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MASTER THESIS

Stochastic Integral Representation of Functionals with Option-like Structure

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Abstract

In this thesis we will study the martingale representation theorem and its application to mathematical finance. Specifically, we will find the stochastic integral representations explicitly in this theorem for a number of functionals, most of which are inspired by the structure of options in mathematical finance. Malliavin calculus turns out to be a powerful tool for finding these stochastic integral representations and we will one of our main tools.

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1 Introduction

In this thesis we will study the martingale representation theorem and its application to mathematical finance. Specifically, we will find the stochastic integral representations explicitly in this theorem for a number of functionals, most of which are inspired by the structure of options in mathematical finance. Malliavin calculus turns out to be a powerful tool for finding these stochastic integral representations and we will one of our main tools.

2 Martingale representation theorem

Let B=$(B_t)_{t\geq 0}$ be Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $(\mathcal{F}_t)_{t\geq 0} \subseteq \mathcal{F}$ the filtration generated by this Brownian motion $\mathcal{F}_t = \sigma(B_s : 0 \leq s \leq t)$. Let $(M_t)_{0\leq t\leq T}$ be a martingale with respect to this filtration. The *martingale representation theorem* ([4], [5]) says that there exists an adapted process $(Y_t)_{0\leq t\leq T}$ with $\mathbf{P}[\int_0^T Y_t^2\, dt < \infty] = 1$, such that

$$M_t = M_0 + \int_0^t Y_s\, dB_s \quad \text{for } 0 \leq t \leq T. \tag{1}$$

Moreover, if F is an integrable $\mathcal{F}_T$-measurable random variable, then we have

$$F = \mathbf{E}[F] + \int_0^T Y_t\, dB_t. \tag{2}$$

Equation (1) is the so called *martingale representation* of M and we will call (2) the *stochastic integral representation* of F, or as is often done in the literature, a martingale representation of F. Our focus in this thesis will be to find the explicit form of the process Y in (2) for a variety of choices of functionals F of Brownian motion.

Finding the integral representation (2) of a random variable explicitly is a very difficult task in general. However, in some cases this problem can be solved using a few simple tricks:

Example 1. For the time average of Brownian motion we have

$$\frac{1}{T}\int_0^T B_t\, dt = \frac{1}{T}\int_{t=0}^T \int_{s=0}^t dB_s dt = \frac{1}{T}\int_{s=0}^T \int_{t=s}^T dt dB_s = \frac{1}{T}\int_{s=0}^T (T-s)\, dB_s = \int_{s=0}^T \left(1 - \frac{s}{T}\right) dB_s. \tag{3}$$

Since $\mathbf{E}[T^{-1}\int_0^T B_t\, dt] = 0$ (3) is our stochastic integral representation of $T^{-1}\int_0^T B_t\, dt$. #

Example 2. Consider the Black-Scholes asset price model $S_T = \mathrm{e}^{(r-\sigma^2/2)T+\sigma B_T}$, where $\sigma > 0$ is the so called volatility and $r \in \mathbb{R}$ the interest rate. By application of Itô's formula we get

$$\begin{aligned} \mathrm{e}^{-\sigma^2 T/2+\sigma B_T} &= \mathrm{e}^0 + \int_0^T \sigma\, \mathrm{e}^{-\sigma^2 t/2+\sigma B_t}\, dB_t - \int_0^T \frac{\sigma^2}{2}\, \mathrm{e}^{-\sigma^2 t/2+\sigma B_t}\, dt + \frac{1}{2}\int_0^T \sigma^2\, \mathrm{e}^{-\sigma^2 t/2+\sigma B_t}\, dt \\ &= 1 + \int_0^T \sigma\, \mathrm{e}^{-\sigma^2 t/2+\sigma B_t}\, dB_t. \end{aligned}$$

Upon multiplication of both sides by e^{rT} we arrive at

$$S_T = \mathrm{e}^{(r-\sigma^2/2)T+\sigma B_T} = \mathrm{e}^{rT} + \sigma\,\mathrm{e}^{rT}\int_0^T \mathrm{e}^{-\sigma^2 t/2+\sigma B_t}\,dB_t,$$

which is our stochastic integral representation (2) of S_T. #

Example 3. To find the stochastic integral representation of Brownian motion raised to an integer power B_T^k for $k \geq 0$ we make repeated use of Itô's formula.. We start by applying Itô's formula to B_T^k, which gives

$$B_T^k = k\int_0^T B_t^{k-1}\,dB_t + \frac{k(k-1)}{2}\int_0^T B_t^{k-2}\,dt. \tag{4}$$

The first integral in (4) is of the form required for the stochastic integral representation, but the second integral is not. Aspiring to transform that second integral to our desired form we apply Itô's formula again, this time to B_t^{k-2}, which gives

$$B_t^{k-2} = (k-2)\int_0^t B_s^{k-3}\,dB_s + \frac{(k-2)(k-3)}{2}\int_0^t B_s^{2k-4}\,ds.$$

Integrating B_t^{k-2} with respect to t we thus get

$$\begin{aligned}
\int_0^T B_t^{k-2}\,dt &= (k-2)\int_{t=0}^T\int_{s=0}^t B_s^{k-3}\,dB_s dt + \frac{(k-2)(k-3)}{2}\int_{t=0}^T\int_{s=0}^t B_s^{k-4}\,dsdt \\
&= (k-2)\int_{s=0}^T\int_{t=s}^T B_s^{k-3}\,dtdB_s + \frac{(k-2)(k-3)}{2}\int_{s=0}^T\int_{t=s}^T B_s^{k-4}\,dtds \\
&= (k-2)\int_0^T (T-s)B_s^{k-3}\,dB_s + \frac{(k-2)(k-3)}{2}\int_0^T (T-s)B_s^{k-4}\,ds. \qquad (5)
\end{aligned}$$

Inserting (5) into (4) we get

$$\begin{aligned}
B_T^k = k\int_0^T B_t^{k-1}\,dB_t &+ \frac{k(k-1)(k-2)}{2}\int_0^T (T-s)B_t^{k-3}\,dB_t \\
&+ \frac{k(k-1)(k-2)(k-3)}{4}\int_0^T (T-t)B_t^{k-4}\,dt.
\end{aligned}$$

We see that after having done one more "iteration" of Itô's formula we are left with another integral of a non-desirable form in shape of the right-most member on the right-hand side. However, we also see that after sufficiently many iterations that problem will vanish, so that we get our stochastic integral representation desired. #

We will stop our calculations here. They turn out to be less tedious using the Clark-Ocone formula, which we will introduce shortly. We will return to Example 3 in a later section thus equipped.

3 Malliavin calculus and the Clark-Ocone formula

In this section we give a short introduction to Malliavin calculus inspired by [1].

Let the probability space Ω be the space $C_0([0,T])$ of continuous functions $\omega : [0,T] \to \mathbb{R}$ such that $\omega(0) = 0$. This space is called the *Wiener space*. Note that any sample path of a Brownian motion can be identified with an element of $C_0([0,T])$.

Choose a function $g \in L^2([0,T])$ (the space of square integrable functions on $[0,T]$) and consider the integral

$$\gamma(t) = \int_0^t g(s)\,ds \quad \text{for } 0 \le t \le T. \tag{6}$$

As $L([0,T]) \subseteq L^2([0,T])$ it is clear that γ is well-defined and belongs to Ω. The space of $\gamma \in \Omega$ of the type (6) is called the *Cameron-Martin space*.

Let $F : \Omega \to \mathbb{R}$ be a random variable. The *directional derivative* of F in a direction γ (6) of the Cameron-Martin spacecan is defined as

$$D_\gamma F(\omega) = \frac{d}{d\varepsilon} F(\omega + \varepsilon\gamma)\Big|_{\varepsilon=0} = \lim_{\varepsilon\to 0} \frac{F(\omega+\varepsilon\gamma) - F(\omega)}{\varepsilon}$$

whenever this derivative exists.

Now assume that F is such that its directional derivative exists and belongs to $L^2(\Omega)$ for all directions γ of the Cameron-Martin space. Further, assume that there exists a function $\Psi(t,\omega) \in L^2([0,T] \times \Omega)$ such that

$$D_\gamma F(\omega) = \int_0^T \Psi(t,\omega) g(t)\,dt. \tag{7}$$

(Note that the L^2-integrability of g and Ψ makes this integral well-defined.) Then we say that F is *Malliavin differentiable* and define the *Malliavin derivative* of F to be

$$D_t F(\omega) = \Psi(t,\omega). \tag{8}$$

We denote the set of all Malliavin differentiable random variables $\mathcal{D}_{1,2}$. The notation $\mathcal{D}_{1,2}$ stems from the fact that we are looking at Malliavin derivatives of the first order that are L^2-integrable.

Example 4. Suppose that the random variable F is given by

$$F(\omega) = \int_0^T f(s)\,dB_s = \int_0^T f(s)\,d\omega(s) \quad \text{for an } f \in L^2([0,T]).$$

With γ given by (6) belonging to the Cameron-Martin space we then have

$$F(\omega + \epsilon\gamma) = \int_0^T f(s)\,(d\omega(s) + \epsilon\,d\gamma(s)) = \int_0^T f(s)\,d\omega(s) + \epsilon \int_0^T f(s)g(s)\,ds,$$

so that

$$\frac{F(\omega + \epsilon\gamma) - F(\omega)}{\epsilon} = \int_0^T f(s)g(s)\,ds$$

exists in $L^2(\Omega)$. We thus have the expression (7) for the directional derivative of F with $\Psi(t,\omega) = f(t)$. This means that $F \in \mathcal{D}_{1,2}$ and $D_t F = f(t)$ for $t \in [0,T]$. This is a key result. In the special case when $f(t) = 1_{[0,t_1]}(t)$ we get

$$F = \int_0^T 1_{[0,t_1]}(s)\,dB_s = B_{t_1} \quad \text{and} \quad D_t B_{t_1} = 1_{[0,t_1]}(t). \quad \#$$

Let us look at random variables of a slightly more complicated form: Let $\mathbb{P}$ be the class of *Wiener polynomials*, that is, the class of random variables of the form $\varphi(\theta_1,...,\theta_n)$, where $\varphi(x) = \varphi(x_1,x_2,...,x_n)$ is a polynomial in n variates and $\theta_i = \int_0^T f_i(s)\,dB_s$ for some functions $f_1,\ldots,f_n \in L^2([0,T])$. It can be shown (see [1]) that

$$\varphi(\theta_1,...,\theta_n) \in \mathcal{D}_{1,2} \quad \text{with} \quad D_t\varphi(\theta_1,...,\theta_n) = \sum_{i=1}^n \frac{\partial\varphi}{\partial x_i}(\theta_1,...,\theta_n)\,f_i(t).$$

We introduce the norm $\|\cdot\|_{1,2}$ on $\mathcal{D}_{1,2}$ through

$$\|F\|_{1,2} = \|F\|_{L^2(\Omega)} + \|D_t F\|_{L^2([0,T]\times\Omega)}.$$

Let $\mathbb{D}_{1,2}$ be the topological completion of $\mathcal{D}_{1,2}$ with respect to the norm $\|\cdot\|_{1,2}$. The Malliavin derivative of an $F \in \mathbb{D}_{1,2}$ is defined as $D_t F = \lim_{n\to\infty} D_t F_n$ whenever $\{F_n\}_{n=1}^\infty \subset \mathcal{D}_{1,2}$ satisfies $F_n \to F$ in $\mathbb{D}_{1,2}$. (See [1] on the details of this construction.)

3.1 Properties of Malliavin derivatives

Let us now highlight some further properties of Malliavin derivatives that will be useful to find stochastic integral representations in the upcoming sections (see [6] for details).

If $g : \mathbb{R}^m \to \mathbb{R}$ is continuously differentiable or with bounded partial derivatives or Lipschitz continuous and $F = (F_1,...,F_m) \in (\mathbb{D}_{1,2})^m$, then the *chain rule* says that

$$g(F) \in \mathbb{D}_{1,2} \quad \text{and} \quad D_t(g(F)) = \sum_{i=1}^n \frac{\partial g(F)}{\partial x_i} D_t F_i.$$

If $F, G \in \mathbb{D}_{1,2}$ with $FG, D_t(F)\,G, F\,D_t(G) \in \mathbb{D}_{1,2}$, then the *product rule* says that

$$D_t(FG) = D_t(F)\,G + F\,D_t(G).$$

Let $\nabla g = (\partial_x g, \partial_y g, \partial_z g)$ denote the gradient of a function $g : \mathbb{R}^3 \to \mathbb{R}$ and $\mathrm{div}\,(g) = \partial_x g + \partial_y g + \partial_z g$ the divergence of g, whenever they are well-defined. Similarly, let $\mathrm{div}_{x,y}(g) = \partial_x g + \partial_y g$, and so on. The following theorem proved in [2] will be useful in the upcoming chapters.

Theorem 1. Let $m_t = \min_{0 \leq s \leq t} B_s$ and $M_t = \max_{0 \leq s \leq t} B_s$. Further write $X_t^\theta = X_t + \theta t$ for any process $(X_t)_{t \geq 0}$ and constant $\theta \in \mathbb{R}$. If $g : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable with bounded derivatives or Lipschitz continuous, then the functional $g(B_T^\theta, m_T^\theta, M_T^\theta)$ has the stochastic integral representation

$$g(B_T^\theta, m_T^\theta, M_T^\theta) = \mathbf{E}[g(B_T^\theta, m_T^\theta, M_T^\theta)] + \int_0^T f(B_t^\theta, m_t^\theta, M_t^\theta; t)\, dB_t,$$

where

$$\begin{aligned} f(a,b,c;t) \;=\; & \mathrm{e}^{-\frac{1}{2}\theta^2\tau}\, \mathbf{E}[\mathrm{div}\,(g)(B_\tau + a, m_\tau + a, M_\tau + a)\, \mathrm{e}^{\theta B_\tau} 1_{(-\infty, b-a]}(m_\tau) 1_{[c-a,\infty)}(M_\tau)] \\ & + \mathrm{e}^{-\frac{1}{2}\theta^2\tau}\, \mathbf{E}[\mathrm{div}_{x,y}(g)(B_\tau + a, m_\tau + a, c)\, \mathrm{e}^{\theta B_\tau} 1_{(-\infty, b-a]}(m_\tau) 1_{(-\infty, c-a]}] \\ & + \mathrm{e}^{-\frac{1}{2}\theta^2\tau}\, \mathbf{E}[\mathrm{div}_{x,z}(g)(B_\tau + a, b, M_\tau + a)\, \mathrm{e}^{\theta B_\tau} 1_{[b-a,\infty)}(m_\tau) 1_{[c-a,\infty)}(M_\tau)] \\ & + \mathrm{e}^{-\frac{1}{2}\theta^2\tau}\, \mathbf{E}[\partial_x g(B_\tau + a, b, c)\, \mathrm{e}^{\theta B_\tau} 1_{[b-a,\infty)}(m_\tau) 1_{(-\infty, c-a]}(M_\tau)] \end{aligned}$$

for $b < a < c, b < 0, c > 0$ and $\tau = T - t$.

3.2 Clark-Ocone formula

The reason that we introduced the Malliavin derivative in Section 3 was to be able to present the *Clark-Ocone formula*, which is a very useful theorem to find the stochastic integral representation for functionals of Brownian motion. In fact, Theorem 1 of the previous Section 3.1 can be derived from this result:

Let $F \in \mathbb{D}_{1,2}$ be an $\mathcal{F}_T$-measurable random variable such that

$$\mathbf{E}[|F|] < \infty \quad \text{and} \quad \int_0^T \mathbf{E}[(D_t F)^2]\, dt < \infty.$$

Then the Clark-Ocone formula says that

$$F = \mathbf{E}[F] + \int_0^T \mathbf{E}[D_t F \mid \mathcal{F}_t]\, dB_t. \tag{9}$$

In other words, for $\mathcal{F}_T$-measurable random variables F that belong to $\mathbb{D}_{1,2}$ the process Y in the stochastic integral representation (2) is given by

$$Y_t = \mathbf{E}[D_t F \mid \mathcal{F}_t] \quad \text{for } 0 \leq t \leq T.$$

For a more general version of this result we use Brownian motion $B^{\mathbf{Q}}$ under a probability measure $\mathbf{Q}$ instead of the $\mathbf{P}$-Brownian motion B to express F. Here $B^{\mathbf{Q}}$ takes the form

$$B_t^{\mathbf{Q}} = \int_0^t \theta_s\, ds + B_t,$$

where θ is an $(\mathcal{F}_t)_{t\geq 0}$-adapted stochastic process. The probability measure $\mathbf{Q}$ is given by

$$d\mathbf{Q}(\omega) = Z(T,\omega)\, d\mathbf{P}(\omega),$$

where $Z(T)$ is a random variable given by

$$Z(T) = \exp\Big\{-\int_0^T \theta_s\, dB_s - \frac{1}{2}\int_0^T \theta_s^2\, ds\Big\}.$$

Expectation under $\mathbf{Q}$ is denoted by $\mathbf{E}_{\mathbf{Q}}$.

Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable. With the notation of the previous paragraph, assume that

$$\mathbf{E}_{\mathbf{Q}}[|F|] < \infty, \quad \int_0^T \mathbf{E}_{\mathbf{Q}}[(D_tF)^2]\, dt < \infty \quad \text{and} \quad \mathbf{E}_{\mathbf{Q}}\left[|F| \int_0^T \left(\int_0^T D_t\theta_s\, dB_s^{\mathbf{Q}}\right)^2 dt\right] < \infty.$$

Then the *generalized Clark-Ocone formula* says that F has the following martingale representation with respect to the $\mathbf{Q}$-Brownian motion $B^{\mathbf{Q}}$:

$$F = \mathbf{E}_{\mathbf{Q}}[F] + \int_0^T \mathbf{E}_{\mathbf{Q}}\left[D_tF - F\int_t^T D_t\theta_s\, dB_s^{\mathbf{Q}} \,\Big|\, \mathcal{F}_t\right] dB_t^{\mathbf{Q}}.$$

4 Application to finance

Assume that we have a market model under the probability measure $\mathbf{P}$ with a *risk-free investment* A_t and a *risky investment* S_t given by

$$dA_t = \rho_t A_t\, dt \quad \text{and} \quad dS_t = \mu_t S_t\, dt + \sigma_t S_t\, dB_t, \tag{10}$$

respectively. Here the *interest rate* ρ, the *drift* μ and the *volatility* σ are supposed to be $(\mathcal{F}_t)_{t\geq 0}$-adapted processes with suitable integrability properties. Examples of safe and risky investments are bonds and stocks, respectively.

A *portfolio* (ξ_t, η_t) is the number of units invested in (A_t, S_t). The value of the portfolio is given by

$$V_t = \xi_t A_t + \eta_t S_t. \tag{11}$$

We will work with *self-financing portfolios*, which means that a change in the value of the portfolio depends solely on a change in A_t and S_t, that is,

$$dV_t = \xi_t dA_t + \eta_t dS_t. \tag{12}$$

By substituting $\eta_t = (V_t - \eta_t S_t)/A_t$ from (11) in (12) and using the assumed price dynamics (10) of A_t and S_t we get

$$dV_t = \rho_t(V_t - \eta_t S_t)\, dt + \sigma_t\eta_t S_t dB_t = (\rho_t V_t + (\mu_t - \rho_t)\eta_t S_t)\, dt + \sigma_t\eta_t S_t\, dB_t. \tag{13}$$

We may want to find a portfolio that has a certain final value V_T. By some manipulations of (13) and applying the generalized Clark-Ocone formula to $G = \exp\{-\int_0^T \rho_s\,ds\}V_T$ (see [1] for the details), we get the following expression for G:

$$G = \mathbf{E}_{\mathbf{Q}}[G] + \int_0^T \mathbf{E}_{\mathbf{Q}}\left[D_t G - G\int_t^T D_t\theta_s\,dB_s^{\mathbf{Q}} \,\middle|\, \mathcal{F}_t\right] dB_t^{\mathbf{Q}}, \tag{14}$$

where $\theta_t = (\mu_t - \rho_t)/\sigma_t$. From this we can easily deduce an expression for V_T. From this in turn (see [1] for the details) it follows that the number of units of the risky investments should be

$$\eta_t = \frac{1}{\sigma_t S_t}\exp\left\{\int_0^t \rho_s\,ds\right\}\mathbf{E}_{\mathbf{Q}}\left[D_t G - G\int_t^T D_t\theta_s\,dB_s^{\mathbf{Q}} \,\middle|\, \mathcal{F}_t\right] \quad \text{for } 0 \le t \le T.$$

This result to determine a portfolio illustrates the importance to find stochastic integral representations in mathematical finance.

4.1 The Black-Scholes model

The expression (14) for G is simplified if $D_t\theta = 0$, which is the case for constant θ. Then we have

$$G = \mathbf{E}_{\mathbf{Q}}[G] + \int_0^T \mathbf{E}_{\mathbf{Q}}[D_t G \mid \mathcal{F}_t]\,dB_t^{\mathbf{Q}}. \tag{15}$$

In the so called *Black-Scholes model* ρ, μ and σ are constants so that θ is a constant. In this case (15) gives us

$$V_T = \mathbf{E}_{\mathbf{Q}}[V_T] + \int_0^T \mathbf{E}_{\mathbf{Q}}[D_t V_T \mid \mathcal{F}_t]\,dB_t^{\mathbf{Q}}, \tag{16}$$

which is actually the Clark-Ocone formula (9) with $\mathbf{P}$-Brownian motion B replaced by $\mathbf{Q}$-Brownian motion $B^{\mathbf{Q}}$. If we want to find the portfolio for an option in the Black-Scholes model (see Section 5 below for more information on such options) it is thus more direct to use the Clark-Ocone formula (9). This we do in the next paragraph.

By changing probability measure from $\mathbf{P}$ to $\mathbf{Q}$ we get that the price process under $\mathbf{Q}$ is given by

$$dV_t = \rho V_t\,dt + \sigma\eta_t S_t\,dB_t^{\mathbf{Q}}.$$

Writing $U_t = \mathrm{e}^{-\rho t}V_t$ it follows that

$$dU_t = \mathrm{e}^{-\rho t}\,dV_t - \rho\,\mathrm{e}^{-\rho t}V_t\,dt = \mathrm{e}^{-\rho t}\sigma\eta_t S_t\,dB_t^{\mathbf{Q}},$$

so that

$$\mathrm{e}^{-\rho T}V_T = V_0 + \int_0^T \mathrm{e}^{-\rho t}\sigma\eta_t S_t\,dB_t^{\mathbf{Q}}. \tag{17}$$

Applying the Clark-Ocone formula (9) to the functional $\mathrm{e}^{-\rho T}V_T$ gives us

$$e^{-\rho T} V_T = \mathbf{E}_{\mathbf{Q}}[e^{-\rho T} V_T] + \int_0^T \mathbf{E}_{\mathbf{Q}}[D_t(e^{-\rho T} V_T) \mid \mathcal{F}_t]\, dB_t^{\mathbf{Q}}. \tag{18}$$

Upon comparing (17) and (18) we see that $V_0 = \mathbf{E}_{\mathbf{Q}}[e^{-\rho T} V_T]$, so that the number of shares of the risky investment in our portfolio should be

$$\eta_t = \frac{e^{\rho(t-T)}}{\sigma S_t} \mathbf{E}_{\mathbf{Q}}[D_t V_T \mid \mathcal{F}_t].$$

In the following we will use the Clark-Ocone formula to find the stochastic integral representation for the final values V_T of various options, some of which really exist while others are fictive. In fact, we will mainly focus on what we call "pseudo-options" which are functionals inspired by the structure of options but with stock prices following other models than the usual Black-Scholes model. The motivation for the development of stochastic integral representations for these kinds of functionals is to gain the skills to be able to find stochastic integral representations for real options. We have researched the field and found that this work has already been done for most (if not all) existing options, where the stochastic integral representations are anaylytically tractable.

The stochastic integral representation has be found already for the call option, the lookback option and the spread lookback option, see e.g., [2], while barrier options and partial barrier options are dealt with in [3]. It should also be mentioned that Shiryaev and Yor [7] and [8] use an alternative method to the Clark-Ocone formula approach based on Itô's formula to find the stochastic integral representations of some other functionals of Brownian motion, including $\max_{t \le T} B_t$, $\max_{t \le T_{-a}} B_t$ and $\max_{t \le g_T} B_t$, where $T_a = \inf\{t : B_t = -a\}$ and $g_T = \sup\{t : B_t = 0\}$.

5 Options based on the Black-Scholes model

Recall that under the probability measure $\mathbf{P}$ the price dynamics of the Black-Scholes market model are given by

$$dA_t = \rho A_t\, dt \quad \text{and} \quad dS_t = \mu S_t\, dt + \sigma S_t\, dB_t,$$

where A_t is the risk-free investment and S_t the risky investment, respectively. From (16) we have the stochastic integral representation of an option with Black-Scholes price dynamics. Here $\mathbf{Q}$ is the unique equivalent martingale measure, which is to say that under $\mathbf{Q}$ we have

$$dS_t = r S_t\, dt + \sigma S_t\, dB_t^{\mathbf{Q}} \quad \text{and} \quad S_t = S_0 e^{(r-\sigma^2/2)t + B_t^{\mathbf{Q}}}.$$

5.1 The call option $(S_T - K)^+$

The stochastic integral representation for a call option is given without a proper derivation in a number of articles, see e.g., [2]. Although it seems clear that this derivation is done in detail somewhere, we have not been able to find it. Therefore we found it motivated to do that derivation here.

To calculate the Malliavin derivative of our option we apply the chain rule from Section 3.1 to the Lipschitz continuous function $\phi(x) = (x-K)^+ = (x-K)1_{(K,\infty)}(x)$. To that end recall that

$$D_t B_T^{\mathbf{Q}} = D_t \int_0^T dB_s^{\mathbf{Q}} = 1_{(-\infty,T]}(t).$$

Since

$$\phi'(x) = (x-K)\,\delta(x-K) + 1_{(K,\infty)}(x) = 1_{(K,\infty)}(x)$$

and

$$D_t S_T = D_t S_0\, \mathrm{e}^{(r-\sigma^2/2)t+B_t^{\mathbf{Q}}} = \sigma S_T,$$

it follows that

$$D_t(S_T-K)^+ = \phi'(S_T-K)\, D_t S_T = 1_{(K,\infty)}(S_T)\,\sigma S_T.$$

The conditional distribution of $(r-\sigma^2/2)T+\sigma B_T^{\mathbf{Q}}$ given $\mathcal{F}_t$ is $\mathrm{N}((r-\sigma^2/2)T+\sigma B_t^{\mathbf{Q}}, \sigma^2(T-t))$. Hence we have

$$\mathbf{E}[1_{(K,\infty)}(S_T) \mid \mathcal{F}_t] = \mathbf{E}[\mathrm{e}^Y 1_{(K,\infty)}(\mathrm{e}^Y)], \tag{19}$$

where Y is $\mathrm{N}((r-\sigma^2/2)T+\sigma B_t^{\mathbf{Q}}, \sigma^2(T-t))$-distributed under $\mathbf{Q}$. No write $\mu \equiv (r-\sigma^2/2)T+\sigma B_t^{\mathbf{Q}}$ and $o^2 \equiv \sigma^2(T-t)$ for simplicity. Then we have $\mathbf{E}[\mathrm{e}^Y] = \mathrm{e}^{\mu+o^2/2}$. We use a change of measure to calculate the expectation (19). Define the likelihood Λ by

$$\Lambda = \frac{d\mathbf{Q}_1}{d\mathbf{Q}} = \frac{S_T}{\mathbf{E}[S_T]} = \frac{\mathrm{e}^Y}{\mathbf{E}[\mathrm{e}^Y]}.$$

Then we have

$$\mathbf{E}[\mathrm{e}^Y 1_{(K,\infty)}(\mathrm{e}^Y)] = \mathbf{E}[\mathrm{e}^Y]\,\mathbf{E}[\Lambda\, 1_{(K,\infty)}(\mathrm{e}^Y)] = \mathbf{E}[\mathrm{e}^Y]\,\mathbf{E}_{\mathbf{Q}_1}[1_{\mathrm{e}^Y>K}] = \mathbf{E}[\mathrm{e}^Y]\,\mathbf{Q}_1(\mathrm{e}^Y > K).$$

Since $d\mathbf{Q}_1/d\mathbf{Q} = \mathrm{e}^{Y-\mu-o^2/2}$ we get that Y is $\mathrm{N}(\mu+o^2, o^2)$-distributed under $\mathbf{Q}$, so that

$$\mathbf{E}[\mathrm{e}^Y 1_{(K,\infty)}(\mathrm{e}^Y)] = \mathrm{e}^{\mu+o^2/2}\,\Phi\Big(\frac{\mu+o^2-\ln K}{o}\Big) = S_t\,\Phi\bigg(\frac{\ln(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\bigg).$$

Hence we have

$$\mathbf{E}[D_t(S_T-K)^+ \mid \mathcal{F}_t] = \sigma S_t\,\Phi\bigg(\frac{\ln(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\bigg),$$

so that finally by the Clark-Ocone formula

$$(S_T-K)^+ = \mathbf{E}[(S_T-K)^+] + \int_0^T \sigma S_t\,\Phi\bigg(\frac{\ln(S_t/K)+(r+\sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}\bigg)\,dB_t^{\mathbf{Q}}.$$

This completes the derivation of the stochastic integral representation for the call option.

5.2 The fictive option $F = (\max S_T - K)^+$

In this section we will find the stochastic integral representation of the fictive option $F = (\max_{0\le t\le T} S_t - K)^+ = (\max_{0\le t\le T} S_t - K)1_{(K,\infty)}(\max_{0\le t\le T} S_t)$ by application of Theorem 1 from Section 3.1. To our knowledge this derivation has not been done before.

In the terminology of Theorem 1 we have $\max_{0\le t\le T} S_t = \mathrm{e}^{\sigma M_T^\theta}$, $g(B_T^\theta, m_T^\theta, M_T^\theta) = (\mathrm{e}^{\sigma M_T^\theta} - K)1_{(K,\infty)}(\mathrm{e}^{\sigma M_T^\theta})$ and

$$f(a,b,c;t) = \mathrm{e}^{-\theta^2\tau/2}\,\mathbf{E}[\sigma\,\mathrm{e}^{\sigma(M_\tau+a)}1_{(K,\infty)}(\mathrm{e}^{\sigma(M_\tau+a)})\,\mathrm{e}^{\theta B_\tau}1_{[c-a,\infty)}(M_\tau)],$$

where $\tau = T-t$. Using the well-known expression for the joint density of maximum of Brownian motion and Brownian motion, see e.g., [4], Theorerm 3.21, this equals

$$\begin{aligned}
&\mathrm{e}^{-\theta^2\tau/2}\sigma\,\mathrm{e}^{\sigma a}\int_{y=\max(c-a,\ln(K/\sigma)-a)}^{\infty}\int_{x=-\infty}^{y}\frac{2(2y-x)}{\tau\sqrt{2\pi\tau}}\,\mathrm{e}^{\sigma y}\mathrm{e}^{\theta x}\mathrm{e}^{\theta x-\theta^2\tau/2-(2y-x)^2/(2\tau)}\,dxdy\\
&= \frac{\mathrm{e}^{-\theta^2\tau}\sigma\,\mathrm{e}^{\sigma a}}{\sqrt{2\pi\tau}}\int_{y=\max(c-a,\ln(K/\sigma)-a)}^{\infty}\mathrm{e}^{\sigma y}\left(\mathrm{e}^{-y^2/(2\tau)} - 2\theta\,\mathrm{e}^{y\theta+\theta^2/2}\sqrt{2\pi\tau}\,\Phi\Big(\frac{-y-2\theta\tau}{\sqrt{\tau}}\Big)\right)dy.
\end{aligned}$$

By Theorem 1 the stochastic integral representation of $(\max_{0\le t\le T} S_t - K)^+$ is thus given by

$$\begin{aligned}
(\max_{0\le t\le T} S_t - K)^+ &= \mathbf{E}[(\max_{0\le t\le T} S_t - K)^+]\\
&\quad + \int_0^T \frac{\mathrm{e}^{-\theta^2\tau}\sigma\,\mathrm{e}^{\sigma a}}{\sqrt{2\pi\tau}}\int_{y=\max(c-a,\ln(K/\sigma)-a)}^{\infty}\\
&\qquad \mathrm{e}^{\sigma y}\left(\mathrm{e}^{-y^2/(2\tau)} - 2\theta\,\mathrm{e}^{y\theta+\theta^2/2}\sqrt{2\pi\tau}\,\Phi\Big(\frac{-y-2\theta\tau}{\sqrt{\tau}}\Big)\right)dydB_t.
\end{aligned}$$

6 Pseudo-options on the Bachelier model

In this section we study functionals of stochastic processes following the Bachelier model that have an option-like appearance.

6.1 The functional $(X-K)^+$ for $X = (\overline{\mu T + \sigma B_T}) = \frac{1}{T}\int_0^T(\mu t + \sigma B_t dt)$

In this section we use the Clark-Ocone formula (9) to find the stochastic integral representation for the pseudo-option $(X-K)^+$, where X is the time average of Brownian motion with drift. This is to our knowledge achieved for the first time here. This functional is an analogue of the Asian average call option $(\frac{1}{T}\int_0^T S_t\,dt - K)^+$, where S_t is the stock price in the Black-Scholes model.

Of course, the distribution of X is Gaussian. In order to calculate the integrand $\mathbf{E}[(X-K)^+ \mid \mathcal{F}_t]$ in the Clark-Ocone formula we will first calculate the expectation and variance of X under the $\mathcal{F}_t$-filtration, as

$$
\begin{aligned}
\mathbf{E}[X \mid \mathcal{F}_t] &= \frac{1}{T}\left(\frac{\mu T^2}{2} + \sigma \int_0^T \mathbf{E}[B_s \mid \mathcal{F}_t]\, ds\right) \\
&= \frac{1}{T}\left(\frac{\mu T^2}{2} + \sigma\left(\int_0^t B_s\, ds + (T-t)B_t\right)\right) \\
&= \frac{\mu T}{2} + \frac{\sigma}{T}\left(\int_0^t B_s\, ds + (T-t)B_t\right),
\end{aligned}
$$

and

$$
\mathbf{Var}[X \mid \mathcal{F}_t] = \mathbf{Var}\left[\frac{1}{T}\int_0^T B_s\, ds \,\middle|\, \mathcal{F}_t\right] = \underbrace{\mathbf{E}\left[\Big(\frac{1}{T}\int_0^T B_s\, ds\Big)^2 \,\middle|\, \mathcal{F}_t\right]}_{I} - \underbrace{\mathbf{E}\left[\frac{1}{T}\int_0^T B_s\, ds \,\middle|\, \mathcal{F}_t\right]^2}_{II}.
$$

Since

$$
I = \frac{1}{T^2}\int_{y=0}^T \int_{s=0}^T \mathbf{E}[B_y B_s \mid \mathcal{F}_t]\, dsdy = \frac{1}{T^2}\int_{y=0}^T \int_{s=0}^T \mathbf{Cov}[B_y, B_s \mid \mathcal{F}_t]\, dsdy + II,
$$

we see that II cancels out, leaving us with

$$
\begin{aligned}
\mathbf{Var}\left[\frac{1}{T}\int_0^T B_s\, ds \,\middle|\, \mathcal{F}_t\right] &= \frac{1}{T^2}\int_{y=0}^T \int_{s=0}^T \mathbf{Cov}[B_y, B_s \mid \mathcal{F}_t]\, dsdy \\
&= \frac{2}{T^2}\int_{y=t}^T \int_{s=t}^y (s-t)\, dsdy \\
&= \frac{2}{T^2}\int_{y=t}^T \Big(\frac{y^2}{2} - ty - \frac{t^2}{2} + t^2\Big)\, dy \\
&= \frac{(T-t)^3}{3T^2}.
\end{aligned}
$$

From (3) in Section 2 we have that

$$
D_t\left(\int_0^T B_s\, ds\right) = D_t\left(\int_0^T (T-s)\, dB_s\right) = T - t,
$$

so that $D_t X = \sigma\,(1 - t/T)$ and $D_t(X-K)^+ = 1_{(K,\infty)}(K)\,\sigma\,(1-t/T)$. And so we have

$$
\begin{aligned}
\mathbf{E}[D_t(X-K)^+ \mid \mathcal{F}_t] &= \sigma\Big(1 - \frac{t}{T}\Big)\mathbf{E}[1_{(K,\infty)}(X) \mid \mathcal{F}_t] \\
&= \sigma\Big(1 - \frac{t}{T}\Big)\,\Phi\left(\frac{\mu T/2 + \sigma(\int_0^t B\, sds + (T-t)B_t)/T - K}{\sigma(T-t)^3/2\sqrt{3}T}\right).
\end{aligned}
$$

The stochastic integral representation of the pseudo-option $(X-K)^+$ is thus

$$
\begin{aligned}
&(X-K)^+ \\
&= \mathbf{E}[(X-K)^+] + \int_0^T \sigma\Big(1 - \frac{t}{T}\Big)\,\Phi\left(\frac{\mu T/2 + \sigma(\int_0^t B_s\, ds + (T-t)B_t)/T - K}{\sigma(T-t)^3/2\sqrt{3}T}\right) dB_t.
\end{aligned}
$$

6.2 The functional $(\overline{B_T} - KB_T)^+$

Another functional that as far as we know have not been investigated before is given by $(\overline{B_T} - KB_T)^+$. Like the pseudo-option in Section 6.1 this functional is also inspired by an Asian option, namely the random strike option $(S_T - \overline{S})^+$. We have generalized this option by adding the factor K.

Using that $D_t(\int_0^T B_s\, ds/T) = 1 - t/T$ from Section 6.1, we have

$$D_t(\overline{B_T} - KB_T)^+ = (D_t\overline{B_T} - KD_tB_T)1_{(KB_T,\infty)}(\overline{B_T}\,) = \left(1 - \frac{t}{T} - K\right)1_{(KB_T,\infty)}(\overline{B_T}\,). \tag{20}$$

To calculate $\mathbf{E}[D_t(\overline{B_T} - KB_T)^+ \mid \mathcal{F}_t]$ we need to find the joint density function of $\overline{B_T}$ and B_T conditioned on the filtration $\mathcal{F}_t$. Here $X \equiv B_T \mid \mathcal{F}_t$ is $\mathrm{N}(B_t, T-t)$-distributed, while we have from Section 6.1 that

$$Y \equiv \overline{B_T} \mid \mathcal{F}_t \sim \mathrm{N}\left(\frac{1}{T}\left(\int_0^t B_s\, ds + (T-t)B_t\right), \frac{(T-t)^3}{3T^2}\right).$$

In order to use the formula for the density function of a bivariate normal distribution

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{\det(\Sigma)}}\, \mathrm{e}^{-(z-\mu)\Sigma^{-1}(z-mu)^T/2}, \tag{21}$$

where $z = (x,y)$, $\mu = (\mathbf{E}[X], \mathbf{E}[Y])$ and Σ is the covariance matrix of (X,Y). The covariance between X and Y is given by

$$\mathbf{Cov}[X,Y] = \underbrace{\mathbf{E}[\overline{B_T}B_T \mid \mathcal{F}_t]}_{I} - \underbrace{\mathbf{E}[\overline{B_T} \mid \mathcal{F}_t]\,\mathbf{E}[B_T \mid \mathcal{F}_t]}_{II},$$

where

$$\begin{aligned} I &= \frac{1}{T}\int_0^T \mathbf{E}[B_sB_T \mid \mathcal{F}_t]\, ds \\ &= \frac{1}{T}\int_0^T \mathbf{Cov}[B_s, B_T \mid \mathcal{F}_t]\, ds - \frac{1}{T}\int_0^T \mathbf{E}[B_s \mid \mathcal{F}_t]\,\mathbf{E}[B_T \mid \mathcal{F}_t]\, ds \\ &= \frac{1}{T}\int_t^T (s-t)\, ds + II, \end{aligned}$$

so that

$$\mathbf{Cov}[X,Y] = \frac{1}{T}\int_t^T (s-t)\, ds = \frac{(T-t)^2}{2T}.$$

Our covariance matrix and its inverse are thus given by

$$\Sigma = \begin{pmatrix} T-t & \dfrac{(T-t)^2}{2T} \\ \dfrac{(T-t)^2}{2T} & \dfrac{(T-t)^3}{3T^2} \end{pmatrix} \quad \text{and} \quad \Sigma^{-1} = \frac{12T^2}{(T-t)^4}\begin{pmatrix} \dfrac{(T-t)^3}{3T^2} & -\dfrac{(T-t)^2}{2T} \\ -\dfrac{(T-t)^2}{2T} & T-t \end{pmatrix}.$$

For the exponent in (21) we therefore have

$$
\begin{aligned}
&-\frac{1}{2}(z-\mu)\Sigma^{-1}(z-\mu)^T \\
&= -\frac{1}{2}\frac{12T^2}{(T-t)^4}\,(x-\mathbf{E}[X], y-\mathbf{E}[Y]) \begin{pmatrix} \dfrac{(T-t)^3}{3T^2} & -\dfrac{(T-t)^2}{2T} \\ -\dfrac{(T-t)^2}{2T} & T-t \end{pmatrix} \begin{pmatrix} x-\mathbf{E}[X] \\ y-\mathbf{E}[Y] \end{pmatrix} \\
&= -\frac{6T^2}{(T-t)^4}\left(\frac{(T-t)^3}{3T^2}(x-\mathbf{E}[X])^2 - 2\,\frac{(T-t)^2}{2T}\,(x-\mathbf{E}[X])(y-\mathbf{E}[Y]) + (T-t)(y-\mathbf{E}[Y])^2\right).
\end{aligned}
$$

We arrive at the following expression for the joint density function of X and Y:

$$
\begin{aligned}
f_{X,Y}(x,y) = \frac{\sqrt{12T^2}}{2\pi\sqrt{(T-t)^4}} \exp\Bigg\{ -\frac{6T^2}{(T-t)^4}\bigg(\frac{(T-t)^3}{3T^2}(x-\mathbf{E}[X])^2 - 2\,\frac{(T-t)^2}{2T} \\
\times (x-\mathbf{E}[X])(y-\mathbf{E}[Y]) + (T-t)(y-\mathbf{E}[Y])^2\bigg)\Bigg\}.
\end{aligned}
$$

Insering this density we can now calculate our required conditional expectation as

$$
\mathbf{E}[D_t(\overline{B_T}-KB_T)^+ \mid \mathcal{F}_t] = \mathbf{P}[\overline{B_T} > KB_T \mid \mathcal{F}_t] = \mathbf{P}[Y > KX] = \int_{x=-\infty}^{\infty}\int_{y=Kx}^{\infty} f_{X,Y}(x,y)\,dxdy. \tag{22}
$$

Using (20) and (22) together with the fact that

$$
\mathbf{E}[(\overline{B_T}-KB_T)^+] = \frac{\sqrt{\mathbf{Var}[\overline{B_T}-KB_T]}}{\sqrt{2\pi}} = \sqrt{\frac{T}{2\pi}\left(\frac{1}{3}-K+K^2\right)}
$$

(using the covariance matrix Σ with $t=0$ in an elementary Gaussian calculation) we may now apply the Clark-Ocone formula to get the stochastic integral representation for our pseudo-option $(\overline{B_T}-KB_T)^+$

$$
(\overline{B_T}-KB_T)^+ = \sqrt{\frac{T}{2\pi}\left(\frac{1}{3}-K+K^2\right)} + \int_{t=0}^{T}\left(1-\frac{t}{T}-K\right)\int_{x=-\infty}^{\infty}\int_{y=Kx}^{\infty} f_{X,Y}(x,y)\,dxdydB_t.
$$

6.3 The functional B_T^k

Let us return to Example 3 of Section 2 where we studied the stochastic integral representation of Brownian motion raised to an integer k by repeated application of using Itô's formula. With the help of Malliavin calculus this work becomes much swifter and is to our knowledge done for the first time here. Although this functional does not have an option-like structure, we still find the result interesting.

Let us look at the Malliavin derivative $D_t B_T^k = k B_T^{k-1}$ of B_T^k. In order to appkly the Clark-Ocone formula and we use the binomial theorem to calculate

$$\begin{aligned}
\mathbf{E}[B_T^{k-1} \mid \mathcal{F}_t] &= \int_{-\infty}^{\infty} x^{k-1} \exp\left\{-\frac{(x-B_t)^2}{2(T-t)}\right\} dx \\
&= \int_{-\infty}^{\infty} (y+B_t)^{k-1} \exp\left\{-\frac{y^2}{2(T-t)}\right\} dy \\
&= \int_{-\infty}^{\infty} \sum_{j=0}^{k-1} \binom{k-1}{j} y^j B_t^{k-1-j} \exp\left\{-\frac{y^2}{2(T-t)}\right\} dy \\
&= \sum_{j=0,2,\ldots}^{k-1-(k-1)mod2} \binom{k-1}{j} B_t^{k-1-j} \frac{2\,(j-1)!!\,(2(T-t))^{j/2}}{2^{j/2+1}} \sqrt{2\pi(T-t)} \\
&= \sum_{p=0}^{(k-1)/2-(k-1)/2 \bmod 2} \binom{k-1}{2p} B_t^{k-1-2p}\,(2p-1)!!\,(T-t)^p \sqrt{2\pi(T-t)},
\end{aligned}$$

where !! is the semifactorial $(2p-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2p-1)$. By the Clark-Ocone formula the stochastic integral representation of B_T^k is therefore

$$B_T^k = \mathbf{E}[B_T^k] + \int_0^T k \sum_{p=0}^{(k-1)/2-(k-1)/2 \bmod 2} \binom{k-1}{2p} B_t^{k-1-2p}\,(2p-1)!!\,(T-t)^p \sqrt{2\pi(T-t)}\; dB_t.$$

Here elementary Gaussian calculations show that $\mathbf{E}[B_T^k] = 0$ for k odd, while

$$\mathbf{E}[B_T^k] = \frac{(2T)^{k/2}\,\Gamma((1+k)/2)}{\sqrt{\pi}} \quad \text{for } k \text{ even.}$$

7 Conclusion

We have found the stochastic integral representation for a number of pseudo-options, that is to say, simplified forms of real world options that do not contain all the features of a real option.

Hopefully, our work can be used as a peace meal step forward to developing stochastic integral representations for more complicated functionals which are closer to real world options.

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