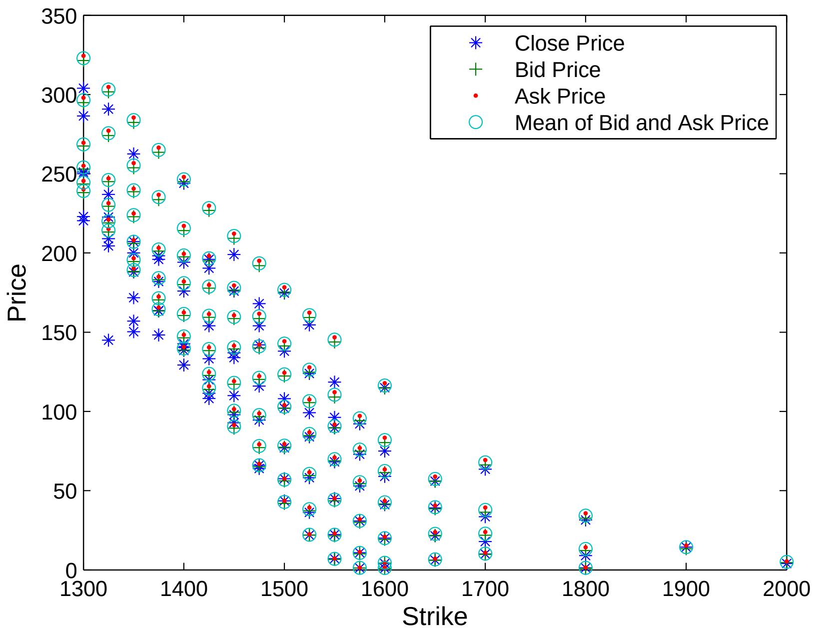

Figure 3.1: Option data set

Although very few paper discuss the selection of data set, it is crucial for the calibration results. In particular, for some option with low volume, there are big difference between the close prices and bid ask prices. This can be explained by that for frequently traded options the bid and ask prices match, while for some little traded options the last trade date might be long time ago, so that the last trade does not express the true value of option.

From the above figure we see that the close prices have a lot of outliers compare to bid and ask. Thus we conclude that the bid and ask prices are better to use than the close prices. The following table show the results of the calibrations of NIG using the bid, the ask as well as the mean of bid and ask:

NIG	Bid	Ask	Mean of bid and ask
APE	0.0176	0.0137	0.0140
AAE	2.2634	1.7956	1.8120
ARPE	0.1089	0.0840	0.0894
RMSE	0.9162	0.3381	0.1501

Table 3.1: NIG Calibration Statistics

From the above table we conclude that calibration using mean of bid and ask is slightly better than bid and ask. All of these three in turn are much better than using close prices. Thus we will use the mean of bid and ask as our option market data to calibrate models.

3.4.2 Calibration results

the following tables show the calibrated model parameters together with the corresponding values of APE, AAE, ARPE and RMSE.

Models	Parameters			
Normal	σ			
	0.1531			
CGMY	C	G	M	Y
	0.0156	0.0767	7.5500	1.2996
NIG	α	β	θ	
Meixner	5.0364	-3.3199	0.0881	
	α	β	θ	
	0.3400	-1.4900	0.2900	

Table 3.2: Calibration results

	Normal	CGMY	Meixner	NIG
APE	0.0575	0.0121	0.0120	0.0140
AAE	7.4543	1.5632	1.5553	1.8120
ARPE	0.3093	0.0793	0.0846	0.0894
RMSE	1.0244	0.0026	0.0426	0.1501

Table 3.3: APE, AAE, ARPE and RMSE for calibrations

The calibrations for CGMY, NIG and Meixner are quite similar in quality and all perform much better than calibration for Normal. Hence we can get improvements if we employ more general Lévy processes that Brownian motion in option pricing.

We remark here that the model parameters we got from our calibrations are different from those obtained by the more conventional calibration method to fit the exponential Lévy process asset price model to real world asset prices underlying the option.

The following four figures show the theoretical option prices from our calibrations together with the corresponding market option prices.

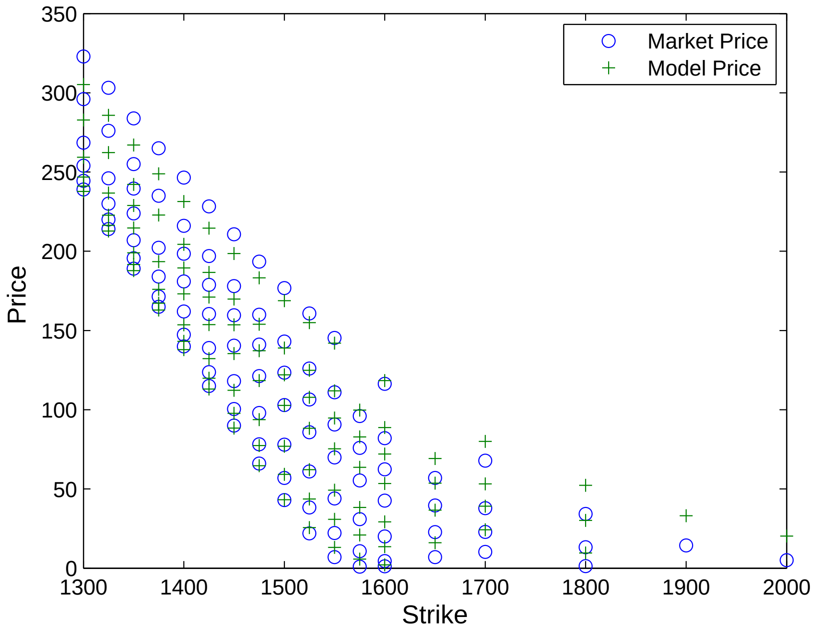

Figure 3.2: Normal calibration to S&P 500 option prices

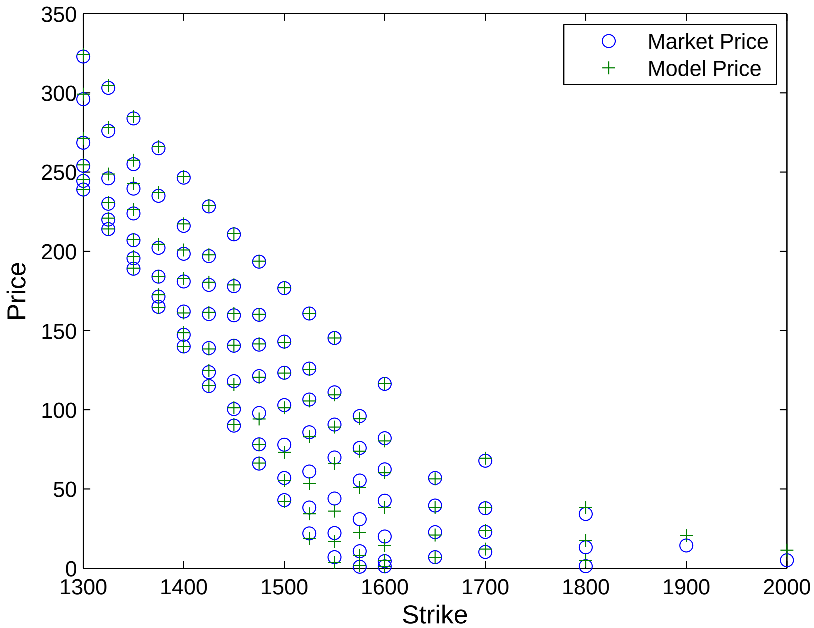

Figure 3.3: NIG calibration to S&P 500 option prices

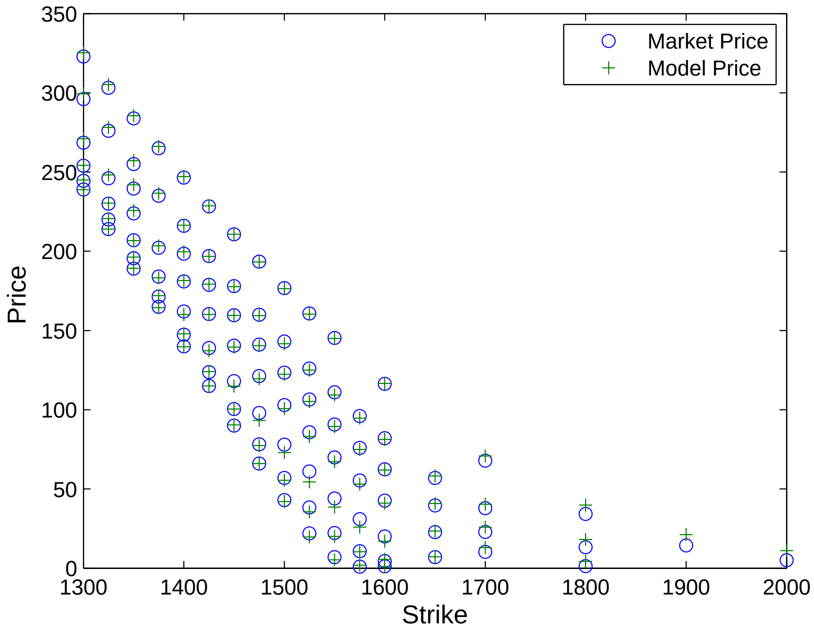

Figure 3.4: CGMY calibration to S&P 500 option prices

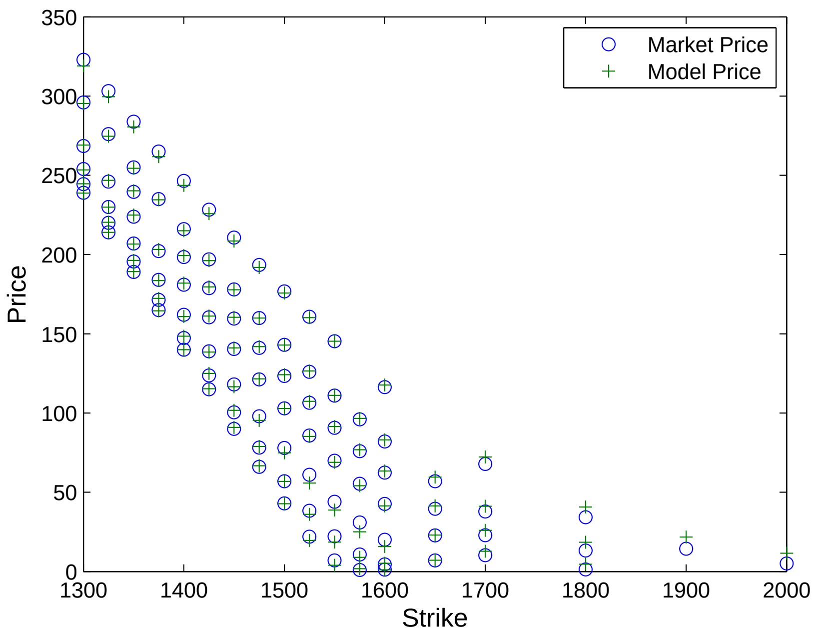

Figure 3.5: Meixner calibration to S&P 500 option prices

Chapter 4

Exotic option pricing

The option we have discussed until now is the vanilla option which means the payoff function only depend on terminal value. However, *path-dependent options* have become popular in the OTC market in the last twenty years. Barrier option and lookback option are two important examples of such so called *exotic options*.

In this chapter we will consider barrier options and discuss their pricing methods based on our previous calibration results. Before discussing the pricing methods we describe the barrier option in more detail.

4.1 Exotic options

4.1.1 Barrier option

The holder of a *barrier option* has the right to buy or sell an asset at a specific price at the end of the contract. The payoff function of a barrier option depends on whether the price of the underlying asset crossses a given threshold *the barrier* before maturity.

There are two types barrier option, namely *knock-in options* and *knock-out options*. A knock-in options is activated only when the underlying asset touches the barrier, while a knock-out option is instead deactivated when it touches the barrier. For each type of barrier option, there is an *up option* and *down option* version opf it. Thus we have four types of barrier call options: down-and-out barrier call, down-and-in barrier call, up-and-in barrier call and up-and-out barrier call.

Let us assume that the duration of the contract is T. Define the maximum and minimum asset price process up til time $t \in [0, T]$ as

$$M(t) = \sup\{S(u) : 0 \leq u \leq t\} \quad \text{and} \quad m(t) = \inf\{S(_u) : 0 \leq u \leq t\},$$

respectively. See Schoutens [15, 16] on more details.

Up-and-in barrier call

The *up-and-in barrier call option* is a standard European call option with strike K when its maximum lies above the barrier H, while it is worthless otherwise. The initial price is give by

$$C_{\mathrm{UI}} = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\big[(S(T) - K)^{+}1_{M(T) \geq H}\big].$$

Up-and-out barrier call

The *up-and-out barrier call option* is a standard European call option with strike K when its maximum lies below the barrier H, while it is worthless otherwise. The initial price is give by

$$C_{\mathrm{UO}} = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\big[(S(T)-K)^{+}1_{M(T)<H}\big].$$

Down-and-in barrier call

The *down-and-in barrier call option* is a standard European call option with strike K when its minimum lies below the barrier H, while it is worthless otherwise. The initial price is give by

$$C_{\mathrm{DI}} = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\big[(S(T)-K)^{+}1_{m(T)\leq H}\big].$$

Down-and-out barrier call

The *down-and-out barrier call option* is a standard European call with strike K when its lies above some barrier H, while it is worthless otherwise. The initial price is give by

$$C_{\mathrm{DO}} = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\big[(S(T)-K)^{+}1_{m(T)>H}\big].$$

4.1.2 Lookback option

The paper by Nguyen-Ngo [11] treats exotic options for underlying exponential Lévy process asset price models. For exotic option base on exponential Brownian motion, see Shreve [17].

There are two types of *lookback options*, namely *fixed and floating* strike lookback option. Here we only consider the fixed strike option.

The payoff function of *fixed strike lookback option* is the difference between the stock terminal value and its lowest value during the option lifetime. The price is given by

$$C_{\mathrm{L}} = \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\big[S(T)-m(T)\big].$$

4.2 Pricing methods

The main problem with barrier option pricing is to find the distribution of minimum and maximum processes m and M. It is possible to obtain the explicit proce formluas in Black-Scholes normal framework. However, the distribution of minima and maxima of more general Lévy processes is usually not known explicitly. We will therefore use the Monte Carlo simulation techniques to estimate the exotic option process. See Schoutens [15, 16] on more details.

4.2.1 Black-Scholes formula

The closed-form solution for the Brownian motion framework attributed to Merton, Reiner and Rubinstein gives the formulas for the types of exotic we consider as

$$\begin{aligned} C_{\mathrm{DO}} &= S(0)\Phi(x_1)\,\mathrm{e}^{-qT} - K\,\mathrm{e}^{-rT}\Phi(x_1-\sigma\sqrt{T} - S(0)\,\mathrm{e}^{-qT}\Big(\frac{H}{S(0)}\Big)^{2\lambda}\Phi(y_1) \\ &\quad +K\,\mathrm{e}^{-rT}\Big(\frac{H}{S(0)}\Big)^{2\lambda-2}\Phi(y_1-\sigma\sqrt{T}\,) \end{aligned}$$

$$
\begin{aligned}
C_{\mathrm{UI}} &= S(0)\Phi(x_1)\,\mathrm{e}^{-qT} - K\,\mathrm{e}^{-rT}\Phi(x_1-\sigma\sqrt{T}\,) - S(0)\,\mathrm{e}^{-qT}\Big(\frac{H}{S(0)}\Big)^{2\lambda}(\Phi(-y)-\Phi(-y_1))\\
&\quad +K\,\mathrm{e}^{-rT}\Big(\frac{H}{S(0)}\Big)^{2\lambda-2}\big(\Phi(-y+\sigma\sqrt{T}\,)-\Phi(y_1+\sigma\sqrt{T}\,)\big)\\
C_{\mathrm{DI}} &= \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\big[(S(T)-K)^+\big] - C_{\mathrm{DO}}\\
C_{\mathrm{UO}} &= \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\big[(S(T)-K)^+\big] - C_{\mathrm{UI}}
\end{aligned}
$$

for $H > K$, while

$$
\begin{aligned}
C_{\mathrm{DI}} &= S(0)\,\mathrm{e}^{-qT}\Big(\frac{H}{S(0)}\Big)^{2\lambda}\Phi(y) - K\,\mathrm{e}^{-rT}\Big(\frac{H}{S(0)}\Big)^{2\lambda-2}\Phi(y-\sigma\sqrt{T}\,)\\
C_{\mathrm{UO}} &= 0\\
C_{\mathrm{DO}} &= \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\big[(S(T)-K)^+\big] - C_{\mathrm{DI}}\\
C_{\mathrm{UI}} &= \mathrm{e}^{-rT}\mathbb{E}^{\mathbb{Q}}\big[(S(T)-K)^+\big]
\end{aligned}
$$

for $H \leq K$, where

$$
\begin{aligned}
\lambda &= \frac{1}{\sigma^2}\Big(r-q+\frac{\sigma^2}{2}\Big),\\
y &= \frac{1}{\sigma\sqrt{T}}\log\Big(\frac{H^2}{S(0)K}\Big)+\lambda\sigma\sqrt{T},\\
x_1 &= \frac{1}{\sigma\sqrt{T}}\log\Big(\frac{S(0)}{H}\Big)+\lambda\sigma\sqrt{T},\\
y_1 &= \frac{1}{\sigma\sqrt{T}}\log\Big(\frac{H}{S(0)}\Big)+\lambda\sigma\sqrt{T}.
\end{aligned}
$$

Moreover, for the lookback option in the Brownian motion framework, we have

$$
C_{\mathrm{L}} = S(0)\,\mathrm{e}^{-qT}\Big(\Phi(a_1)-\frac{\sigma^2}{2(r-q)}\,\Phi(-a_1)\Big) - S(0)\,\mathrm{e}^{-rT}\Big(\Phi(a_2)-\frac{\sigma^2}{2(r-q)}\,\Phi(-a_2)\Big),
$$

where

$$
a_1 = \frac{1}{\sigma}\Big(r-q+\frac{\sigma^2}{2}\Big)\sqrt{T} \quad \text{and} \quad a_2 = \frac{1}{\sigma}\Big(r-q-\frac{\sigma^2}{2}\Big)\sqrt{T}\,.
$$

4.2.2 Monte Carlo pricing using Lévy processes

It is not possible to get the closed-form exotic option prices for the more general Lévy process than Brownian motion we consider. Hence we will use Monte Carlo methods to find these option prices.

The Monte Carlo Pricing procedure goes as follows:

1. Calibrate the model on the vanilla option prices available in the market (S&P 500 call option in our case) and find the risk-neutral parameters of the model. (This procedure has already been carried out in a previous chapter.)

2. Simulate N trajectories of the calibrated Lévy process based models.

3. Calculate the payoff function p_i for each trajectory, $i = 1, \ldots, N$.

4. Calculate the sample mean payoff to get the estimated payoff $\hat{p} = \sum_{i=1}^{N} p_i/N$.

5. Discount the estimated payoff at the risk-free rate and get the derivative price $\mathrm{e}^{rT}\hat{p}$.

4.2.3 Simulation techniques

Normal inverse Gaussian processes can be described as time changed Brownian motions, which is they key to simulate them, see the books by Cont and Tankov [8] and Schoutens [15].

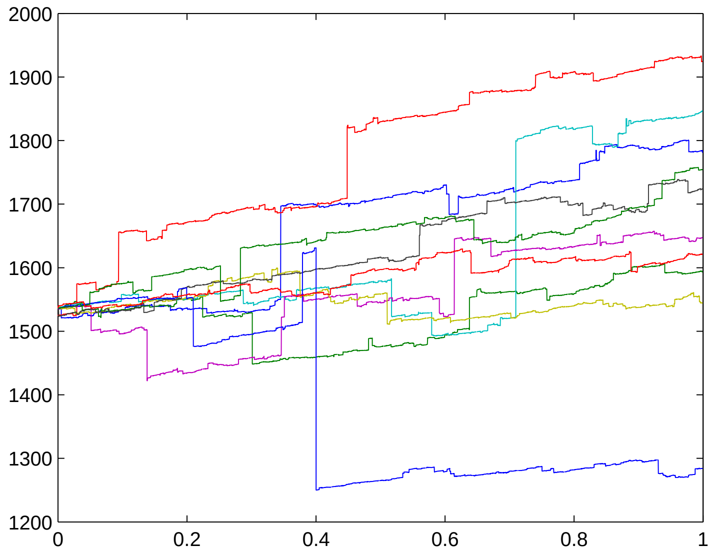

Figure 4.1: Ten simulated NIG process paths.

As for CGMY processes, they can be simulated by methods developed by Madan and Yor [10] and Poirot and Tankov [12][2].

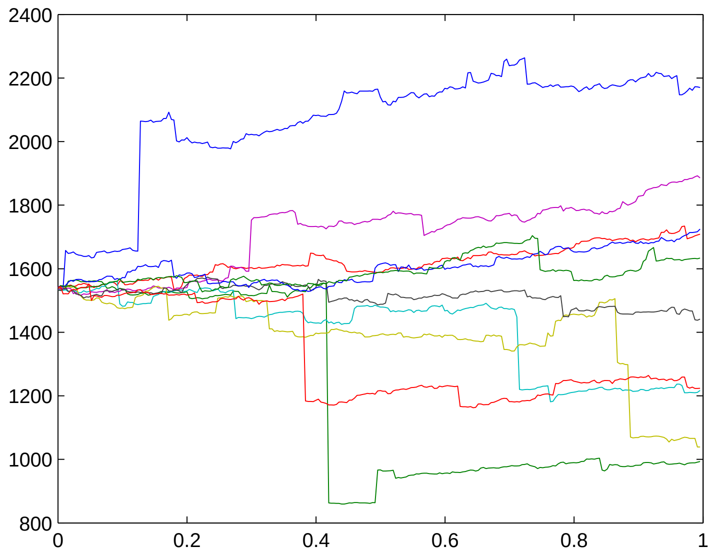

Figure 4.2: Ten simulated CGMY paths

[2]We are grateful to Peter Tankov for providing us with a copy of this paper.

We will not discuss simulation techniques for Meixner processes, and thus not consider exotic option pricing based on Meixner processes.

4.3 Results

Before pricing the exotic options we checked the accuracy of our simulation approach by means of pricing the European vanilla option using Monte Carlo simulations for the maturity date June 20, 2008, and compare the simulated prices with the corresponding real world market prices from Appendix A. The results were as follows:

Strike	BS	NIG	CGMY	Mean of bid and ask	Bid	Ask	Close
1300	282.9	298.8	299.7	296.4	294.9	297.9	286.5
1325	262.2	277.9	278.5	275.6	274.1	277.1	145.0
1350	242.2	257.3	257.5	255.2	253.7	256.7	150.3
1375	222.9	237.0	237.0	235.2	233.7	236.7	182.0
1400	204.3	217.1	216.9	215.6	214.1	217.1	140.5
1425	186.6	197.6	197.3	196.6	195.1	198.1	190.4
1450	169.8	178.7	178.3	178.0	176.5	179.5	176.0
1475	153.9	160.3	159.9	160.1	158.6	161.6	154.0
1500	138.9	142.6	142.3	142.8	141.3	144.3	138.0
1525	124.9	125.6	125.6	126.3	124.8	127.8	124.0
1550	111.9	109.5	109.8	110.6	109.1	112.1	96.3
1575	99.8	94.4	95.2	95.7	94.2	97.2	92.2
1600	88.7	80.5	81.7	81.9	80.4	83.4	75.0
1650	69.3	56.6	58.6	57.4	55.9	58.9	56.0
1700	53.3	38.5	40.6	37.9	36.4	39.4	33.6
1800	30.2	17.5	18.1	13.2	12.2	14.2	9.0

Table 4.1: Monte Carlo price for European Vanilla and Market Prices

From the above table we see that the Monte Carlo simulations give very satisfactory results (for the mean of bid and ask price).

Next we apply the Monte Carlo approach to exotic option pricing. We selected the maturity time $T = 1.0521$ and the $K = 1500$, while the barrier levels ranged from $0.5\,S(0)$ to $1.5\,S(0)$. We used $N = 100000$ simulated trajectories. The results from NIG and CGMY simulations are preceeded by those from the closed-form formulas fot the Black-Scoles Brownian motion framework. Note that results for NIG processes and CGMY processes above are very similar, while results for normal Brownian motions are a little bit different.

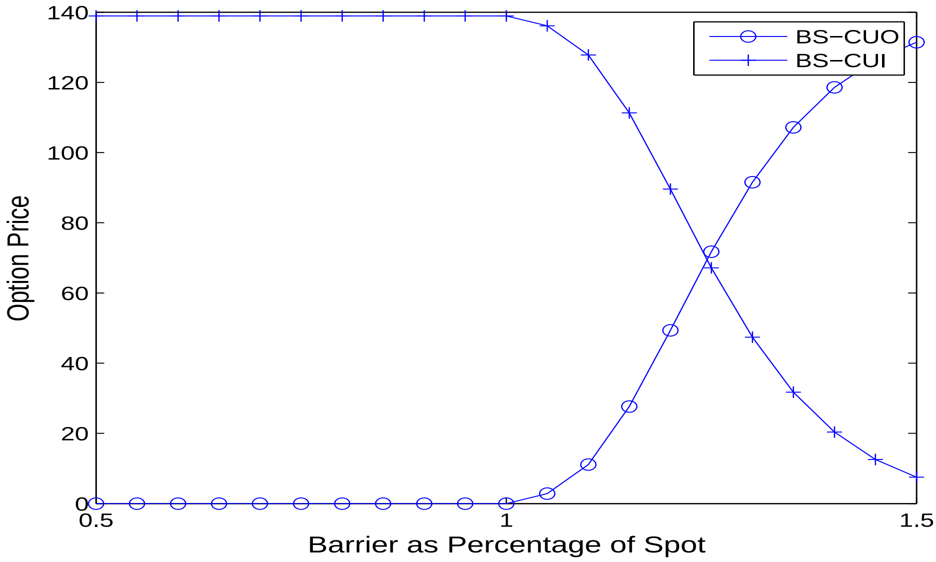

Figure 4.3: Brownian motion barrier as percentage of spot

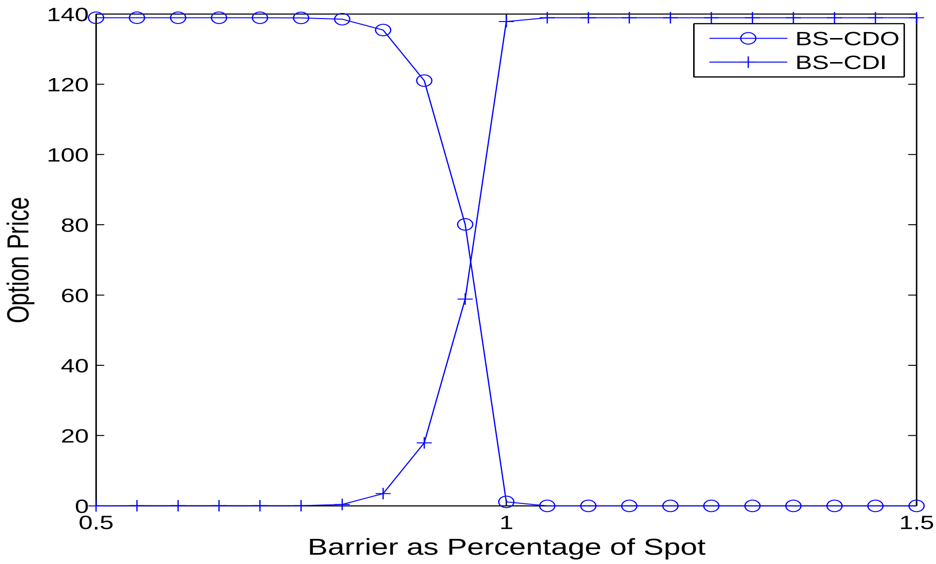

Figure 4.4: Brownian motion barrier as percentage of spot

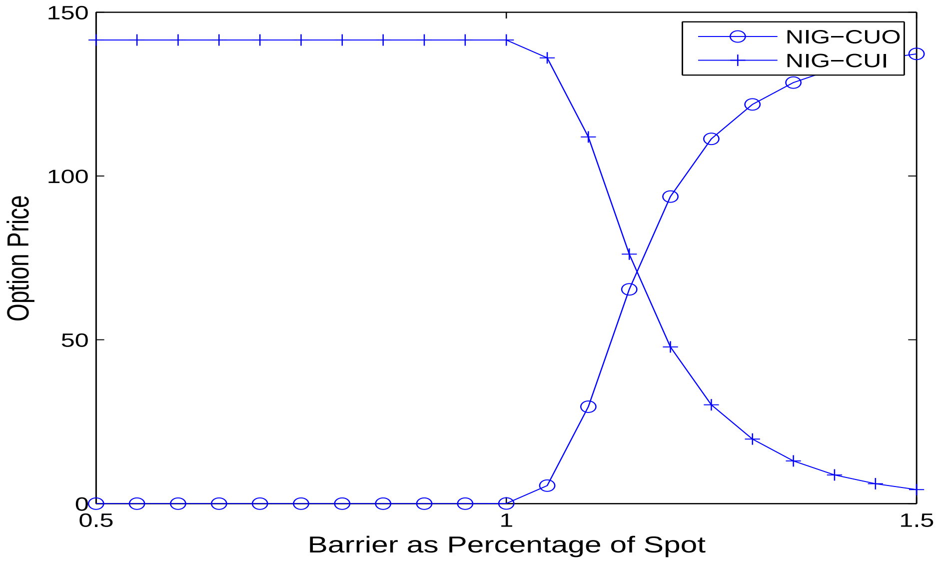

Figure 4.5: NIG barrier as percentage of spot

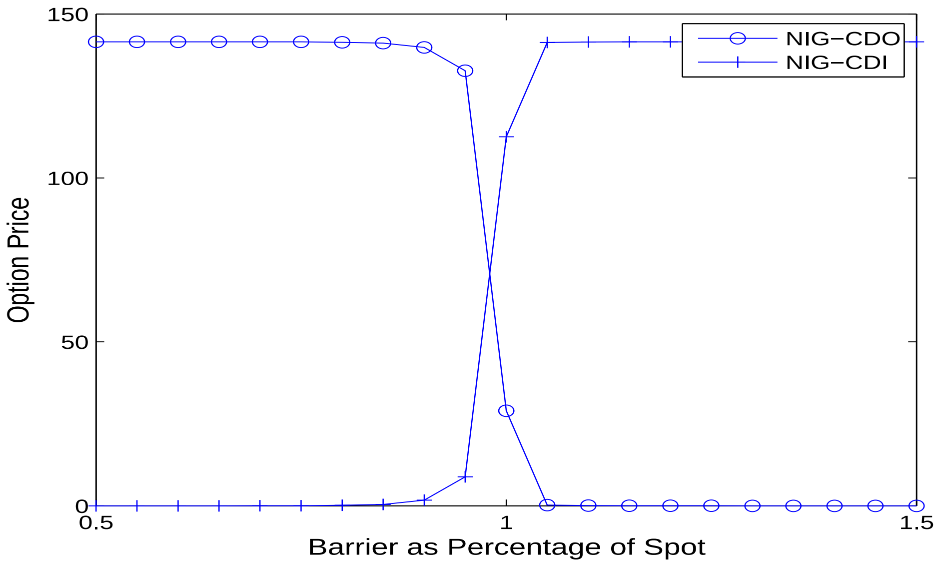

Figure 4.6: NIG barrier as percentage of spot

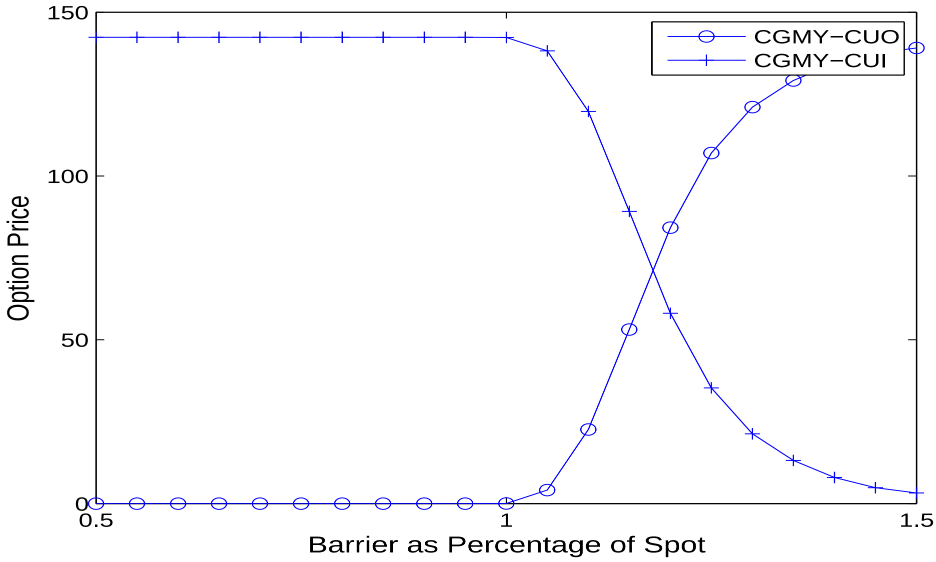

Figure 4.7: CGMY barrier as percentage of spot

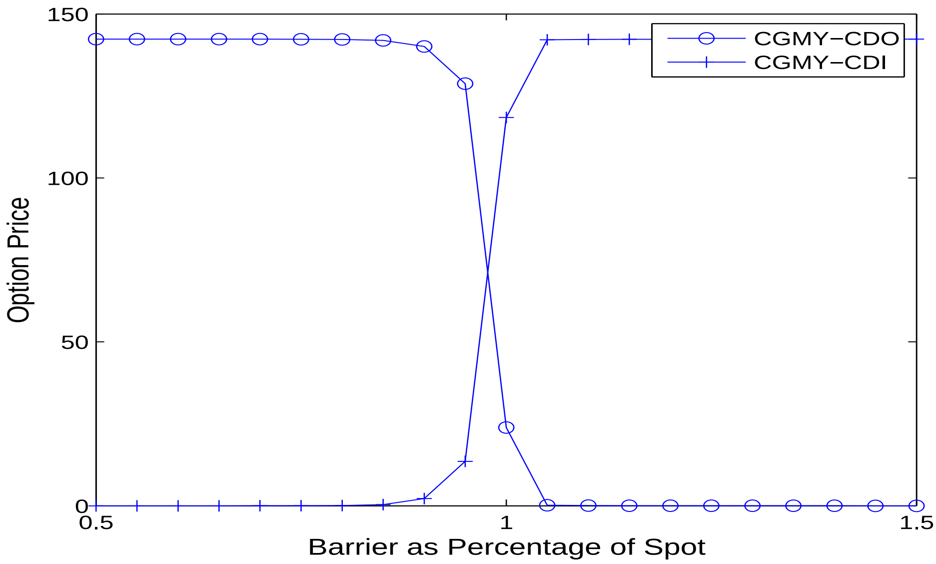

Figure 4.8: CGMY barrier as percentage of spot

Chapter 5

Conclusion

In this master thesis, we focus on the Option Pricing using Lévy processes. We started with definition of Lévy process and the examples of Lévy process.

We use Maximum-likelihood estimation to estimate the parameters and show that Lévy process fit the log-returns of of S&P 500 historical data better than Brownian Motion by means of both graphical and quantitative tests. However, for the distribution which do not have closed-form, this method cannot be used. Secondly, we price the option price using fast fourier transformation for Lévy process since it is easy to find the characteristic function for most of the Lévy process.The calibration results show that non-Gaussian Lévy processes describes the market price better than Brownian motion. At last, we use the calibration result to price exotic option using Monte Carlo simulation.

We also test the selection of dataset. The results show that using mean of bid and ask is slightly better than bid and ask. All of these three in turn are much better than using close prices.

In the future work, it would be interested to calibrate inverse problem with more option data. We also need to consider the Lévy process with stochastic volatility.

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Appendix A

S&P 500 call option prices

We collected 100 call option prices for the S&P 500 index at the close of market on Jun,1,2007 from Yahoo Finance. The closed index price is $S_0 = 1536.34$. We selected the risk free interest rate 0.05 and dividend yield 0.019. The depicted prices are the mean of bid and ask prices that we used for our calibrations.

Strike	Jun 15 2007	Jul 20 2007	Sep 21 2007	Dec 21 2007	Mar 21 2008	Jun 20 2008	Dec 19 2008
1300	239.1	244.5	254.0	268.5		296.4	322.9
1325	214.2	220.0	230.4	246.0		275.6	303.2
1350	189.3	195.6	207.1	223.9	239.6	255.2	283.9
1375	164.5	171.4	184.1	202.2		235.2	265.0
1400	139.7	147.4	161.5	181.0	198.5	215.6	246.5
1425	114.9	123.7	139.4	160.4	178.8	196.6	228.3
1450	90.4	100.5	118.1	140.4	159.6	178.0	210.7
1475	66.05	78.2	97.7	121.2	141.1	160.1	193.5
1500	42.85	56.9	78.4	103.0	123.4	142.8	176.8
1525	22.25	38.3	60.6	85.8	106.5	126.3	160.8
1550	6.95	22.25	44.4	69.9	90.7	110.6	145.3
1575	1.275	10.75	31.0	55.4	75.9	95.7	
1600	1.15	4.5	20.1	42.6	62.4	81.9	116.3
1650			6.6	22.8	39.6	57.4	
1700				10.3	22.9	37.9	67.8
1800				1.25		13.2	34.2
1900							14.4
2000							5.0