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MASTER'S THESIS

Lévy process based option pricing on Swedish markets

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Abstract

In this Master's thesis we investigate how well some different Lévy processes perform in the modelling of price movements and option prices on the Swedish stock market. The asset used as reference for our investigation is the OMXS30 index.

In the first part of the thesis we calibrate the different models to the log returns of the OMXS30 index. We also evaluate the fit of each model by applying some frequently used statistical tests.

In the second part of the thesis we discuss and compare the distributional properties of two different risk-neutral probability measures, the Mean Corrected measure and the Esscher measure. We also look briefly at the concept of risk-neutral option pricing and possible calibration methods in order to predict future market prices of this type of derivative securities. Two calibration methods are compared against observed market prices for the OMXS30 put and call options. It is shown that Lévy process based models perform slightly better than the Black-Scholes model in the pricing of options. We also conclude that it might be better to calibrate against historical option prices than against historical asset prices, at least when only a single type of option is under consideration.

Keywords: Black-Scholes mode, calibration, Esscher measure, European option, Lévy process, Mean Corrected measure, option pricing, risk-neutral measure, vanilla option.

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Göteborg in November 2006

Andreas Hansson

Contents

Chapter 1

Introduction

The area of mathematical finance has had a strong development during the last four decades, beginning with the Bachelier-Samuelson approach to the modelling of financial assets. This was followed in 1973 by the famous and frequently used option pricing model of Black, Scholes and Merton.

During the last decade a variety of asset price models have been proposed. One important example of such a model is the exponential Lévy process. Since the assumptions behind the derivation of the Black-Scholes formula are independent of the underlying model, the prices of contingent claims might as well be derived under the adoption of more general stochastic processes driving the underlying entity. The drawback of this approach is that the option prices, which are given by integrals, have to be calculated numerically.

An important issue, discussed in most recent literature on derivative pricing, is the choice of calibration method to estimate the parameters of the underlying asset price model. Different authors suggest different methods based on their personal experience and no general benchmark has been established. In this thesis calibration of three different families of exponential Lévy processes models to the Swedish OMXS30 index is carried out by applying different goodness of fit tests. Further, two different risk-neutral martingale measures are used and compared for the employed Lévy processes. We also investigate the issue whether calibration should of the underlying asset price process should be done against historical asset prices or against observed option prices. Finally, theoretical option prices for the two calibration methods, under the different Lévy models, are compared to future market prices of the OMXS30 put and call options.

Chapter 2

Lévy models

2.1 Considered Lévy processes

In the thesis we will work under the assumption that financial asset prices S_t develop as a stochastic process $S_t = S_0 e^{X_t}$, where X_t is a Lévy process[1]. This model implies independence and identical distributions of consecutive daily log-returns, which in turn is not completely true for most financial assets. However, it is well-known to be a good approximative model.

2.2 The Normal process (Brownian Motion)

The *Normal process* (Brownian Motion with drift) is the Lévy process X_t such that X_t has a Normal distribution with probability density function

$$f_{\text{Normal}}(x;\mu t,\sigma^2 t) = \frac{1}{\sqrt{2\pi\sigma^2 t}} \exp\left\{-\frac{(x-\mu t)^2}{2\sigma^2 t}\right\} \quad \text{for } x \in \mathbb{R},$$

where $\mu \in \mathbb{R}$ and $\sigma^2 \geq 0$ are parameters.

2.3 The Meixner process

The *Meixner process* is the Lévy process X_t such that X_t has a Meixner distribution with probability density function

$$f_{\text{Meixner}}(x;a,b,dt,mt) = \frac{(2\cos(b/2))^{2dt}}{2a\pi\Gamma(2dt)} \exp\left\{\frac{b(x-mt)}{a}\right\} \left|\Gamma\left(dt + \frac{i(x-mt)}{a}\right)\right|^2 \quad \text{for } x \in \mathbb{R},$$

where Γ is the Gamma function and $a > 0$, $b \in (-\pi,\pi)$, $d > 0$ and $m \in \mathbb{R}$ are parameters.

[1] We say that $\{X_t\}_{t\geq 0}$ is a Lévy process if X_t has stationary and independent increments and satisfy $X_0 = 0$. See [8] for more information.

The Meixner process has the following first four moments:

$$adt\tan\left(\frac{b}{2}\right)+mt \qquad \text{(Mean)} \tag{2.1}$$

$$\frac{1}{2}\frac{a^2dt}{\cos^2(b/2)} \qquad \text{(Variance)} \tag{2.2}$$

$$\sin\left(\frac{b}{2}\right)\sqrt{\frac{2}{dt}} \qquad \text{(Skewness)} \tag{2.3}$$

$$3+\frac{2-\cos(b)}{dt} \qquad \text{(Kurtosis)} \tag{2.4}$$

The Meixner process has no continuous Brownian motion part. Since the jump part of the Meixner process can model a large number of small jumps, the Brownian motion does not seem to be required.

2.4 The Normal Inverse Gaussian process (NIG)

The *Normal Inverse Gaussian process* (NIG) is the Lévy process X_t which is such that X_1 has a Normal Inverse Gaussian distribution with probability density function

$$f_{\text{NIG}}(x;\alpha,\beta,\delta t,\mu t)=\frac{\alpha\delta t}{\pi}\exp\left\{\delta t\sqrt{\alpha^2-\beta^2}+\beta(x-\mu t)\right\}\frac{K_1\left(\alpha\sqrt{\delta^2t^2+(x-\mu t)^2}\right)}{\sqrt{\delta^2t^2+(x-\mu t)^2}} \quad \text{for } x\in\mathbb{R},$$

where

$$K_\nu(z)=\frac{1}{2}\int_0^\infty u^{\nu-1}\exp\left\{-\frac{z}{2}\left(u+\frac{1}{u}\right)\right\}du$$

is the modified Bessel function of the third kind, and $\alpha>0$, $\beta\in(-\alpha,\alpha)$, $\delta>0$ and $\mu\in\mathbb{R}$ are parameters.

The Meixner process has the following first four moments:

$$\frac{\beta\delta t}{\sqrt{\alpha^2-\beta^2}}+\mu t \qquad \text{(Mean)} \tag{2.5}$$

$$\frac{\alpha^2\delta t}{(\alpha^2-\beta^2)^{3/2}} \qquad \text{(Variance)} \tag{2.6}$$

$$\frac{3\beta}{\alpha\sqrt{\delta t}(\alpha^2-\beta^2)^{1/4}} \qquad \text{(Skewness)} \tag{2.7}$$

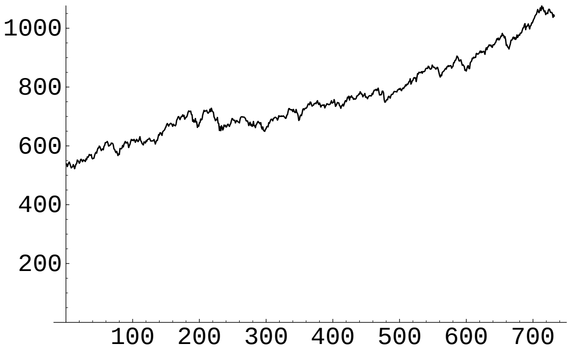
$$3\left(1+\frac{\alpha^2+4\beta^2}{\delta t\alpha^2\sqrt{\alpha^2-\beta^2}}\right) \qquad \text{(Kurtosis)} \tag{2.8}$$

The NIG process has no continuous Brownian motion part. Since the jump part of the NIG process can model a large number of small jumps, the Brownian motion does not seem to be required.

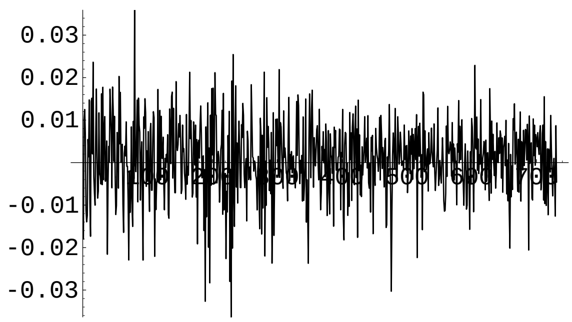
Chapter 3

Modelling of the OMXS30 index

The data set we chosed, the OMXS30 index[2], spans from the 12 June, 2003, to 3 May, 2006, a total of 732 trading days. The data set itself and the corresponding log-returns are depicted in Figure 3.1 below.



The OMXS30 index.



Log-returns of the OMXS30 index.

Figure 3.1: The OMXS30 index from 12 June, 2003, to 3 May, 2006.

3.1 Independence of the log-returns

First we check if the assumption of independence between the daily log-returns is reasonably correct. To that end, the empirical autocorrelation function (ACF) is studied.

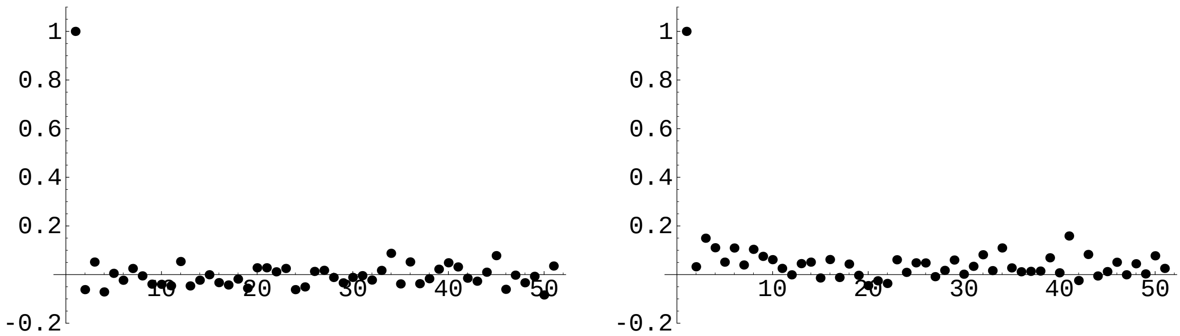
The empirical ACF at lag j for a data set $x = \{x_1, x_2, ..., x_n\}$ is defined as

$$\rho_n(j) = \frac{\gamma_n(j)}{\gamma_n(0)}, \quad \text{where} \quad \gamma_n(j) = \frac{1}{n}\sum_{t=1}^{n-j}\left(x_t - \overline{x}\right)\left(x_{t+j} - \overline{x}\right).$$

In Figure 3.2 below we have plotted the ACF for our log-returns, as well as that for the squared log-returns. This is a customary practice, as log-returns are well-known to

[2]OMX Stockholm 30. The 30 most traded stocks on the Stockholm stock market.

usually be uncorrelated, while squared log-returns usually will be correlated, if there is any dependence.



Empirical ACF for the log-returns. Empirical ACF for the squared log-returns.

Figure 3.2: Empirical auto correlation function for the OMXS30 log-returns.

Clearly, there are no indications of correlation between log-returns. On the other hand, there seems to be a slight correlation between squared log-returns for lags of a few days, indicating a non-constant volatility, but no correlation between squared log-returns further apart. However, as we will only be interested in option prices for 30 days, or longer periods, based on Figure 3.2, it seems very reasonable to assume that log-returns are sufficiently independent for our use of exponential Lévy process models.

3.2 The distribution of the OMXS30 log returns

We fit the log returns of the OMXS30 data set to the increments of Normal, Meixner and NIG Lévy processes. The parameters for the different distributions are estimated by the Maximum Likelihood method.

We apply QQ-plot methodology to get a graphical illustration of the fit for the examined Lévy process models in Figures 3.3-3.5 below. We see that the Meixner and NIG processes give better fits that the Normal process. This is not unexpected, as the former two processes, contatry to the later, can accomodate a variable skewness and kurtosis, as well as semi-heavy tails. This in turn, argubly, are well-known features of log-returns of stock prices.

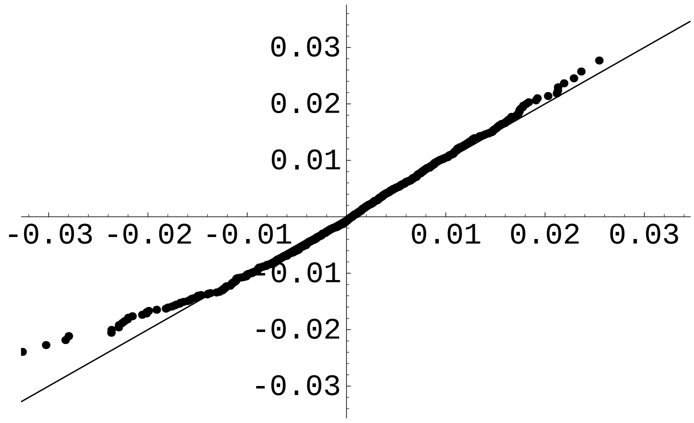


Figure 3.3: Normal QQ-plot for the OMXS30 index.

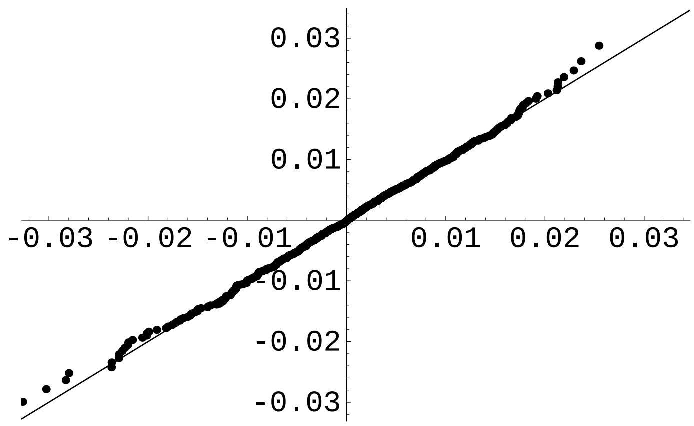


Figure 3.4: Meixner QQ-plot for the OMXS30 index.

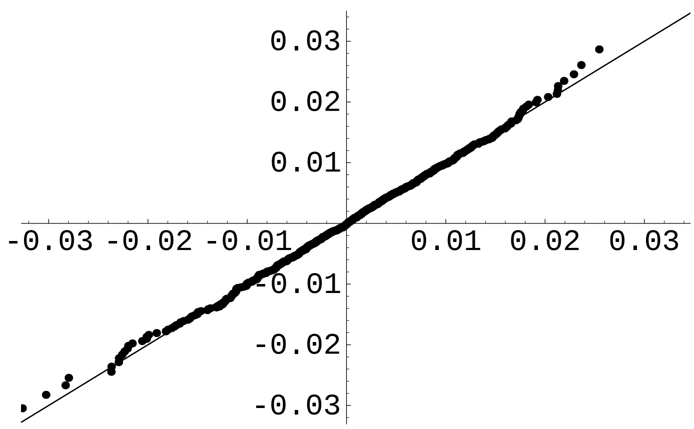


Figure 3.5: NIG QQ-plot for the OMXS30 index.

To get a quantitative measure of the fit of the different Lévy models, we employ the Kolmogorov-Smirnov (KS) and the Anderson-Darling (AD) test statistics. (See e.g, [12] for an explanation of these well-known tests.) The observed values for the KS and AD test statistics are given in Table 3.1 below.

	Normal	Meixner	NIG
KS	0.03970	0.01325	0.01332
AD	0.34366	0.04801	0.04558

Table 3.1: KS and AD test statistics for the fit of OMXS30 log returns.

The relative difference between the KS and AD test statistics for the Meixner and NIG models, and those for the Normal model, Δ_{KS} and Δ_{AD}, respectively, are defined as

$$\Delta_{\text{KS}} = \frac{\text{KS}_{\text{Normal}} - \text{KS}_{\text{Lévy}}}{\text{KS}_{\text{Lévy}}} \quad \text{and} \quad \Delta_{\text{AD}} = \frac{\text{AD}_{\text{Normal}} - \text{AD}_{\text{Lévy}}}{\text{AD}_{\text{Lévy}}},$$

where Lévy = Meixner and Lévy = NIG. The observed values for Δ_{KS} and Δ_{AD} are given in Table 3.2 below.

	Meixner	NIG
Δ_{KS}	2.00	1.98
Δ_{AD}	6.16	6.54

Table 3.2: Test statistics for the OMXS30 log returns.

We see that the Meixner and NIG models 2 times better that the Normal model for the KS test, and 6-6.5 better than the Normal model for the AD test. In addition, as can be seen from well-known KS test theory, the KS test statistic values for the Meixner and NIG models are non-significant, while the KS statistic for the Normal model indicates a significant deviation from such a model.

In conclusion, the Normal distribution cannot model the log-returns for the OMXS30 price process as well as the Meixner and NIG distributions.

3.3 Replicating the OMXS30 price movements

It is important to be able to replicate the assumed model for an asset price. Fortunately, there is a number of ways in which a Lévy model with known parameters can be simulated. See [9] or [1] for information on simulation algorithms.

In this section we will look at some simulated trajectories of the calibrated models. This is in order to see if it is possible to tell any difference between the simulated processes and the OMXS30 data set.

3.3.1 Modelling the OMXS30 index using the Normal model

In Figure 3.6 below the log-returns for the OMXS30 data set (the upper right plot) is depicted together with three plots of the increments of the calibrated Normal process (the upper left plot and the lower plots). Possibly, one can argue that the OMXS30 data display somewhat more really large log-returns, than does the Normal process increments, but it is really hard to tell a difference.

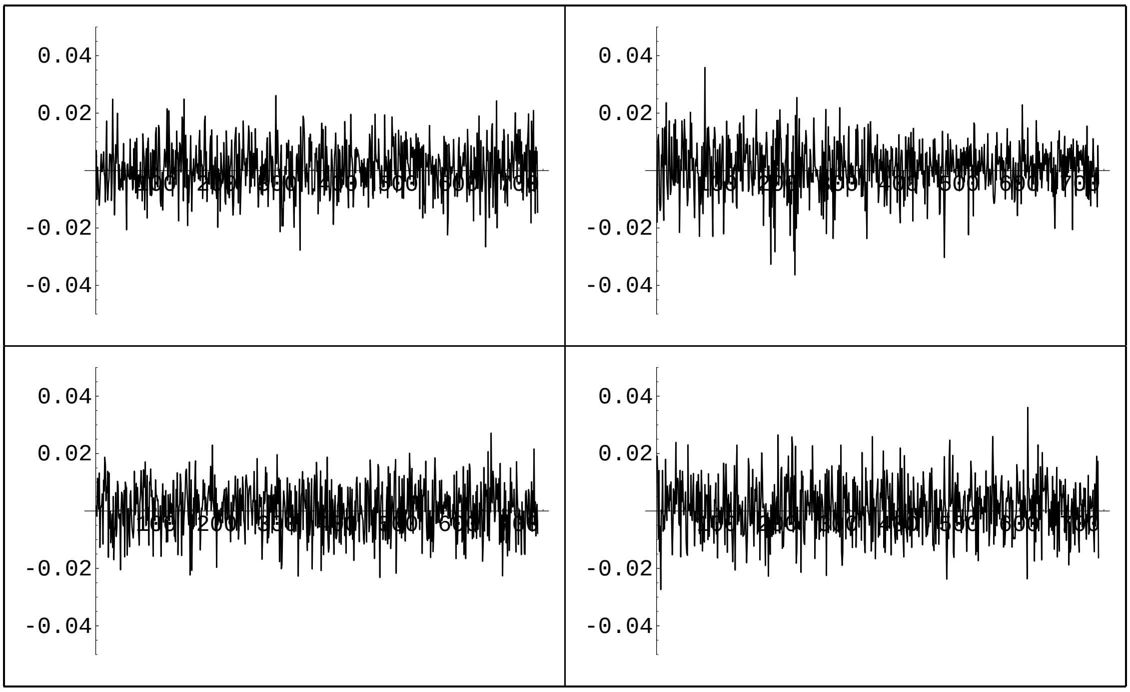


Figure 3.6: Log-returns of the OMXS30 data set (upper right plot) together with three plots of increments of calibrated Normal process.

3.3.2 Modelling the OMXS30 index using the Meixner model

In Figure 3.7 below the log-returns for the OMXS30 data set (the upper right plot) is depicted together with three plots of the increments of the calibrated Meixner process (the upper left plot and the lower plots). It seems that it is now hard to find any difference between the plot based on OMXS30 data and the simulated ones.

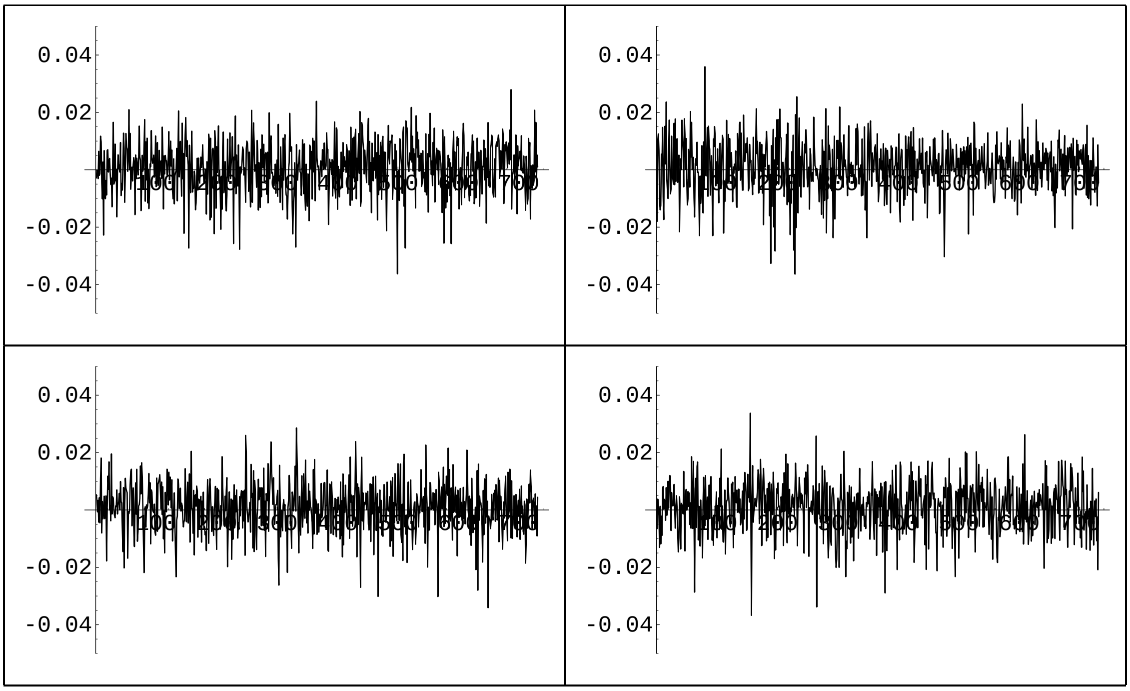


Figure 3.7: Log-returns of the OMXS30 data set (upper right plot) together with three plots of increments of calibrated Meixner process.

3.3.3 Modelling the OMXS30 index using the NIG model

In Figure 3.8 below the log-returns for the OMXS30 data set (the upper right plot) is depicted together with three plots of the increments of the calibrated NIG process (the upper left plot and the lower plots). It seems that it is again hard to find any difference between the plot based on OMXS30 data and the simulated ones.

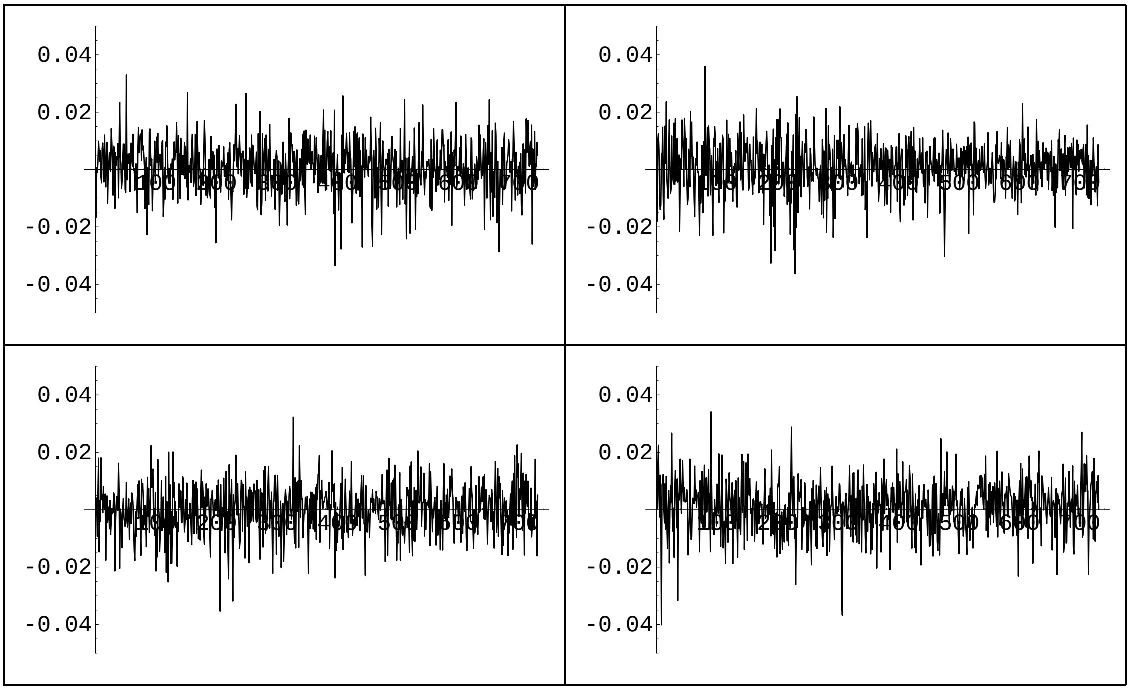


Figure 3.8: Log-returns of the OMXS30 data set (upper right plot) together with three plots of increments of calibrated NIG process.

3.4 Comments

We have found that there seem to be no strong dependencies between log-returns of the OMXS30 index.

After a calibration of exponential Normal, Meixner and NIG Lévy process models to the OMXS30 data set, we found that the Meixner and NIG models can capture the distribution of the OMXS30 log-returns better than the Normal model.

That Meixner and NIG models are better to model stock prices and stock indices, that are the Normal model, has already been shown in previous work, see for example [3] or [12], so this is no surprise. In our work the main reason for calibrating the Normal, Meixner and NIG models to data is not to show that Meixner and NIG are better yet again, but to get the right parameters in order to be able to derive the risk-neutral distributions used in latter sections on option pricing.

Chapter 4

Arbitrage and risk-neutral market models

We begin with the fundamental definition of the concept of *arbitrage.*

Definition *An* arbitrage *is a portfolio value process (strategy)* Y_t *with* $Y_0 = 0$ *such that*

$$\mathbb{P}[Y_T \geq 0] = 1 \quad \text{and} \quad \mathbb{P}[Y_T > 0] > 0 \quad \text{for some} \quad T > 0.$$

Hence an arbitrage is a strategy that allows the making of profit out of nothing without taking any risk. This would make it impossible for the market to be in equilibrium. The question we thus have to ask ourselves is: Doen there exist an arbitrage free market model? The answer to this question is given in the following theorem (see e.g, [10]), that is crucial in the theory of financial derivatives.

Theorem [First Fundamental Theorem of Asset Pricing] *If a market model has a* risk-neutral probability measure*, then it does not admit arbitrage.*

4.1 Risk-neutral measures

We will not treat the concept of risk-neutrality including the construction of risk-neutral measures.

A *risk-neutral probability measure* is an equivalent probability measure[3] that makes no distinction between putting the money on the bank or investing them in some risky asset. Hence, in order to make sure that there are no arbitrage opportunities, we must find a probability measure $\mathbb{Q}$, equivalent to the real probability measure $\mathbb{P}$, such that

$$S_0 \mathrm{e}^{rt} = \mathbb{E}^{\mathbb{Q}}[S_t | \mathcal{F}_0], \tag{4.1}$$

[3] See e.g, [10] and [4] on more about equivalent probability measures, absolutely continuous distributions and Radon-Nikodym derivatives.

where $S_0 > 0$ is the initial value of the risky asset, $\mathcal{F}_0$ is the trivial σ-field, $r > 0$ is the risk free interest rate and S_t the value of the risky asset at time t. We then say that the discounted stock price process is a *martingale*, which is a very basic property in the subject of stochastic calculus.

If we assume that $S_t = S_0 e^{X_t}$, where X_t is a Lévy process independent of S_0, under the real probability measure $\mathbb{P}$, the *taking out what is known* property in martingale theory means that (4.1) is equivalent to the problem of finding a $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that

$$\mathrm{e}^{rt} = \mathbb{E}^{\mathbb{Q}}[\mathrm{e}^{X_t}|\mathcal{F}_0]. \tag{4.2}$$

There are a number of ways in which a $\mathbb{Q}$ satisfying (4.2) can be constructed, see e.g, [9] and [3]. We will consider two such ways, namely the *Mean Correcting Measure* and the *Esscher Measure.*

4.1.1 The Mean Correcting Measure (MCM)

One way to construct a risk neutral measure is to require that the law of X_t under $\mathbb{Q}$ is the same as that of $(r - \omega)t + X_t$ under $\mathbb{P}$, for a suitable ω. Because then (4.2) becomes

$$\mathrm{e}^{rt} = \mathbb{E}[\mathrm{e}^{(r-\omega)t+X_t}|\mathcal{F}_0] \implies \mathrm{e}^{\omega t} = \mathbb{E}[\mathrm{e}^{X_t}|\mathcal{F}_0] = \left(\mathbb{E}[\mathrm{e}^{X_1}|\mathcal{F}_0]\right)^t,$$

so that

$$\omega = \log\left(\mathbb{E}[\mathrm{e}^{X_1}|\mathcal{F}_0]\right). \tag{4.3}$$

MCM for the Normal process

If $X_t \sim \mathrm{N}(\mu t, \sigma^2 t)$ is a Normal process, then a measure that makes $X_t \sim \mathrm{N}((r-\sigma^2/2)t, \sigma^2 t)$ is risk-free. The exponential of this process is known as the *Black-Scholes model.*

MCM for the Meixner process

If $X_t \sim \mathrm{Meixner}(a, b, dt, mt)$ is a Meixner process, then a measure that makes $X_t \sim \mathrm{Meixner}(a, b, dt, (r-\omega)t)$ is risk-free, where by (4.3)

$$\omega = \log\left(\mathbb{E}[\mathrm{e}^{X_1 - m}|\mathcal{F}_0]\right) = 2d\log(\cos(b/2)) - 2d\log(\cos[(a+b)/2]).$$

MCM for the NIG process

If $X_t \sim \mathrm{NIG}(\alpha, \beta, \delta t, \mu t)$ is a NIG process, then a measure that makes $X_t \sim \mathrm{NIG}(\alpha, \beta, \delta t, (r-\omega)t)$ is risk-free, where

$$\omega = \log\left(\mathbb{E}[\mathrm{e}^{X_1 - m}|\mathcal{F}_0]\right) = \delta\sqrt{\alpha^2 - \beta^2} - \delta\sqrt{\alpha^2 - (1+\beta)^2}.$$

4.1.2 The Esscher Measure (EM)

The Esscher transform method is based upon considering the probability density function

$$f_{X_t}^{(\theta)}(x) = \frac{e^{\theta x} f_{X_t}(x)}{\mathbb{E}[e^{\theta X_t}|\mathcal{F}_0]}. \tag{4.4}$$

Under this density function (4.2) becomes

$$e^{rt} = \mathbb{E}^{\theta}[e^{X_t}|\mathcal{F}_0] = \frac{\mathbb{E}[e^{(1+\theta)X_t}|\mathcal{F}_0]}{\mathbb{E}[e^{\theta X_t}|\mathcal{F}_0]} = \left(\frac{\mathbb{E}[e^{(1+\theta)X_1}|\mathcal{F}_0]}{\mathbb{E}[e^{\theta X_1}|\mathcal{F}_0]}\right)^t,$$

so that

$$r = \frac{\log\left(\mathbb{E}[e^{(1+\theta)X_1}|\mathcal{F}_0]\right)}{\log\left(\mathbb{E}[e^{\theta X_1}|\mathcal{F}_0]\right)}. \tag{4.5}$$

If we can a θ the solves (4.5), then we have a probability measure $\mathbb{Q} = \mathbb{P}^{\theta}$ such that (4.2) is fulfilled.

EM for the Normal process

If $X_t \sim \mathrm{N}(\mu t, \sigma^2 t)$ is a Normal process, then $X_t \sim \mathrm{N}((r - \sigma^2/2)t, \sigma^2 t)$ under the risk-free EM measure, which is again the Black-Scholes model, as for MCM.

EM for the Meixner process

If $X_t \sim \mathrm{Meixner}(a, b, dt, mt)$ is a Meixner process, then $X_t \sim \mathrm{Meixner}(a, a\theta + b, dt, mt)$ under the risk-free EM measure, where the θ that solves (4.5) is given by

$$\theta = -\frac{1}{a}\left(b + 2\arctan\left(\frac{e^{(m-r)/(2d)} - \cos(\frac{1}{2}a)}{\sin(\frac{1}{2}a)}\right)\right). \tag{4.6}$$

EM for the NIG process

If $X_t \sim \mathrm{NIG}(\alpha, \beta, \delta t, \mu t)$ is a NIG process, then $X_t \sim \mathrm{NIG}(\alpha, \beta + \theta, \delta t, \mu t)$ under the risk-free EM measure, where θ can be calculated numerically from the equation

$$r = \delta\sqrt{\alpha^2 - (\beta + \theta)^2} - \delta\sqrt{\alpha^2 - (\beta + \theta + 1)^2} + m. \tag{4.7}$$

4.2 Examination of the risk-neutral measures

Having discussed two methods to find risk-neutral measures for our exponential Lévy process models Normal, Meixner and NIG, we will now take a closer look at the calibrated risk-neutral distributions for the OMXS30 log-returns.

From now on we assume that the risk-free interest rate r is 2.085%, which is the average 2 month STIBOR[4] interest rate for the first quarter 2006.

[4] Stockholm Interbank Offered Rate.

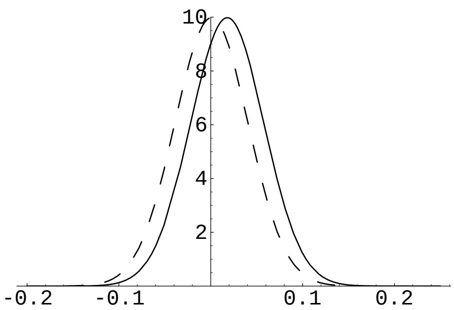
4.2.1 Comparing measures for the Normal distribution

In Table 4.1 below we display the calibrated Normal distribution together with the corresponding risk-neutral Normal distribution for the daily OMXS30 log-returns.

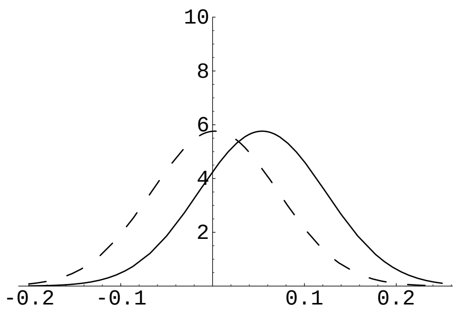
Original distribution	N(0.00090, 0.00894)
Risk-neutral distribution	N(0.000042, 0.00894)

Table 4.1: Original and risk-neutral Normal distribution

A graphical illustration of the corresponding Normal process probability density functions (PDF) are given in Figure 4.1 below, for the times $t = 20$ and $t = 60$ days. Since the drift of the risk-neutral distribution is much smaller than that of the originally calibrated distribution, the difference between the processes will increase with time.



PDF for $t = 20$ trading days.



PDF for $t = 60$ trading days.

Figure 4.1: Originally calibrated (full line) and risk-neutral (dashed line) Normal process distributions.

The standard deviation is an often used indication of risk, as it gives a measure of the amount of randomness when *looking ahead*. The standard deviation for the risk-neutral Normal process is plotted in Figure 4.2 below.

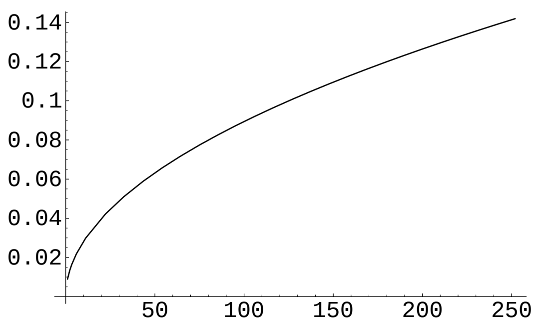


Figure 4.2: The standard deviation against time (trading days) for the calibrated Normal process.

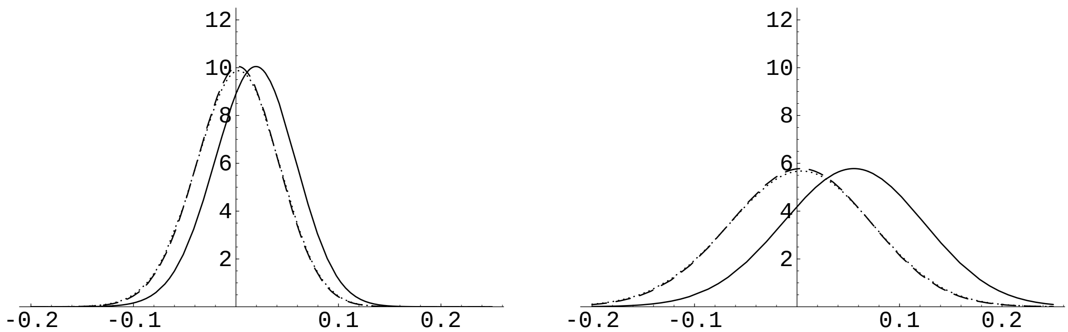
4.2.2 Comparing measures for the Meixner distribution

In Table 4.2 below we display the calibrated Meixner distribution together with the corresponding risk-neutral Meixner distributions for the daily OMXS30 log-returns.

Original distribution	Meixner(0.01154, -0.51005, 1.12090, 0.00427)
MCM distribution	Meixner(0.01154, -0.51005, 1.1209, 0.00341)
EM distribution	Meixner(0.01154, -0.63238, 1.1209, 0.00427)

Table 4.2: Originally calibrated and risk-neutral Meixner distributions.

A graphical illustration of the correponding Meixner process PDF's are given in Figure 4.3 below, for the times $t = 20$ and $t = 60$ days. Note that the difference between the two risk-neutral distribution is very small. We will later investigate to what extent these small differences affect option prices.



PDF for $t = 20$ trading days. PDF for $t = 60$ trading days.

Figure 4.3: Originally calibrated (full line), Mean Corrected (dashed line) and Esscher transformed (dotted line) Meixner process distributions.

Figure 4.4 below shows how the mean [recall (2.1)] for the two risk-neutral Meixner distributions increase with time.

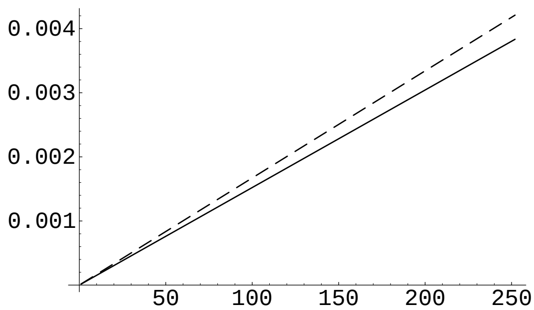


Figure 4.4: The mean for the Mean Corrected (dashed line) and Esscher transformed (full line) Meixner distributions against time (trading days).

The standard deviations of the risk-neutral Meixner processes [recall (2.2)] are plotted in Figure 4.5 below. Note that there is a slight difference between the two proceses, that increases with time.

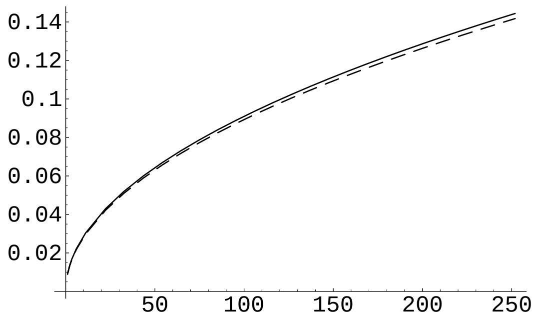


Figure 4.5: The standard deviation for the Mean corrected (dashed line) and Esscher transformed (full line) Meixner distribution against time (trading days).

In Figure 4.6 below the skewness [recall (2.3)] for the risk-neutral Meixner processes is depicted. Note that the skeweness is always negative skew, but that it tends to zero for large times for both of the processes.

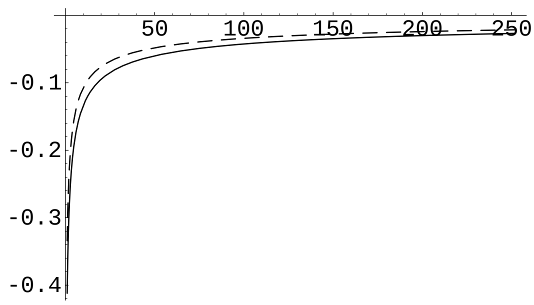


Figure 4.6: The skewness for the Mean corrected (dashed line) and Esscher transformed (full line) Meixner distributions against time (trading days).

Figure 4.7 below shows how the kurtosis [recall (2.4)] for the risk-neutral Meixner processes changes with time. Note that the kurtosis initially is *leptokurtic* (high peaked), but that it settles down with time to get more and more *mesokurtic* (Normal).

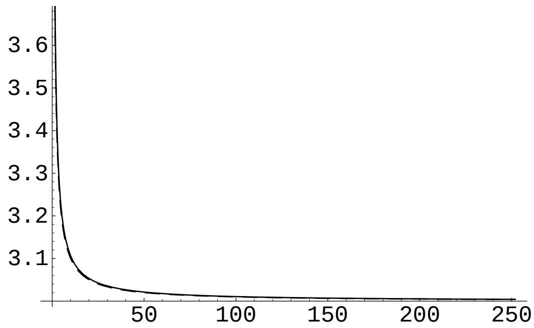


Figure 4.7: The kurtosis for the Mean corrected (dashed line) and Esscher transformed (full line) Meixner distributions against time (trading days).

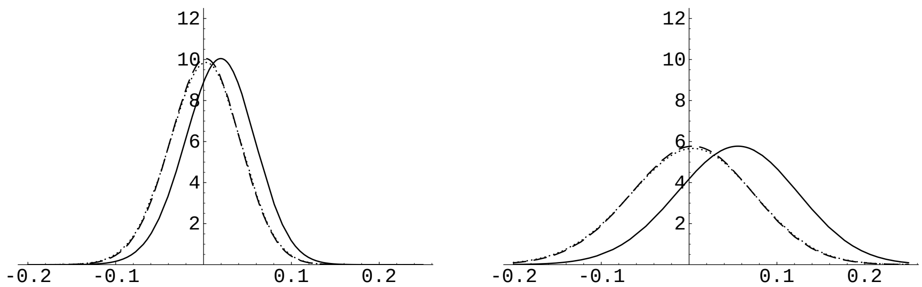
4.2.3 Comparing measures for the NIG distribution

In Table 4.3 below we display the calibrated NIG distribution together with the corresponding risk-neutral NIG distributions for the daily OMXS30 log-returns.

Original distribution	NIG(209.28075, -45.17271, 0.01556, 0.00434)
MCM distribution	NIG(209.28075, -45.17271, 0.01556, 0.00348)
EM distribution	NIG(209.28075, -55.7409, 0.01556, 0.00434)

Table 4.3: Originally calibrated and risk-neutral NIG distributions.

A graphical illustration of the correponding NIG process PDF's are given in Figure 4.8 below, for the times $t = 20$ and $t = 60$ days. As for the Meixner model, it is hard to tell any difference between the two risk-neutral distributions.



PDF for $t = 20$ trading days. PDF for $t = 60$ trading days.

Figure 4.8: Originally calibrated (full line), Mean corrected (dashed line) and Esscher transformed (dotted line) NIG process distributions.

Figure 4.9 below shows how the mean [recall (2.5)] for the two risk-neutral NIG distributions increase with time.

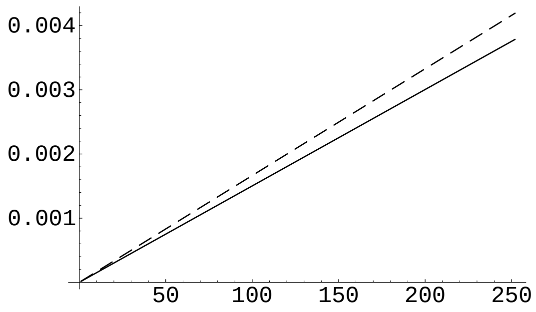


Figure 4.9: The mean for the Mean Corrected (dashed line) and Esscher transformed (full line) NIG distributions against time (trading days).

The standard deviations of the risk-neutral NIG processes [recall (2.6)] are plotted in Figure 4.10 below. Note that there is a slight difference between the two proceses, that increases with time.

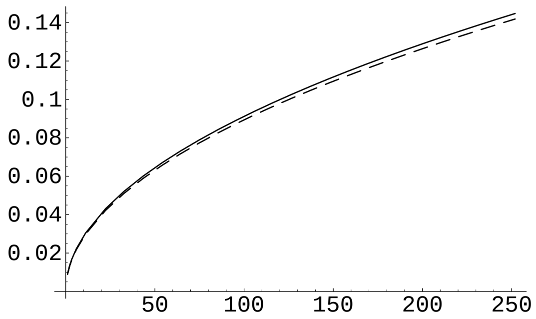


Figure 4.10: The standard deviation for the Mean corrected (dashed line) and Esscher transformed (full line) NIG distribution against time (trading days)..

In Figure 4.11 below the skewness [recall (2.7)] for the risk-neutral NIG processes is depicted. As for the Meixner model, the skewness is always negative, and tends to zero for large times.

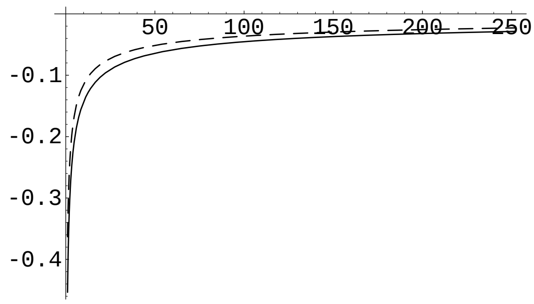


Figure 4.11: The skewness for the Mean corrected (dashed line) and Esscher transformed (full line) NIG distributions against time (trading days).

Figure 4.12 below shows how the kurtosis [recall (2.8)] for the risk-neutral NIG processes changes with time. As for the Meixner model they are leptokurtic in the beginning, but get more and more mesokurtic as time runs.

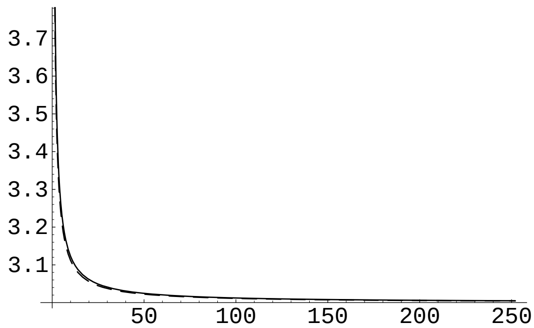


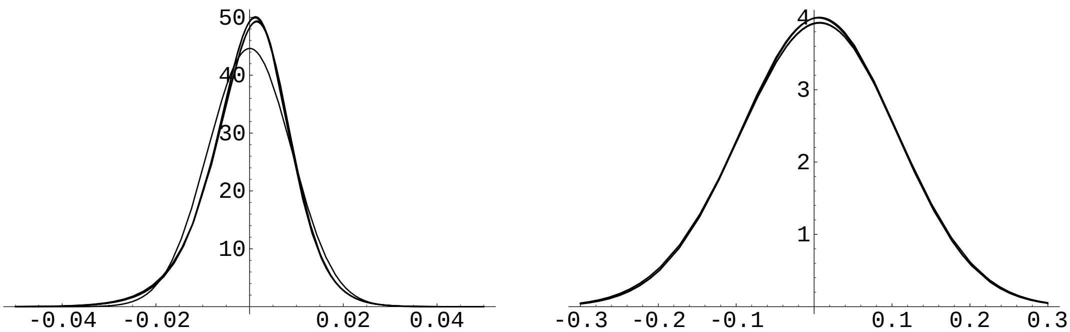
Figure 4.12: The kurtosis for the Mean corrected (dashed line) and Esscher transformed (full line) NIG distributions against time (trading days).

4.3 Comments on the risk-neutral distributions

From Figures 4.3 and 4.8 it seems that the risk-neutral distributions for the Meixner and NIG models are quite similar. Hence prices of contingent claims might not differ in any dramatic way for the two risk-neutral measures considered, under these models. However, we cannot a priori ignore the fact that there is a difference, that might affect the price of a derivative. Hence some kind of comparison has to be made. Such a comparison will be carried out in the following chapters.

An interesting observation is that the risk-neutral NIG and Meixner distributions seem to converge towards Normal distributions as time increases, see the next paragraph. One might speculate in what is the parameters of these Normal distributions, and if they coincide, as is indicated by Figure 4.13 below.

In Figure 4.13 below the risk-neutral NIG and Meixner distributions are depicted together with the Black-Scholes Normal distribution. It really seems as if the other distributions converge rather fast towards the Normal distributionis as time increases. This would imply that the prices of a contingent claim for long maturities are approximately equal to the Black-Scholes prices, at least for the models analyzed in this thesis. In the next chapter a more general result concerning the asymptotical prices of European options is discussed.



PDF's for $t = 1$ day. PDF's for $t = 125$ days.

Figure 4.13: Normal, Meixner and NIG process risk-neutral distributions.

Chapter 5

Risk-neutral option pricing

Many different options are traded on the markets. On large stock exchanges, such as the ones in London and New York, almost every imaginable option can be bought or sold. However, at smaller echanges, like the Stockholm bourse, the number of traded options is much lower.

In the following section we will look at the simplest type of options, namely the *European call and put options*. The options we are intersted in traded with the OMXS30 index as underlying asset are of this type.

5.1 Options

Anoption that may be expired prior to by the buyer maturity is called an *American option*, while an option that can only be expired on its maturity date is called a *European option*. European options are sometimes also referred to as *vanilla options* because of their simple design.

5.1.1 The European call option

An *European call option* gives the buyer the right, but not the obligation to buy the underlying asset S_t for the *strike price* K at the maturity date T.

The value of the European call option at time t is given by

$$c(S_t; K, T) = \max(S_t - K, 0).$$

If the value of the call option at the maturity date is greater than zero the buyer of the contract buys the asset for the strike price, while if this is not the case the contract expires without any trade being carried out.

5.1.2 The European put option

An *European put option* gives the buyer the right, but not the obligation to sell the underlying asset S_t for the strike price K at the maturity date T.

The value of the European put option at time t is given by

$$p(S_t; K, T) = \max(K - S_t, 0).$$

If the value of the put option at the maturity date is greater than zero the buyer of the contract sells the asset for the strike price, while if this is not the case the contract expires without any trade being carried out.

5.1.3 The put-call parity

A very useful result that gives a relation between the values of Europoean put options and a European call options is the *put-call parity*, which is easy to derive, and takes the form

$$p(S_t; K, T) + S_t = c(S_t; K, T) + K\mathrm{e}^{-r(T-t)}. \tag{5.1}$$

By the put-call parity, when comparing option pricing strategies, it is enough to consider one of the above type of European options, as $p(S_t; K, T)$ and $c(S_t; K, T)$ are determined by each other.

5.2 Pricing under the risk-neutral measure

The fair price Π_t at time t of a derivative Φ_t with fixed exercise date T based on a risk-neutral measure $\mathbb{Q}$ is given by

$$\Pi_t = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}\Phi_T \,\middle|\, \mathcal{F}_t\right].$$

In particular the fair prices at time t for the European call option is given by

$$c_t = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}c(S_T; K, T) \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}\max(S_T - K, 0) \,\middle|\, \mathcal{F}_t\right],$$

while the fair prices at time t for the European put option is

$$p_t = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}p(S_T; K, T) \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}\max(K - S_T, 0) \,\middle|\, \mathcal{F}_t\right].$$

In the pricing of derivatives which can be exercised prior to the maturity date other formulas have to be considered, but we will not encounter such derivatives in our work.

5.2.1 Price asymptotics for European call and put options

It is interesting to see what price a fair European option price converges to when the time to maturity T grows. In [5] it is proved that the price for a European call option converges towards the underlying asset price under any risk-neutral measure, that is

$$c_t = \mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-r(T-t)}\max(K - S_T, 0) \,\middle|\, \mathcal{F}_t\right] \to S_t \quad \text{as } T \to \infty.$$

In order to see what price a European put option converges to, we take conditional expectations on both sides of the put-call parity (5.1)

$$p_t+S_t = \mathbb{E}^{\mathbb{Q}}\left[p(S_t;K,T)+S_t \,\middle|\, \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[c(S_t;K,T)+K\mathrm{e}^{-r(T-t)} \,\middle|\, \mathcal{F}_t\right] = c_t+K\mathrm{e}^{-r(T-t)}. \quad (5.2)$$

As $c_t \to S_t$ and $Ke^{-r(T-t)} \to 0$, it follows that

$$p_t \to 0 \quad \text{as } T \to \infty.$$

The put-call parity is independent of the underlying risk-neutral measure. Hence, the Black-Scholes put price also converges to zero. Recall that we found that different risk-neutral distributions look much the same for long times in Figure 4.13. Since the Black-Scholes prices are easy to handle, we might consider to use the for options with long time to maturity, instead of prices based on more complicated exponential Lévy process models.

Chapter 6

Calibration methodology

When pricing options under the assumption of Lévy distributed log-returns, we have the possibility to choose either historical prices of the underlying entity or historical market option prices to calibrate the Lévy process. We will compare these approaches against true market prices[5] for the OMXS30 option.

6.1 Calibration using historical asset prices

This calibration is rather straightforward: We calibrate the model parameters for the log-returns and then change them (in a way specified by the chosen risk-neutral measure), to obtain risk-freeness. As have been mentioned, we will use both the mean corrected and Esscher risk-neutral measures.

6.2 Calibration using historical option prices

This calibration is frequently mentioned in the recent literature on derivative pricing. It is based on the idea that the markets view on the underlying asset is built on the option prices. Hence, the optimal parameters can be found without any direct knowledge of the history of the underlying asset. Given that there is a model which can replicate the option prices perfectly we want to solve the following problem:

Calibration Problem 1

Given observed call prices c_i for maturities T_i and strike prices K_i, $i \in I$, find the Lévy process X_t such that

$$c_i = \mathbb{E}\big[e^{-rT_i}\big(S_{T_i} - K_i\big)^+ \,\big|\, \mathcal{F}_0\big] = \mathbb{E}\big[e^{-rT_i}\big(S_0 \mathrm{e}^{X_{T_i}} - K_i\big)^+ \,\big|\, \mathcal{F}_0\big] \quad \textit{for } i \in I.$$

[5]Shared to the author by Jan Engvall, OMX Financial Training.

However, usually Lévy models cannot replicate every observed option price perfectly, so that Calibration Problem 1 has no solution. This might be handeled by minimizing the Root Mean Square Error between the theoretical pricing formula and the observed market prices, see [9]. A drawback with this approach is that options with higher prices are more influential on the calibration than are ones with lower price. To avoid this drawback we will use a slightly different calibration approach, which is discussed in [3] and [11]. It is stated in the following way:

Calibration Problem 2

Given observed call prices c_i for maturities T_i and strike prices K_i, $i \in I$, find the Lévy process X_t that minimizes

$$\sum_{i\in I} \omega_i \left|\mathbb{E}\left[e^{-rT_i}\left(S_0 \mathrm{e}^{X_{T_i}} - K_i\right)^+ \middle| \mathcal{F}_0\right] - c_i\right|^2$$

for some selection of weights $\omega_i > 0$.

The choice of the weights ω_i in Calibration Problem 2 is discussed in [3] and [11]. This choice should reflect our confidence in each option, which is given by the liquidity. Both [3] and [11] propose different choices depending on how much information we have on the market option prices. However, the optimal choice for the weights according to [3] is

$$\omega_i = \frac{1}{|c_i^{\text{bid}} - c_i^{\text{ask}}|^2},$$

where $|c_i^{\text{bid}} - c_i^{\text{ask}}|$ is the bid-ask spread for option c_i.

The bid-ask spread is by most actors on the market considered a good measure on the liquidity of the option, where a small spread answers to a high liquidity and a big spread to a smaller liquidity. Since the bid-ask spread for the options considered in this thesis is known, it is natural for us to use these weights for the calibration method.

Of course, in a practical consideration of Calibration Problem 2, one has to narrow down the class of Lévy processes considered to some parametric families, such as our selection of Normal, Meixner and NIG processes.

The optimal values of the objective function in the minimization procedure turned out to be 1.21829, 1.11047 and 1.10633, for the Normal, Meixner and NIG processes, respectively.

Chapter 7

Comparison of different calibration methods for the OMXS30 index

In order to compare different calibration methods we compare the theoretical European option prices for the OMXS30 index with the known option prices on 4 May, 2006. For the we use both graphical illustrations and quantitative mathematical measures to get a feeling for the differences and similarities. To find the put prices from the call prices we use the put-call parity (5.2).

7.1 Differences between the distributions

The distribution of a risk-neutral asset price model can in many cases tell us what to expect from the theoretical option prices. If two risk-neutral measures do not agree, then the corresponding option prices may differ significantly. Hence a study of the calibrated distributions obtained with the different calibration methods should give a good feeling for possible differences betweem the corresponding option prices.

7.1.1 Comparing the Normal distributions

In Figure 7.1 below the Black-Scholes Normal PDF is depicted for the two calibration methods.

We note a larger variance for the index calibrated model than for the option calibrated one. This might indicate that the actors on the market, although agreeing on the present option prices, expect a lower standard deviation in the near future than at present. Recall that in Figure 3.1 we found that the standard deviation of the log-returns was higher during 2003 and 2004 than during 2006.

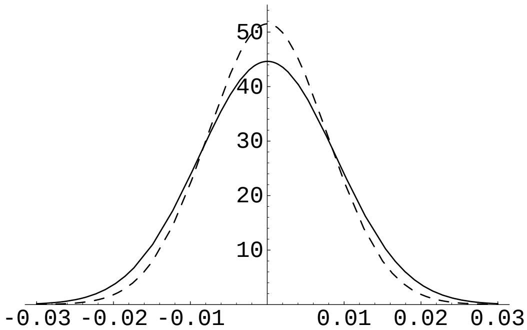


Figure 7.1: Index calibrated Black-Sholes PDF (full line) and option calibrated Black-Scholes PDF (dashed line).

7.1.2 Comparing the Meixner distributions

In Figure 7.2 below the two index calibrated risk-neutral Meixner PDF's are depicted together with the option calibrated Meixner PDF.

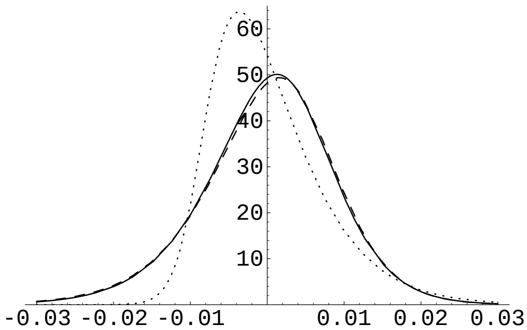


Figure 7.2: Index calibrated MCM PDF (full line), index calibrated Esscher transformed PDF (dashed line) and option calibrated PDF (dotted line) for the Meixner distribution.

The PDF of the index calibrated Meixner distributions have already been discussed in Section 4.2.2. As for the PDF of the option calibrated Meixner distribution, it shows a rather unusual feature for a distribution featuring in matematical finance, namely a light left tail and heavy right tail, indicating a greater probability for extremly high gains than for extreme losses. Nevertheless, it is this distribution achieved from calibration by call option prices that do in some sense answers to the markets view on the OMXS30 index.

It is important to understand that there may exist many different solutions to Calibration Problem 2. However, we assume that our solution for the Meixner distribution is the correct one. A rigorous analysis of this problem is carried out [3] and [11].

7.1.3 Comparing the NIG distributions

In Figure 7.3 below the two index calibrated risk-neutral NIG PDF's are depicted together with the option calibrated risk-neutral NIG PDF.

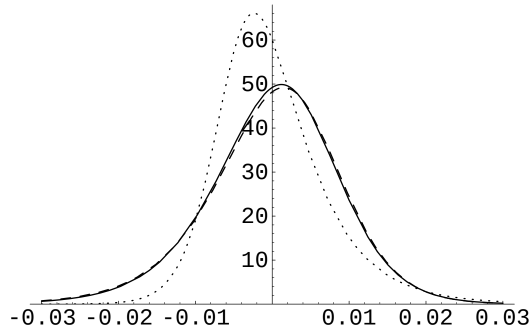


Figure 7.3: Index calibrated MCM PDF (full line), index calibrated Esscher transformed PDF (dashed line) and option calibrated PDF (dotted line) for the NIG distribution.

The PDF of the index calibrated NIG distributions have already been discussed in Section 4.2.3. As for the Meixner model, the PDF of the option calibrated NIG distribution displays the unusual feature of a light left tail and a heavy right tail.

7.1.4 Comments on the calibrated distributions

All of the risk-neutral distributions, fitted against the log-returns for the historical index prices, are either mesokurtic, or will at least tend towards the Black-Scholes Normal distribution rather quickly as time increases. On the other hand, the option calibrated risk-neutral distribution display very unusual features. In the sequel we will see how well the corresponding theoretical option prices replicate true option prices.

7.2 Model prices and market prices

In this section we compare the theoretical call option prices c_{T_i} for the different models against observed market prices c_i. This is done on the 4th of May, 2006. The statistic used is the Average Relative Percentage Error, defined as

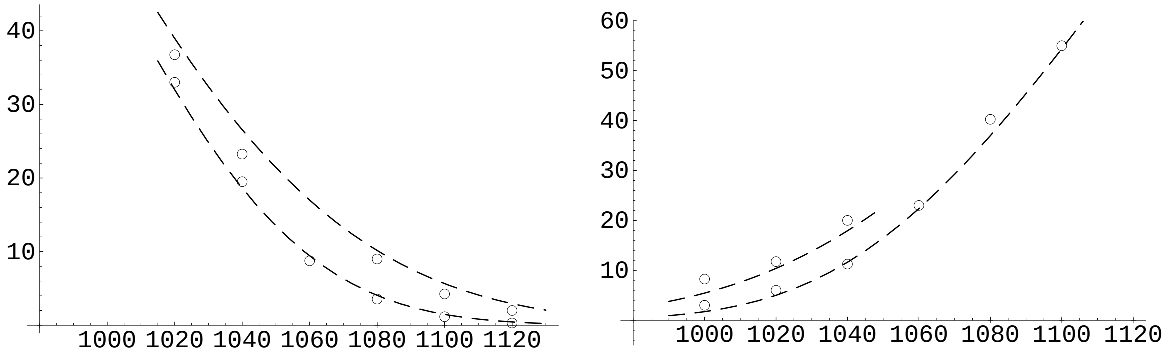
$$\text{ARPE} = \frac{1}{N}\sum_{i=1}^{N}\left|\frac{c_{T_i} - c_i}{c_i}\right|.$$

Each figure in this section show the theoretical option prices for two different maturities. These maturities answers to 16 trading days (26 May, 2006) respectively 35 trading days (22 June, 2006). The upper lines and their surrounding circles in the plots correspond to the option prices for the longer 35 day maturity and the lower ones to thr prices for the shorter 16 day to expiration one.

The call option with 36 days to expiration and a strike price of 1060 kronor on 4 May, 2006, has not been traded. This is also the case for the put options with 1020, 1040 and 1060 kronor strike and 36 days to maturity.

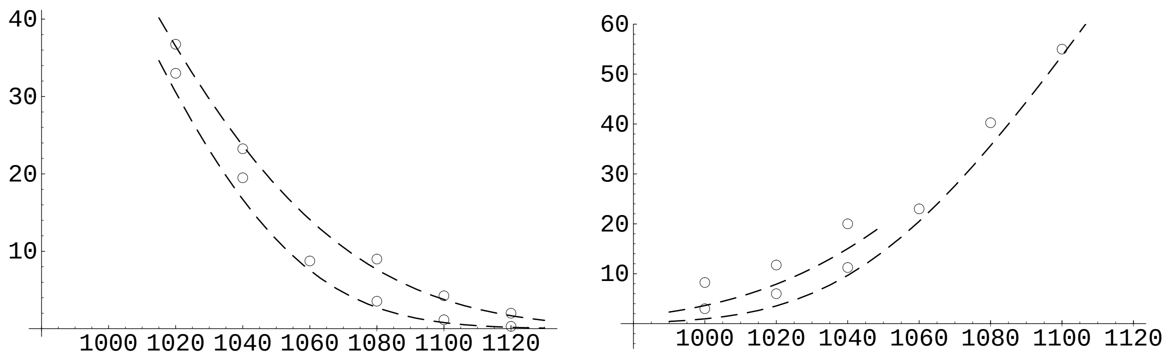
7.2.1 Option prices for the Normal model

Since there are no distinctions between any of the risk-neutral measures for the Normal model all of them give the same option prices. The only thing we can compare for this model is the calibration methods. In Figures 7.4 and 7.5 below we depict the index calibrated Black-Scholes option prices together with their true counterparts, and the option calibrated Black-Scholes option prices together with their true counterparts, respectively, for two different maturities on 4 May, 2006,



Call option prices. Put option prices.

Figure 7.4: Index calibrated Black-Scholes prices (dashed line) and market prices (circles).



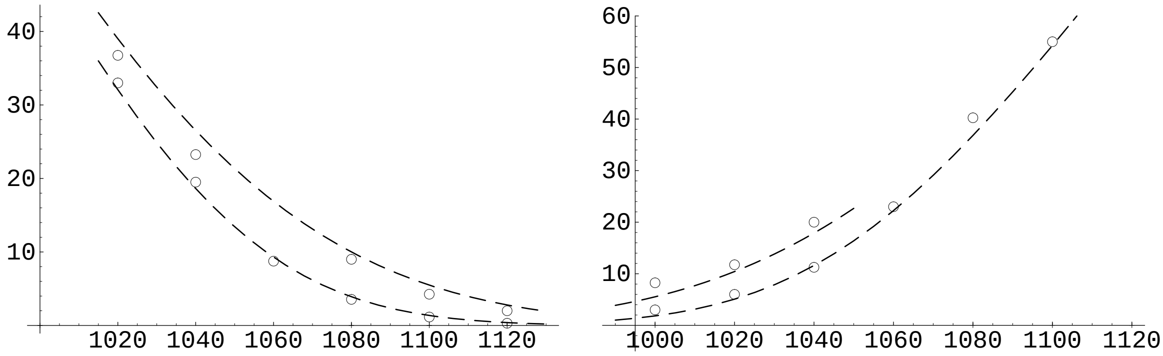
Call option prices. Put option prices.

Figure 7.5: Option calibrated Black-Scholes prices (dashed line) and market prices (circles).

The ARPE between the index calibrated models prices and the market is 0.197664 for the call option and 0.145412 for the put. The ARPE between market and theory for the option calibrated Normal model is 0.160807 and 0.289936 for the call and the put, respectively.

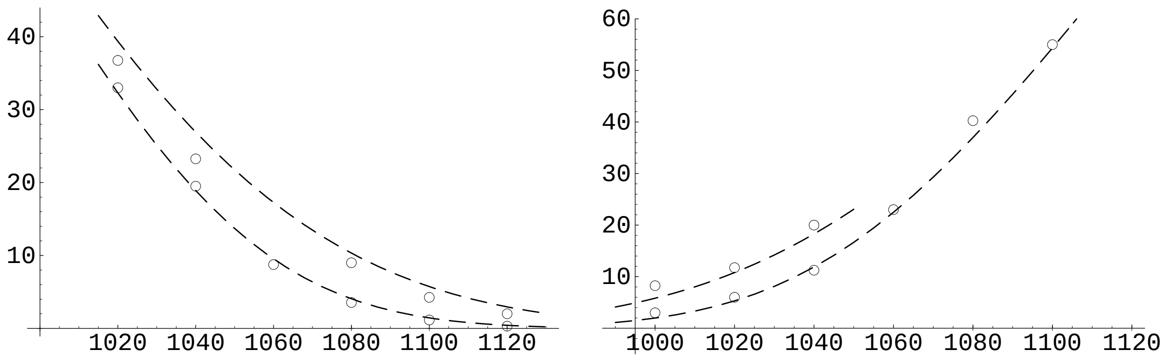
7.2.2 Option prices for the Meixner model

For the Meixner model we compare option prices for three different cases. These are the option price calibrated model and the index calibrated MCM and EM models. In Figure 7.6 below we depict the theoretical option prices for the Mean corrected Meixner model calibrated against historical log-returns and the corresponding market prices. The same type of illustration, but for the index calibrated Esscher transformed model and the option calibrated Meixner model can be viewed in Figures 7.7 and 7.8 below, respectively.



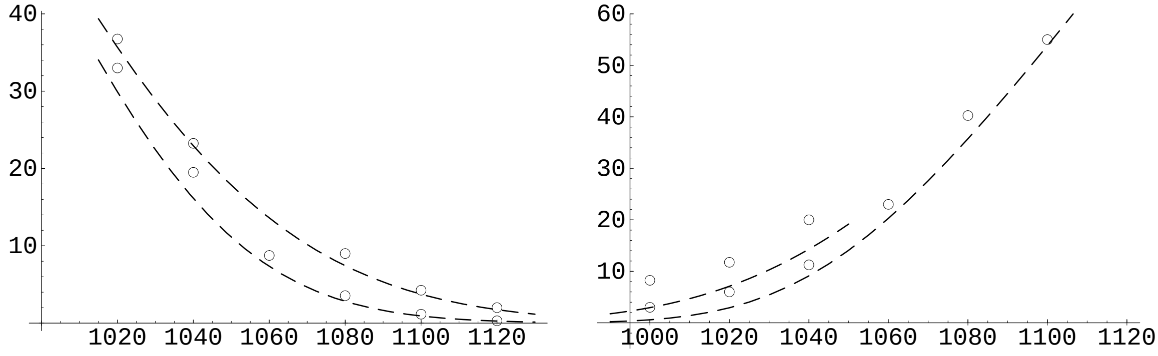
Call option prices. Put option prices.

Figure 7.6: Index calibrated Mean corrected Meixner prices (dashed line) and market prices (circles).



Call option prices. Put option prices.

Figure 7.7: Index calibrated Esscher transformed Meixner prices (dashed line) and market prices (circles).



Call option prices. Put option prices.

Figure 7.8: Option calibrated Meixner prices (dashed line) and market prices (circles).

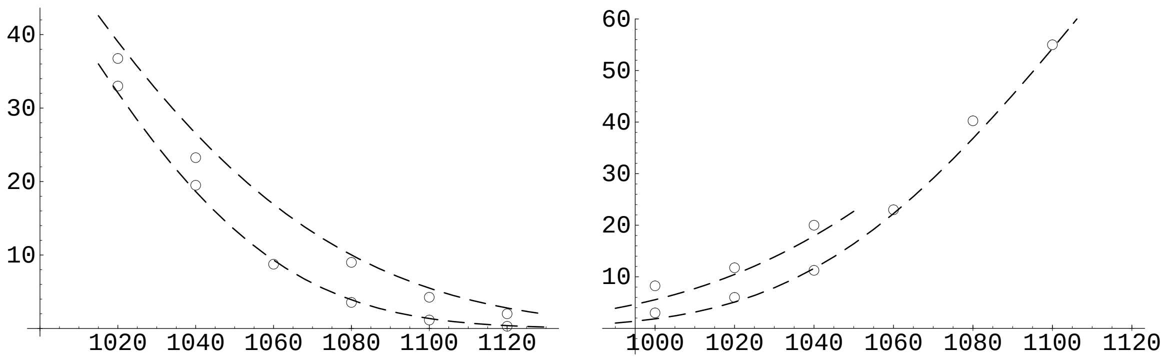
The measured ARPE errors between the market and the calibrated Meixner models are given in Table 7.1 below.

Meixner	Index calibrated MCM	Index calibrated EM	Option calibrated model
Call options	0.155327	0.191952	0.130719
Put options	0.139536	0.121375	0.342852

Table 7.1: ARPE between market and theoretical Meixner model on 4 May, 2006.

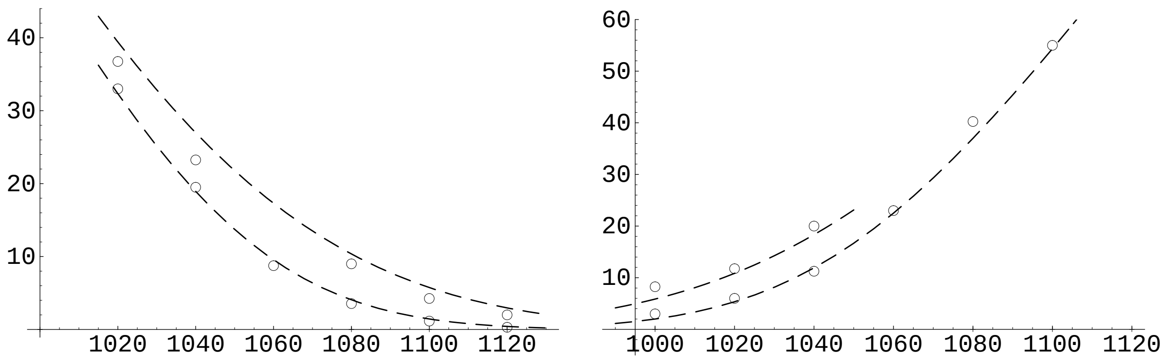
7.2.3 Option prices for the NIG model

Again we compare option prices for the option price calibrated model and the index calibrated MCM and EM models. In Figures 7.9-7.11 below the theoretical option prices are plotted together with the observed market prices on 4 May, 2006.



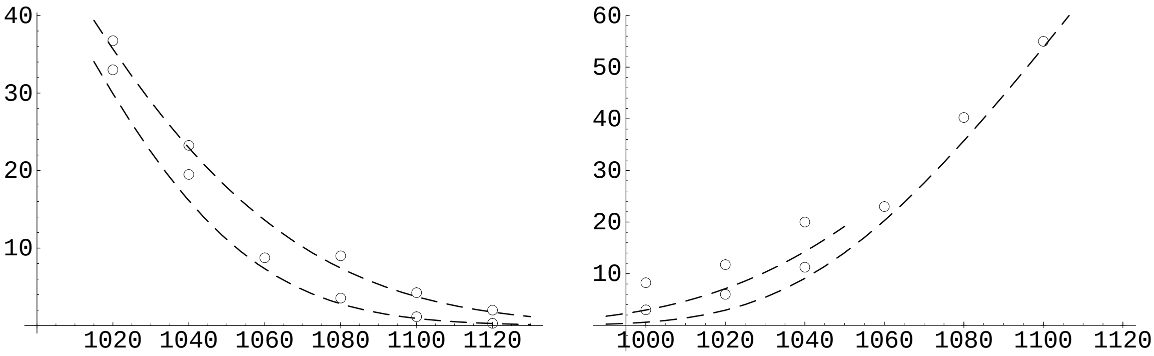
Call option prices. Put option prices.

Figure 7.9: Index calibrated Mean corrected NIG prices (dashed line) and market prices (circles).



Call option prices. Put option prices.

Figure 7.10: Index calibrated Esscher transformed NIG prices (dashed line) and market prices (circles).



Call option prices. Put option prices.

Figure 7.11: Option calibrated NIG prices (dashed line) and market prices (circles).

The ARPE between the market and the calibrated NIG models are given in Table 7.2 below.

NIG	Index calibrated MCM	Index calibrated EM	Option calibrated model
Call options	0.154712	0.194099	0.127118
Put options	0.138267	0.118549	0.342014

Table 7.2: ARPE between market and theoretical NIG model on 4 May, 2006.

7.3 Comments on the fit for the option price models

Analyzing the differences between the market prices and the theoretical prices of the different models, we see that the option calibrated model works best when trying to forecast call option prices. This seems fair, since the thoughts on the market probably will not change that much from one day to the other, and the historical prices of the underlying entity far back will not be considered for relatively short maturities.

An interesting feature is that the option calibrated models fit terribly to the put options one day ahead, while the models fitted against historical asset prices work really well here.

Chapter 8

Conclusions

8.1 Results

We have shown that the Normal Inverse Gaussian process and the Meixner process are better than the Normal process when modelling the OMXS30 index.

Further, we saw indications that the examined calibrated risk-neutral distributions for the OMXS30 index converge quite fast towards the Black-Scholes distribution. The distributions for the option calibrated models had appearances corresponding to a market belief of a smaller variance, as compared with distributions of the index calibrated models. This seems fair, since we could notice a smaller variance for the last months, than for the whole period of the employed data set.

The Lévy models, calibrated against call option prices, were better than the those calibrated against historical index prices, when used to predict future call prices. The best call option predictions were obtained with the option calibrated NIG model, and the worst predictions with the index calibrated Black-Scholes model. The call prices for the index calibrated models are in general higher than the true prices. This in turn is a consequence of the relatively high variance for the whole time period os the data set, as compared with that during the last few months.

In the pricing of put options the option fitted models performed very poorly, by significantly underestimating the option prices. On the other hand, index fitted models performed quite well. More specifically, the Esscher transformed NIG model performed the best and the option calibrated Meixner model performed the worst, in the prediction of put prices. Hence, it seems advisable to treat call options and put options differently, in terms of calibration methodology.

8.2 Further studies

It would be of interest to examine more complex and versatile parametric or semi-parametric Lévy process models than those we have considered, by the use of Fourier methods, see [9] and [3], on the Swedish market. In addition, we would like to investigate calibration by means of the so called Relative Entropy Measure, see [3] and [11].

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