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MASTER'S THESIS

On Long Range Dependence in the Returns of Risky Assets

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Abstract This thesis deals with the property of long range dependence (LRD) in stochastic processes. Two different methods for estimating LRD are evaluated: The classical $\mathcal{R}/\mathcal{S}$-analysis and the Wavelet method. The evaluation is done by applying these methods on simulated time series, namely fractional Gaussian noise (FGN) and linear fractional stable noise (LFSN). We show that the $\mathcal{R}/\mathcal{S}$ estimator displays bias for negatively skewed LFSN.

Three models for risky assets that incorporates LRD in their log returns are evaluated: fractional Brownian motion (FBM), linear fractional stable motion (LFSM) and a process with NIG marginal distributions where the LRD is modeled by a set of Gaussian copulas.

The evaluation involves fitting of the marginal distributions (normal, stable and NIG) and applying the Wavelet method to investigate the presence of LRD in empirical financial data. We show that the stable distribution, with its economically appealing properties, fits the implied density of the log returns somewhat better than the NIG distribution. The normal distribution completely lacks ability to model the extreme values of the log returns. We also show that LRD may be observed in log returns but the high variance of the Wavelet estimator for LFSN makes it difficult to get a reliable result.

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Contents

Chapter 1

Introduction

In [S] A.N. Shiryaev suggests that the price of a risky asset (e.g. stock or portfolio) $S(t)$ has properties that makes the use of fractional Brownian motion (FBM) more appropriate than Brownian motion for modelling its evolvement over time. While the increments of Brownian motion are independent the increments of FBM displays long range dependence (LRD), i.e. has a dependence structure that makes the covariance function decay slowly[1] over time. Many others have also proposed that risky assets should be modeled by a stochastic process with LRD, see [Ø] [WRL].

In [KK] an economic justification for fractional Brownian motion to be used in finance is given. As Brownian motion is a limit of a random walk, fractional Brownian motion is a limit of a Poisson shot noise process. The shot noise model is interpreted as information which enters the price at random Poisson times. The arrival of new information may change the price drastically. The long memory appears because new information may need some time to spread among the market participants. This is thus an argument that relates to inefficient markets.

Both Brownian motion and FBM have normal marginal distributions. It is well-known that the normal distribution lacks properties that are observed in empirical financial data. Our intent is to investigate the presence of LRD under more general assumptions. More precisely, the marginal distributions are allowed to have heavy tails and skewness. The two distributions considered here that fulfill these properties are the economically appealing stable distribution and the flexible normal inverse Gaussian (NIG) distribution.

The diposition of this thesis is as follows: The first chapters gives a theoretical description of the stochastic processes considered. The chapter about fractional Brownian motion includes a brief summary of the recent results about the construction of a fractional market[2] that lacks arbitrage opportunities and is complete. Then two estimators of LRD, $\mathcal{R}/\mathcal{S}$ analysis and wavelet method, are described and evaluated. After that a test is made on the error of the estimated parameters (of the marginal distribution) when treating data with mutual dependence as independent. Finally, empirical financial data are fitted to the normal, NIG and stable distribution and LRD is estimated for these data sets. Simulation algorithms for FBM and LFSM can be found in an concluding appendix.

The computer calulations of this thesis have been done with *Matlab* and *Mathematica*.

All random variables and stochastic processes featuring in this thesis are assumed to be

[1] More precisely *slow enough*, for exact definition, see Chapter 2

[2] The risky assets are modelled by an exponential FBM.

defined on a suitable common filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$.

Chapter 2

Fractional Brownian motion

We start out with defining fractional Brownian motion and discussing some of its properties.

Definition 2.1 *If* $0 < H < 1$ *then the (standard) fractional Brownian motion with Hurst parameter* H *is the Gaussian process* $\{B_H(t)\}_{t\in\mathbb{R}}$, *with* $B_H(0) = 0$, *that has mean*

$$\mathbf{E}\{B_H(t)\} = B_H(0) = 0 \quad \textit{for } t \in \mathbb{R},$$

and covariance function

$$\mathbf{E}\{B_H(s)B_H(t)\} = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right) \quad \textit{for } s, t \in \mathbb{R}.$$

Note that for $H = \frac{1}{2}$ we get the ordinary Brownian motion.

Fractional Brownian motion is a *H-self-similar* stochastic process, i.e.

$$\{B_H(at)\}_{t\geq 0} =_d \{a^H B_H(t)\}_{t\geq 0} \quad \text{for } a > 0,$$

where $=_d$ denotes equality of finite dimensional distributions.

The stationary sequence of increments for FBM,

$$\{X_H(j)\}_{j\in\mathbb{N}} = \{B_H(j) - B_H(j-1)\}_{j\in\mathbb{N}},$$

is called *fractional Gaussian noise*. It turns out that this sequence is strongly correlated for $H \neq \frac{1}{2}$. More precisely, we have the covariance function

$$\rho_H(k) = \mathbf{E}\{X_H(j)X_H(j+k)\} = \frac{1}{2}\left(|k+1|^{2H} - 2|k|^{2H} + |k-1|^{2H}\right) \quad \text{for } k \in \mathbb{Z}, \tag{2.1}$$

so that

$$\rho_H(k) \sim H(2H-1)k^{2H-2} \quad \text{as } k \to \infty.$$

Notice that the covariance is positive $\rho_H(k) > 0$ for $k \neq 0$ when $\frac{1}{2} < H < 1$, while it is negative $\rho_H(k) < 0$ for $k \neq 0$ when $0 < H < \frac{1}{2}$.

For all $0 < H < 1$, the covariance function tends to zero as $k \to \infty$, but when $\frac{1}{2} < H < 1$ it tends to zero so slowly that $\sum_{k=-\infty}^{\infty} \rho_H(k)$ diverges. In this case we say that $\{X_H(j)\}_{j\in\mathbb{N}}$ exhibits *long range dependence* (LRD). From now on we will assume that $\frac{1}{2} \leq H < 1$ unless otherwise is stated.

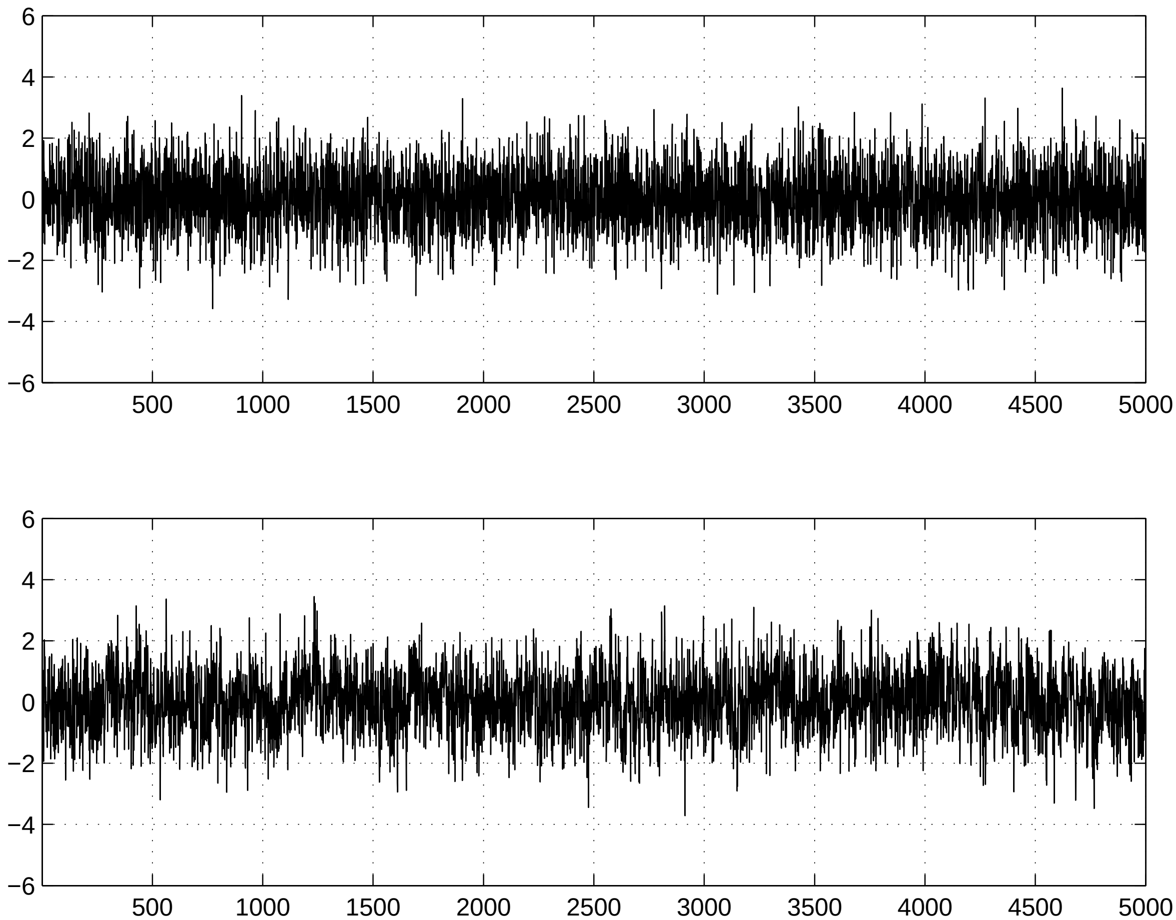

Figure 2.1: Simulated sample paths of fractional Gaussian noise with Hurst parameter $H = 0.5$ (top) and $H = 0.7$ (bottom). Simulation was done using the Davies-Harte algorithm, see Appendix A.

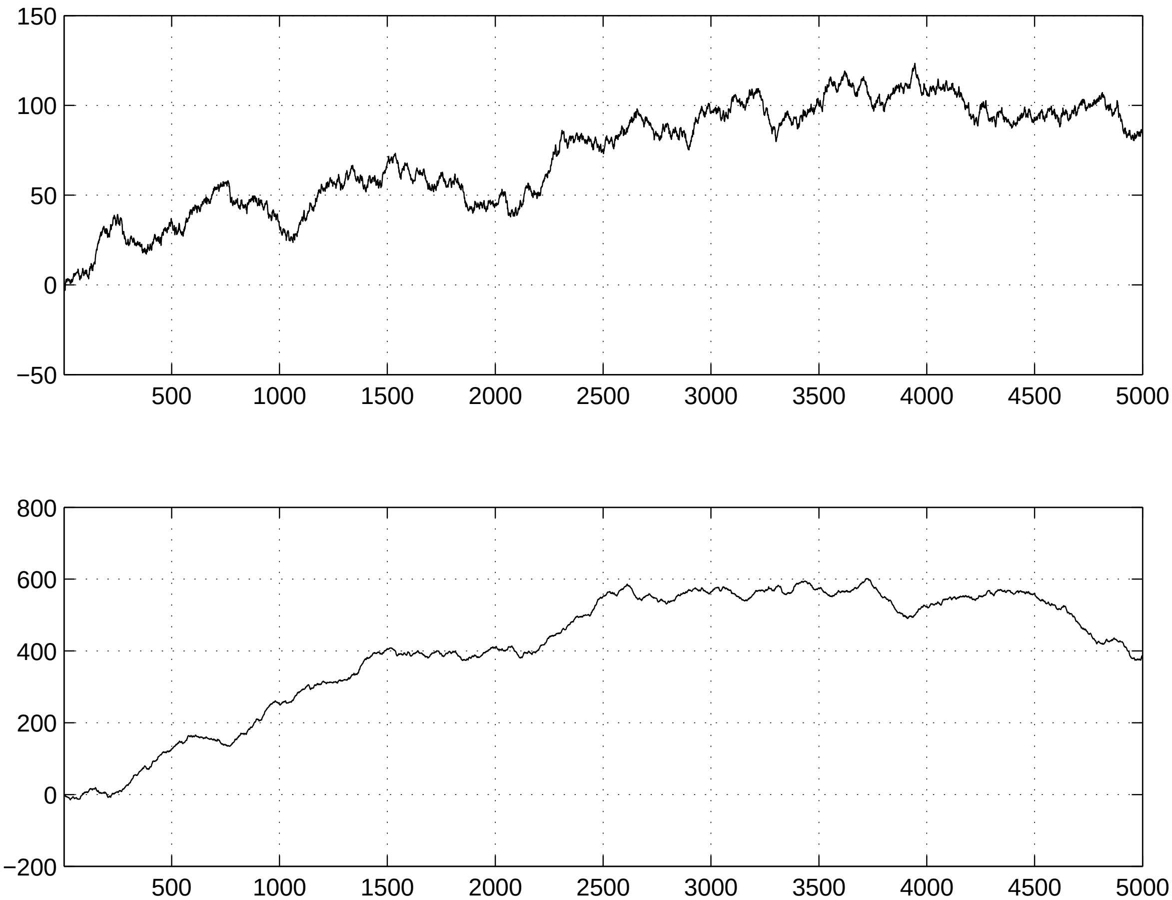

Figure 2.2: Simulated sample paths of fractional Brownian motion with Hurst parameter $H = 0.5$ (top) and $H = 0.7$ (bottom). Simulation was done using the Davies-Harte algorithm, see Appendix A.

2.1 Arbitrage

If a market is modelled by a riskfree asset (e.g. bond) $B(t)$ and a risky asset (e.g. stock or stock index) $S(t)$, where

$$B(t) = B(0)\,\mathrm{e}^{rt},$$

and the risky asset follows an exponential FBM with drift, i.e.

$$S(t) = S(0)\,\mathrm{e}^{\mu t + \sigma B_H(t)},$$

then there exists arbitrage opportunities in the market. This model for the risky asset is the solution to a stochastic differential equation (SDE) driven by FBM. More specifially, it is the solution using *pathwise integration.*

Fractional Brownian motion is not a semimartingale, hence the general stochastic calculus for semimartingales cannot be applied to solve SDE's driven by FBM. This has caused several different stochastic integrals with respect to FBM to be developed. Two of these are the pathwise integral and the *fractional Wick-Itô integral.*

2.2 Pathwise integral

If the integrand $\phi(t,\omega)$ is *caglad*[3], then the pathwise integral can be defined as a limit of Riemann sums

$$\int_0^T \phi(t,\omega)dB_H(t) = \lim_{\Delta t_k \to 0} \sum_{k=0}^{N-1} \phi(t_k)(B_H(t_{k+1}) - B_H(t_k)),$$

if the limit exists. Here $0 = t_0 < t_1 < \ldots < t_N = T$ is partition of $[0,T]$ and $\Delta t_k = t_{k+1} - t_k$.

The pathwise integral obeys *Stratonovich* type integration rules:

Theorem 2.2 *Suppose $H \in (\frac{1}{2}, 1)$ and $f \in C^1(\mathbb{R})$. Let*

$$Y(t) = f(X(t)),$$

where $X(t)$ is given by

$$dX(t) = u(t)dt + v(t)dB_H(t).$$

Then

$$dY(t) = f'(X(t))dX(t).$$

Let us consider a market consisting of a riskfree asset $B(t)$ and a risky asset $S(t)$, where the assets follow

$$dB(t) = rB(t)dt$$

and

$$dS(t) = \mu S(t)dt + \sigma S(t)dB_H(t), \tag{2.2}$$

respectively. Here r, μ and $\sigma > 0$ are constants, and (2.2) should be interpreted in a pathwise sense. Using Theorem 2.2 on (2.2), letting $X(t) = \mu t + \sigma B_H(t)$ and $S(t) = f(X(t))$, we get the solution

$$S(t) = S(0)\,\mathrm{e}^{\mu t + \sigma B_H(t)}.$$

The pathwise integral is intuitively appealing, but when applied to finance it leads to a market with arbitrage, see [C] or [Ø].

[3] Left-continuous with right limits

2.3 Wick-Itô integral

The Wick-Itô (or Skorohod) integral may be defined as follows:

$$\int_0^T \phi(t,\omega)\delta B_H(t) = \lim_{\Delta t_k \to 0} \sum_{k=0}^{N-1} \phi(t_k) \diamond (B_H(t_{k+1}) - B_H(t_k)).$$

Here $\diamond$ denotes the *Wick product*, for further details see e.g. [Ø]. The Wick-Itô analogue of (2.2) is denoted

$$\delta S(t) = \mu S(t)dt + \sigma S(t)\delta B_H(t). \tag{2.3}$$

We use the following result to derive a solution to (2.3):

Theorem 2.3 ([BØSW]) *Let $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ belong to $C^{1,2}(\mathbb{R} \times \mathbb{R})$ and assume that the random variables*

$$f(t, B_H(t)), \quad \int_0^t \frac{\partial f}{\partial s}(s, B_H(s))ds \quad \textit{and} \quad \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_H(s))s^{2H-1}ds$$

all have finite second moments. Then

$$\delta f(t, B_H(t)) = \frac{\partial f}{\partial t}(t, B_H(t))dt + \frac{\partial f}{\partial x}(t, B_H(t))\delta B_H(t) + H\frac{\partial^2 f}{\partial x^2}(t, B_H(t))t^{2H-1}dt.$$

From Theorem 2.3 it follows easily that

$$S(t) = S(0)\, \mathrm{e}^{\sigma B_H(t) + \mu t - \sigma^2 t^{2H}/2} \tag{2.4}$$

is the (unique) solution to (2.3). Note that if $H = \frac{1}{2}$ this solution coincides with that obtained from Itô calculus for Brownian motion. The Wick-Itô integral does in fact behave in many ways like the Itô integral for Brownian motion. The latter is of course a fundamental tool for the classical Bachelier-Samuelsson market ([B], [T]).

The market modelled by (2.2) and (2.3) is in fact arbitrage free and complete, see e.g. [Ø]. Completeness of the market is important for pricing of financial derivatives since the theoretical price is then uniquelly determined. This is for instance not the case in a general semimartingale market.

Chapter 3

Stochastic processes with copulas

We want to create a stochastic process with LRD that has marginal distributions that permits heavy tails and skewness. One possibilty is to consider a process with Gaussian copulas.

3.1 Copulas

Recall the definition of a copula:

Definition 3.1 *A function $C : [0,1]^n \to [0,1]$ is an n-copula if it enjoys the following properties:*

- *$C(1,\dots,1,u,1,\dots,1) = u$ for $u \in [0,1]$;*
- *$C(u_1,\dots,u_n) = 0$ for $u_1,\dots,u_n \in [0,1]$ with $u_i = 0$ for some $i \in \{1,\dots,n\}$;*
- *C is grounded and n-increasing, i.e. the C-volume of every box whose vertices lie in $[0,1]^n$ is positive.*

A copula is hence a multivariate distribution with uniform marginals on $[0,1]$.
We shall use the following important result, known as Sklar's Theorem:

Theorem 3.2 *Let F be an n-dimensional distribution function with margins $F_1,\dots,F_n$. Then there exists an n-copula C such that*

$$F(x_1,\dots,x_n) = C(F_1(x_1),\dots,F_n(x_n)) \quad \text{for } (x_1,\dots,x_n) \in \mathbb{R}^n. \tag{3.1}$$

If $F_1,\dots,F_n$ are all continuous, then the copula C is unique.

By (3.1) it is clear that given a multivariate distribution F with marginals $F_1,\dots,F_n$, the function $C(u_1,\dots,u_n) = F\big(F_1^{-1}(u_1),\dots,F_n^{-1}(u_n)\big)$ is an n-copula. Here F^{-1} denotes the generalized left inverse of the distribution function F, i.e.

$$F^{-1}(u) = \inf\{x : F(x) \geq u\} \quad \text{for } u \in (0,1).$$

Theorem 3.3 (Invariance theorem) *Let $X_1, \ldots, X_n$ be continuous random variables with copula C. Then, for strictly increasing functions $g_1, \ldots, g_n$, the random variables $Y_1 = g_1(X_1)$, $\ldots, Y_n = g_n(X_n)$ have the same copula C.*

Theorem 3.3 shows that the dependence between n random variables is completely captured by the copula, independently of the shape of the marginal distributions. Thus for continuous multivariate distributions the univariate margins and the multivariate dependence can be seperated.

The copula of the standard multivariate normal distribution with correlation matrix Σ is

$$C_\Sigma^{\mathrm{Ga}}(u_1, \ldots, u_n) = \Phi_\Sigma^n(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_n)), \tag{3.2}$$

where Φ_Σ^n denotes the joint distribution function of the n-dimensional standard normal distribution function with correlation matrix Σ, and Φ the univariate standard normal distribution function.

The 'classical' use of copulas is to model the dependence structure between two or more random variables. But copulas may also be used to model stochastic processes [SC]:

Corollary 3.4 (Corollary to Kolmogorovs Theorem) *Let $\mathcal{C} = \{C_{t_1,\ldots,t_n} : t_i \in T, t_1 < \cdots < t_n, n \in \mathbb{N}\}$ be a set of copulas with*

$$\lim_{u_k \nearrow 1} C_{t_1,\ldots,t_n}(u_1, \ldots, u_n) = C_{t_1,\ldots,t_{k-1},t_{k+1},\ldots,t_n}(u_1, \ldots, u_{k-1}, u_{k+1}, \ldots, u_n),$$

and $\mathcal{F} = \{F_t : t \in T\}$ a set of one-dimensional distribution functions. Then there exists a probability space $(\Omega, \mathcal{G}, \mathbf{P})$ and a stochastic process $X = \{X_t\}_{t \in T}$ with

$$\mathbf{P}\{X_{t_1} \le x_1, \ldots, X_{t_n} \le x_n\} = C_{t_1,\ldots,t_n}\big(F_{t_1}(x_1), \ldots, F_{t_n}(x_n)\big) \tag{3.3}$$

for $x_1, \ldots, x_n \in \mathbb{R}$, $t_1, \ldots, t_n \in T$ and $n \in \mathbb{N}$.

From a financial perspective, the new use of copulas indicated by Corollary 3.4 means a switch from the common method of using copulas to model dependence in a portfolio between several assets (see e.g. [ELM]) to instead model the dependence over time for a single asset.

By an copula approach, one may seperate the dependence structure of FBM from its marginal distribution, i.e. the Gaussian. The dependence structure is then modeled by a set of Gaussian copulas.

3.2 Normal inverse Gaussian distribution

To be able to model the possibility of extreme event we need a distribution with heavy tails. It is also well-known that the implied density of log returns are skewed, i.e. one tail is heavier than the other. One usually assumes that the left tail is heavier than the right, due to the so called *leverage effect*: A negative correlation between past stock returns and future volatility.

A distribution that has turned out to be successful in modeling heavy tails and skewness of financial data (see e.g. [B], [BO] and [T]) is the *generalized hyperbolic* distribution (GH):

Definition 3.5 *The GH distribution has probability density function given by*

$$gh(x;\lambda,\alpha,\beta,\delta,\mu) = a(\lambda,\alpha,\beta,\delta)(\delta^2+(x-\mu)^2)^{(\lambda-1/2)/2} \times K_{\lambda-\frac{1}{2}}(\alpha\sqrt{\delta^2+(x-\mu)^2})\mathrm{e}^{\beta(x-\mu)},$$

where

$$a(\lambda,\alpha,\beta,\delta) = \frac{(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^{\lambda}K_{\lambda}(\delta\sqrt{\alpha^2-\beta^2})}.$$

Here K_λ is a modified Bessel function of the third kind with index λ, i.e.

$$K_\lambda(x) = \frac{1}{2}\int_0^\infty y^{\lambda-1}\mathrm{e}^{-\frac{1}{2}x(y^{-1}+y)}dy,$$

and the parameters satisfy $0 \le |\beta| < \alpha$, $\mu,\lambda \in \mathbb{R}$ and $\delta > 0$.

We intend to consider a special case of the GH distribution, namely the *normal inverse Gaussian* distribution (NIG):

Definition 3.6 *Taking $\lambda = -1/2$ in the GH distribution, we get the NIG distribution, with probability density function given by*

$$nig(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta}{\pi}\mathrm{e}^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)}\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}.$$

The parameters satisfy $0 \le |\beta| < \alpha$, $\mu \in \mathbb{R}$ and $\delta > 0$.

We can now create a stochastic process which has the property described in (3.3), where $\mathcal{F} = \{F_t : t \in T\}$ is a set of NIG distributions, and $\mathcal{C} = \{C_{t_1,\dots,t_n} : t_i \in T, t_1 < \cdots < t_n, n \in \mathbb{N}\}$ is a set of Gaussian copulas described by (3.2), where Φ_Σ^n is the multivariate normal distribution for fractional Gaussian noise.

Chapter 4

Linear fractional stable motion

Often the central limit theorem (CLT) is an argument for using the Gaussian model in finance. Then the randomness observed in risky assets are considered being a result of many small effects so that according to the CLT a Gaussian model would be appropriate.

However, assuming that random effects are heavy-tailed a non-Gaussian stable model may be more accurate [JW]. In contrast to Gaussian distributions non-Gaussian stable distributions are heavy-tailed (always infinite variance) and admit skewness. These are properties which are appropriate for applications in finance.

4.1 Stable distributions

The family of *stable* distributions plays has an important role i probability theory, because of the following defining closedness property:

Definition 4.1 *A random variable X is α-stable if for any $n \geq 2$, there exists a constant $b_n \in \mathbb{R}$ such that*

$$X_1 + X_2 + \cdots + X_n =_d n^{1/\alpha} X + b_n$$

for some $0 < \alpha \leq 2$, where $X_1, X_2, \ldots, X_n$ are independent copies of X.

An definition equivalent to Definition 4.1 is to say that stable distributions are the only distributions that can be obtained as limits of normalized sums of independent identically ditributed (iid.) random variables:

Definition 4.2 *A random variable X is said to have a stable distribution if it belongs to a domain of attraction, i.e. if there is a seqeuence of iid. random variables $Y_1, Y_2, \ldots$ and sequences of numbers $a_n \in \mathbb{R}_+$ and $b_n \in \mathbb{R}$, such that*

$$\frac{Y_1 + Y_2 + \cdots + Y_n}{a_n} + b_n \Rightarrow_d X \quad \text{as } n \to \infty, \tag{4.1}$$

where $\Rightarrow_d$ denotes convergence in distribution.

A stable distribution is characterized by four parameters: The *index of stability* α; the *scale parameter* $\sigma \geq 0$; the *skewness parameter* $\beta \in [-1, 1]$; and the *location parameter* $\mu \in \mathbb{R}$. We write

$$X \sim S_\alpha(\sigma, \beta, \mu)$$

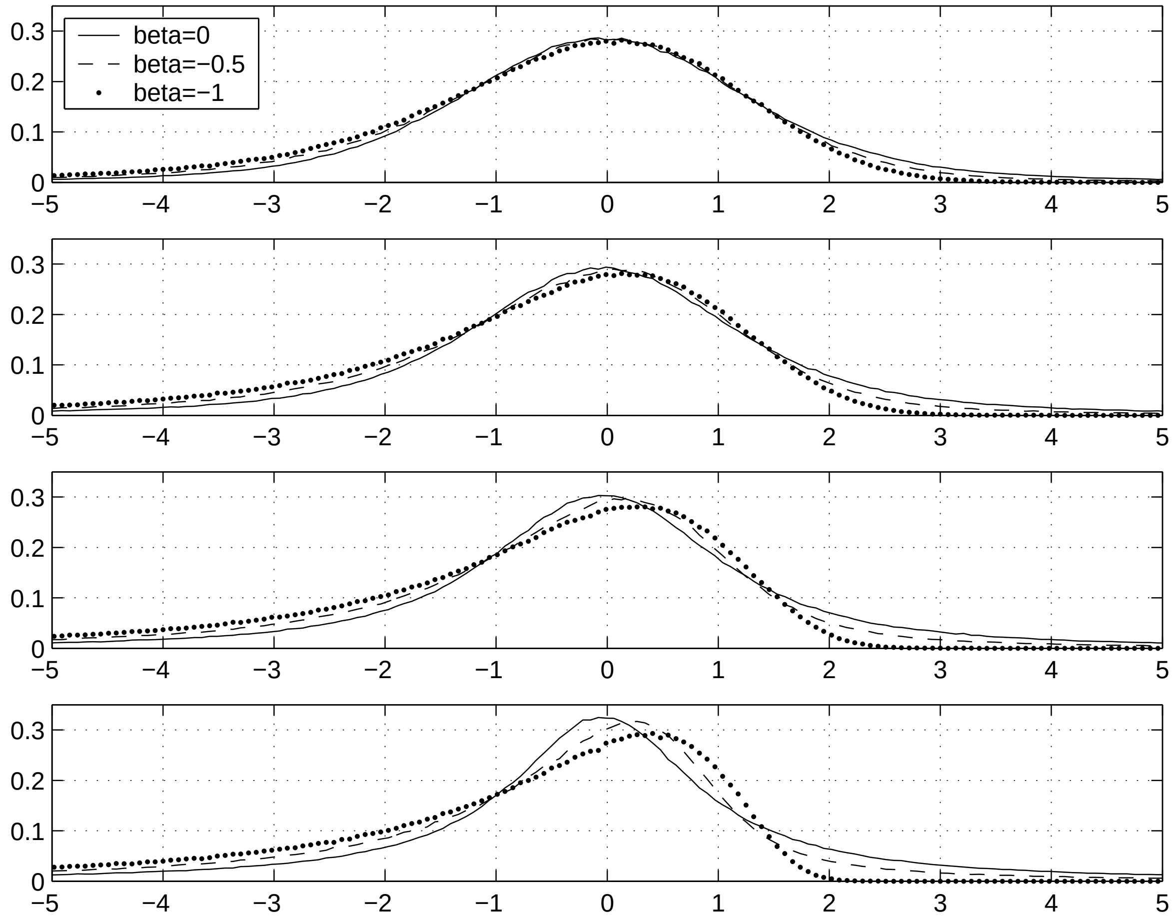

Figure 4.1: Approximative density functions of stable distributions with $\mu = 0$, $\sigma = 1$, $\alpha = 1.6$ (top left), $\alpha = 1.4$ (top right), $\alpha = 1.2$ (bottom left) and $\alpha = 1.0$ (bottom right). The density function are approximated by simulating $1,500,000$ independent stable random variables.

to indicate that X has stable distribution with these parameters.

The parameter α controls the tails of the distribution of a stable random variable X, in the sence that

$$\mathbf{P}\{|X| \geq x\} \sim Cx^{-\alpha} \quad \text{as } x \to \infty,$$

for some constant $C \geq 0$. Greater values of α means lower probability of extreme values for X.

Although FBM displays long range dependence, its marginal distribution is Gaussian and thus concentrate its mass around the mean. Stable distributions with $0 < \alpha < 2$, on the other hand, are heavy-tailed. Note that if X is normal distributed and the Y_i:s have finite variance, then (4.1) is the statement of the CLT. Thus, arguing that the randomness in risky assets is a result of many small effects in the 'the real world', stable marginal distributions should be used to model these random processes. If these effects moreover are heavy-tailed, these marginal distributions should be non-Gaussian $\alpha < 2$ (see Property 4.3 below).

Except for a few particular values of the four parameters the probability density function of a stable distribution is not known explicitly. A random variable with stable distribution is therefore usually described using its characteristic function (chf.):

Property 4.3 *A random variable X satisfies $X \sim S_\alpha(\sigma, \beta, \mu)$ if and only if*

$$\mathbf{E}\{\mathrm{e}^{i\theta X}\} = \begin{cases} \mathrm{e}^{-\sigma^\alpha |\theta|^\alpha [1 - i\beta \,\mathrm{sign}(\theta) \tan(\pi\alpha/2)] + i\mu\theta} & \textit{when } \alpha \neq 1 \\ \mathrm{e}^{-\sigma^\alpha |\theta|^\alpha [1 + i\beta (2/\pi) \,\mathrm{sign}(\theta) \log(|\theta|)] + i\mu\theta} & \textit{when } \alpha = 1 \end{cases} . \tag{4.2}$$

We see that if $\alpha = 2$ in (4.2), then the chf. becomes $\mathrm{e}^{-\sigma^2\theta^2 + i\mu\theta}$, i.e. the chf. for a Gaussian random variable with mean μ and variance $2\sigma^2$.

Definition 4.4 *A stochastic process $\{L(t)\}_{t\in\mathbb{R}}$ such that $L(0) = 0$ a.s. is called an α-stable Lévy motion with skewness β, if the process has independent and stationary increments that satisfy*

$$L(t) - L(s) \sim S_\alpha\left((t-s)^{1/\alpha}, \beta, 0\right) \quad \textit{for } t > s \geq 0.$$

4.2 Totally negatively skewed stable random variables

If one wants to model some risky asset such that the logarithm follows an α-stable Lévy motion with $\alpha < 2$, the fact that α-stable random variables have infinite variance imposes a problem: The expected payoff of an asset might not be finite! In fact, we have the following result:

Property 4.5 *If $X \sim S_\alpha(\sigma, \beta, \mu)$, then the Laplace transform of X is not finite unless $\beta = 1$. When $\beta = 1$, the Laplace transform is given by*

$$\mathbf{E}\{\mathrm{e}^{-\lambda X}\} = \mathrm{e}^{-\lambda\mu - \lambda^\alpha \sigma^\alpha \sec(\alpha\pi/2)} \quad \textit{for } \lambda \geq 0.$$

Property 4.6 *For any $0 < \alpha < 2$, we have*

$$X \sim S_\alpha(\sigma, \beta, \mu) \Leftrightarrow -X \sim S_\alpha(\sigma, -\beta, -\mu).$$

Property 4.5 together with Property 4.6 give us that if X is a *totally negatively skewed* stable random variable, i.e. $\beta = -1$, then $\mathbf{E}\{\mathrm{e}^X\} < \infty$. Carr and Wu [CW] use this fact to model risky assets where the log returns are stable random variables, i.e.

$$S(t) = S(0)\,\mathrm{e}^{X_t},$$

where $X_t \sim S_\alpha(\sigma, -1, \mu)$. Their intent is to produce finite option prices, but as they put it, "our specification has the added attraction of capturing the highly skewed feature of the implied density for log returns".

4.3 General stable random variables

The assumption that log returns from risky assets are stable random variables that are totally negatively skewed is, of course, restrictive. In Chapter 7 of this thesis we investigate just how restrictive this assumption is. In [MC] McCulloch gives a justification for relaxing this assumption. We will make use of the following result:

Property 4.7 *Let X_1 and X_2 be independent random variables with $X_i \sim S_\alpha(\sigma_i, \beta_i, \mu_i)$ for $i = 1, 2$. Then we have*

$$X_1 + X_2 \sim S_\alpha(\sigma, \beta, \mu),$$

where

$$\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}, \quad \beta = \frac{\beta_1\sigma_1^\alpha + \beta_2\sigma_2^\alpha}{\sigma_1^\alpha + \sigma_2^\alpha}, \quad \text{and} \quad \mu = \mu_1 + \mu_2. \tag{4.3}$$

In [MC] the price of an asset at future time T is taken to be

$$S_T = \frac{U_2}{U_1},$$

where U_2 is the random future marginal utility[4] of the asset and U_1 is the random future marginal utility of the numeraire in which the asset is priced.

Now, let $u_1 = \log U_1$ and $u_2 = \log U_2$. Then Properties 4.6 and 4.7 imply that when u_1 and u_2 are (independent and) stable with a common parameter α, then

$$\log S_T = u_2 + (-u_1)$$

will also be stable with the same α. In order to keep $\mathbf{E}\{S_T\}$ finite, Property 4.5 requires that u_1 and u_2 both are totally negatively skewed. However, by this setup, $\log S_T$ itself may still have the general stable distribution

$$\log S_T \sim S_\alpha(\sigma, \beta, \mu),$$

where the skewness β is determined by (4.3), where $\beta_1 = \beta_2 = -1$ and σ_1 and σ_2 are the scale parameters of u_1 and u_2, respectively.

4.4 α-stable stochastic integrals

We state a basic property for the stochastic integral $\int_{x\in\mathbb{R}} f(x)dL(x)$ of a deterministic function $f : \mathbb{R} \to \mathbb{R}$ with respect to α-stable Lévy motion.

Property 4.8 *Let $\{L(t)\}_{t\in\mathbb{R}}$ be an α-stable Lévy motion with skewness β. For measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $\int_{x\in\mathbb{R}} |f(x)|^\alpha dx < \infty$ we have*

$$\int_{x\in\mathbb{R}} f(x)dL(x) \sim S_\alpha\left(\left(\int_{x\in\mathbb{R}} |f(x)|^\alpha dx\right)^{1/\alpha}, \beta\frac{\int_{x\in\mathbb{R}} \operatorname{sign}(f(x))|f(x)|^\alpha dx}{\int_{x\in\mathbb{R}} |f(x)|^\alpha dx}, 0\right).$$

[4] For those unfamiliar with this microeconomic term, see e.g. [V] for definition.

4.5 Linear fractional stable motion

Our intent is to use the suggestions in [CW] and [MC], and add to this a dependence structure in a similar way as a dependence structure is added to the Bachelier-Samuelsson model via FBM. One natural way to do this is to consider the *linear fractional stable motion* (LFSM). LFSM has for instance been proposed as a model for traffic in broadband networks, see e.g. [HHL].

Definition 4.9 *The linear fractional stable motion is the stochastic process given by*

$$L_{\alpha,H}(a,b;t) = \int_{-\infty}^{\infty} f_{\alpha,H}(a,b;t,x)M(dx), \tag{4.4}$$

where

$$f_{\alpha,H}(a,b;t,x) = a\Big(\big((t-x)^+\big)^{H-1/\alpha} - \big((-x)^+\big)^{H-1/\alpha}\Big) + b\Big(\big((t-x)^-\big)^{H-1/\alpha} - \big((-x)^-\big)^{H-1/\alpha}\Big),$$

and where a,b are constants, $|a|+|b|>0$, $0<\alpha<2$, $0<H<1$, $H \neq 1/\alpha$, and M is an α-stable random measure on $\mathbb{R}$ with Lebesgue control measure.

We do not make a formal definition of the α-stable random measure in (4.4) since it can be viewed as an α-stable Lévy motion, see [ST]. This is the setup that we will consider.

Like FBM, linear fractional stable motion is H-self-similar and has stationary increments.

Following [CW] and [MC] we want the log returns to be α-stable random variables with a common skewness parameter β. By Property 4.8 this is ensured if we choose an α-stable Lévy motion with skewness β (In the case of [CW], $\beta=-1$) and $f_{\alpha,H}(a,b;t,x)$ such that $\operatorname{sign}(f_{\alpha,H}(a,b;t,x)) = 1$ for $t>0$. By restricting our attention to the case $H>1/\alpha$ (which is LRD; see below) this is true when $a=1, b=0$. Then (4.4) reduces to

$$L_{\alpha,H}(t) = \int_{-\infty}^{\infty} \Big(\big((t-x)^+\big)^{H-1/\alpha} - \big((-x)^+\big)^{H-1/\alpha}\Big) dL(x). \tag{4.5}$$

From now on we will by LFSM mean the stochastic process described by (4.5), where $L(t)$ is an α-stable Lévy motion with skewness β, unless otherwise is stated.

The sequence of increments for LFSM is the stationary sequence

$$\begin{aligned}\{Y_{\alpha,H}(j)\}_{j\in\mathbb{N}} &= \{L_{\alpha,H}(j) - L_{\alpha,H}(j-1)\}_{j\in\mathbb{N}} \\ &= \left\{\int_{-\infty}^{\infty} \Big(\big((j-x)^+\big)^{H-1/\alpha} - \big((j-1-x)^+\big)^{H-1/\alpha}\Big) dL(x)\right\}_{j\in\mathbb{N}}\end{aligned} \tag{4.6}$$

called *linear fractional stable noise* (LFSN). By analogy with fractional Gaussian noise, $\{Y_{\alpha,H}(j)\}_{j\in\mathbb{N}}$ displays long range dependence when $H>1/\alpha$ and negative dependence when $H<1/\alpha$. Since $H\in(0,1)$ long range dependence is thus only possible when $\alpha>1$.

The resulting proposed model for the price of a risky asset is thus an exponential LFSM with drift, i.e.

$$S(t) = S(0)\,\mathrm{e}^{L_{\alpha,H}(t)+\mu t}.$$

The issue to adjust this model for non-arbitrage goes beyond the scope of this thesis.

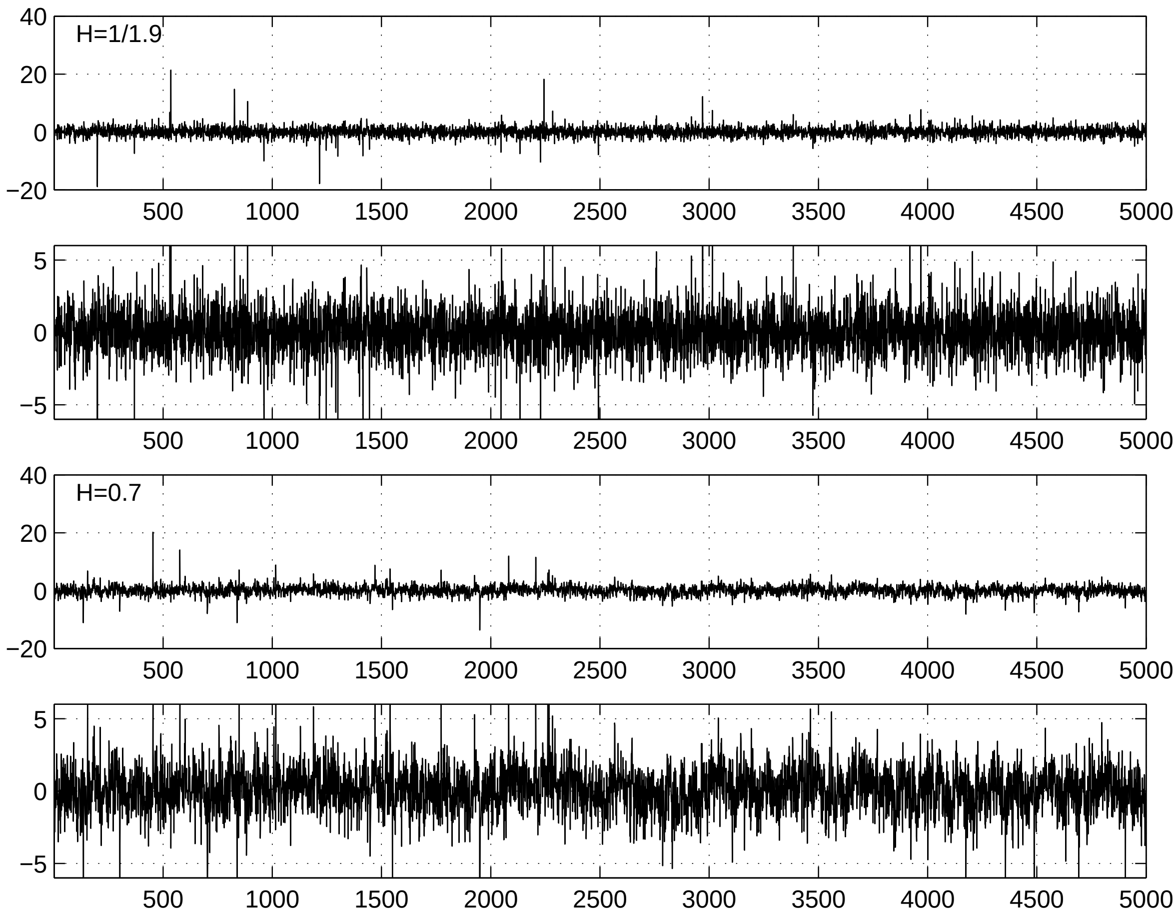

Figure 4.2: Linear fractional stable noise. Two realizations, with $H = 1/\alpha = 0.526$ (top) and $H = 0.7$ (bottom), are displayed with two different scalings of values. The parameters of the driving Lévy motion $L(t)$ are $\alpha = 1.9$ and $\beta = 0$.

Chapter 5

Estimating long range dependence

To estimate the long range dependence in an observed time series is not a trivial matter and many suggestions for methods can be found in the literature. In this chapter two methods to estimate the Hurst parameter H are described and evaluated.

5.1 $\mathcal{R}/\mathcal{S}$-analysis

The phenomena of 'long memory' or 'long range dependence' was discovered by H. Hurst in 1951 when studying annual run-offs from the Nile. This led to creation of the so-called $\mathcal{R}/\mathcal{S}$-analysis: Let $\{X_t\}_{t=1,\ldots,n}$ be an observed time series. Define

$$H_k = \sum_{t=1}^{k} X_t \quad \text{and} \quad \mathcal{R}_k = \max_{j \le k}\Big(H_j - \frac{j}{k}H_k\Big) - \min_{j \le k}\Big(H_j - \frac{j}{k}H_k\Big).$$

Here $H_j - \frac{j}{k}H_k$ is the deviation of H_j from the empirical mean over $1, \ldots, k$, and $\mathcal{R}_k$ characterizes the range of the sequence $H_1, \ldots, H_k$ relative to its empirical mean. Further, let

$$\mathcal{S}_k^2 = \frac{1}{k}\sum_{t=1}^{k} X_t^2 - \big(\frac{1}{k}\sum_{t=1}^{k} X_t\big)^2$$

be the empirical variance. Now, $\mathcal{R}_k/\mathcal{S}_k$ is the normalized range of the cumulative sums $\{H_k\}_{k=1,\ldots,n}$. This is known as the rescaled range statistics ($\mathcal{R}/\mathcal{S}$-statistics).

If $\{X_t\}_{t=1,\ldots,n}$ where independent, then it would be true that (see [S])

$$\frac{\mathcal{R}_k}{\mathcal{S}_k} \sim ck^{1/2} \quad \text{as } k \to \infty.$$

Hurst discovered that instead one may actually have

$$\frac{\mathcal{R}_k}{\mathcal{S}_k} \sim ck^{H} \quad \text{as } k \to \infty, \tag{5.1}$$

where the Hurst parameter H is larger than $\frac{1}{2}$. This suggests that the sequence displays long range dependence.

By (5.1) an estimator of the Hurst parameter H can be obtained by performing a linear regression of $\log(\mathcal{R}_k/\mathcal{S}_k)$ on k for $k = 1, \ldots, n$.

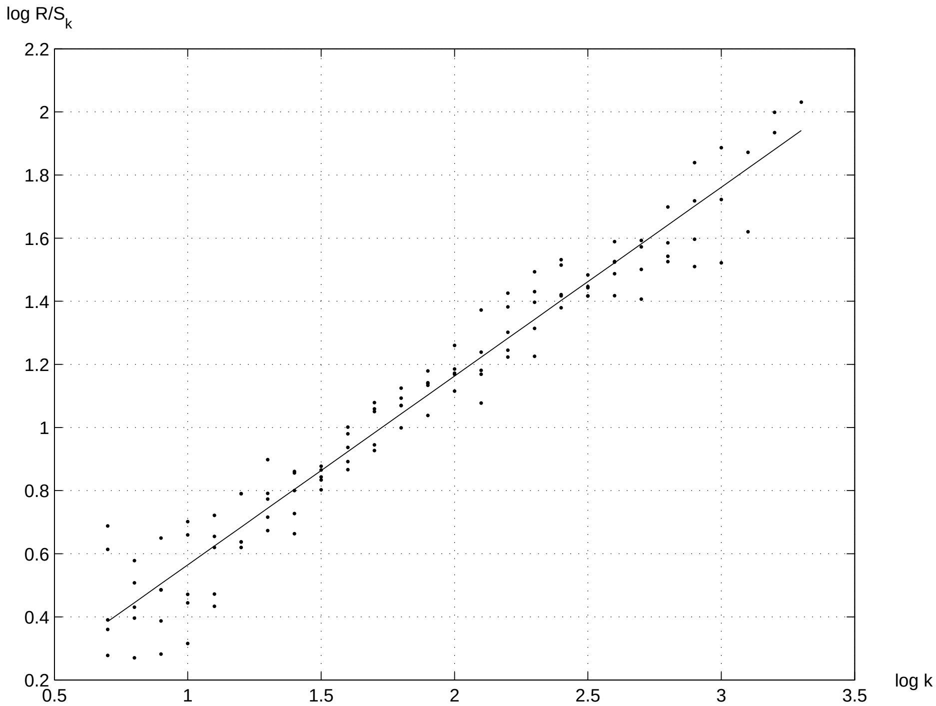

Figure 5.1: $\mathcal{R}/\mathcal{S}$-analysis for FGN with Hurst parameter $H = 0.6$. The fitted regression line is $\log(\mathcal{R}_k/\mathcal{S}_k) = -0.034 + 0.598 \log k$.

5.2 Wavelet method

Another method for estimating long range dependence is based on wavelet analysis. This estimation tool is described in e.g. [AFTV] and [AV]. A brief introduction to wavelets and the properties of the wavelet coefficients for long range dependent processes are given in the next two sections.

5.2.1 Multiresolution analysis and discrete wavelet transform

A multiresolution analysis (MRA) consists of a collection of subspaces $\{V_j\}_{j\in\mathbb{Z}}$ with the following properties:

- $\bigcap_{j\in\mathbb{Z}} V_j = \emptyset$;
- $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $\mathbb{L}^2(\mathbb{R})$;
- $V_j \subset V_{j-1}$ for $j \in \mathbb{Z}$;
- $X(t) \in V_j \iff X(2^j t) \in V_0$ for $j \in \mathbb{Z}$;
- there exists a so called *scaling function* $\phi_0 \in V_0$ such that the collection $\{\phi_0(t-k)\}_{k\in\mathbb{Z}}$ is an unconditional Riesz basis for V_0.

Here the V_j's are approximation subspaces of the space of square integrable functions $\mathbb{L}^2(\mathbb{R})$. The fact that the set of shifted scaling functions $\{\phi_0(t-k)\}$ form a 'Riesz basis' means that they are linearly independent and span the space V_0. The index j is called the *octave*. Notice that it follows that the set of functions

$$\{\phi_{j,k}\}_{k\in\mathbb{Z}} = \{2^{-j/2}\phi_0(2^{-j}t-k)\}_{k\in\mathbb{Z}}$$

is a Riesz basis for V_j.

With the MRA, the process $X(t)$ is successively projected into each of the approximation subspaces V_j

$$\text{approx}_j(t) = \sum_k a_X(j,k)\phi_{j,k}(t) \ \in V_j.$$

Since $V_j \subset V_{j-1}$, $\text{approx}_j(t)$ is a rougher approximation of $X(t)$ than $\text{approx}_{j-1}(t)$. That is, less information about $X(t)$ is contained in $\text{approx}_j(t)$. The idea is thus to study the process by examining its rougher approximations, cancelling more details (high frequencies) from the data.

The information that is lost when going from one approximation to a rougher one is called the *detail*

$$\text{detail}_j(t) = \text{approx}_{j-1}(t) - \text{approx}_j(t).$$

The details can be obtained directly by projecting $X(t)$ onto the collection of subspaces $W_j = V_{j-1} \setminus V_j$. MRA theory shows that there exists a function ψ_0, called the *mother wavelet*, derived from ϕ_0, such that the set of functions

$$\{\psi_{j,k}(t)\}_{k\in\mathbb{Z}} = \{2^{-j/2}\psi_0(2^{-j}t-k)\}_{k\in\mathbb{Z}}$$

is a Riesz basis for W_j. The details are then obtained as

$$\text{detail}_j(t) = \sum_k d_X(j,k)\psi_{j,k}(t).$$

In practical applications one considers some finite range for the octaves $j = 0,\ldots,J$. This means that the approximation of $X(t)$ on the subspace V_0 is obtained as the low-resolution approximation onto the smaller subspace V_J and the collection of details between 0 and J

$$\text{approx}_0(t) = \text{approx}_J(t) + \sum_{j=1}^{J}\text{detail}_j(t) = \sum_k a_X(J,k)\phi_{J,k}(t) + \sum_{j=1}^{J}\sum_k d_X(j,k)\psi_{j,k}(t).$$

Now, given a scaling function ϕ_0 and a mother wavelet ψ_0, the discrete wavelet transform consists of the collection of coefficients

$$X(t) \to \Big\{\{a_X(J,k)\}_{k\in\mathbb{Z}}, \{d_X(j,k)\}_{j=1,\ldots,J,k\in\mathbb{Z}}\Big\},$$

where the coefficients are defined by

$$a_X(j,k) = \langle X,\phi_{j,k}\rangle = \int_{\mathbb{R}} X(t)\phi_{j,k}(t)dt \quad \text{and} \quad d_X(j,k) = \langle X,\psi_{j,k}\rangle = \int_{\mathbb{R}} X(t)\psi_{j,k}(t)dt.$$

The mother wavelet ψ_0 has a number N of vanishing moments:

$$\int t^k\psi_0(t)dt = 0 \quad \text{for } k = 0,1,\ldots,N-1.$$

The value N can be chosen by selecting the appropriate mother wavelet, for instance the Daubechies wavelets indexed by $N = 1,2,\ldots,$ are often used in practice.

5.2.2 Wavelet transform of long range dependent processes

Using properties of wavelet coefficients it can be shown (see [AFTV]), that under some conditions, for a stationary increment process $X(t)$ which is H-self-similar,

$$\mathbf{E}\{d_X(j,k)^2\} \sim 2^{j(2H-1)} c_f C(H,\psi_0) \quad \text{as } j \to \infty. \tag{5.2}$$

Here c_f is a non-zero constant,

$$C(H,\psi_0) = \int |\nu|^{-(2H-1)} |\Psi_0(\nu)|^2 d\nu,$$

and $\Psi_0(\nu)$ is the Fourier transform of ψ_0. Now

$$y_j = \frac{1}{n_j} \sum_{k=1}^{n_j} |d_X(j,k)|^2$$

is an unbiased estimator of the variance of the process $d_X(j,t)$, where n_j is the number of coefficients at octave j.

By (5.2) an estimator $\hat{H}$ of H can be designed from a simple weighted linear regression of $\log_2 y_j$ on $j = j_1, \ldots, j_2$. Define

$$S = \sum 1/\sigma_j^2, \quad S' = \sum j/\sigma_j^2 \quad \text{and} \quad S'' = \sum j^2/\sigma_j^2,$$

where σ_j^2 is an arbitrary weight associated with $\log_2 y_j$. The unbiased estimator $\hat{H}$ of H is derived from

$$2\hat{H} - 1 = \frac{\sum \log_2 y_j (Sj - S')/\sigma_j^2}{SS'' - (S')^2}. \tag{5.3}$$

With the weigths $\sigma_j^2 = \mathbf{Var}\{y_j\}$ in (5.3), one gets the minimum variance unbiased estimator (MVUE) of the intercept and slope (see [AV2]). The minimum value of j_1 is 1, and the maximum value of j_2 is $\mathcal{J}$. In practice, one often chooses a smaller range between j_1 and j_2.

5.3 Simulation

To test the estimation methods we simulate FGN and LFSN. Two different simulation methods where tried out: The Hosking method and the Davies-Harte algorithm, see Appendix A. Both these simulation methods are exact, i.e. the covariance structure is not approximated in any way. When simulating FGN with $n = 2000$ the Hosking method took approximately twice as long time as the Davies-Harte algorithm. Hence, we used the Davies-Harte algorithm for the simulations of FGN. For the LFSN, the algorithm proposed by Stoev and Taqqu [SST] was used, see Appendix A.

5.4 Evaluation of the estimation methods

In this section an investigation is made to single out the best method for estimating the Hurst parameter from a time series. This is done by simulating 40 independent paths of fractional Gaussian noise and linear fractional stable noise for three different values of H. The length of the paths is $n = 5000$. LFSN is simulated for three different values of α and β. On the

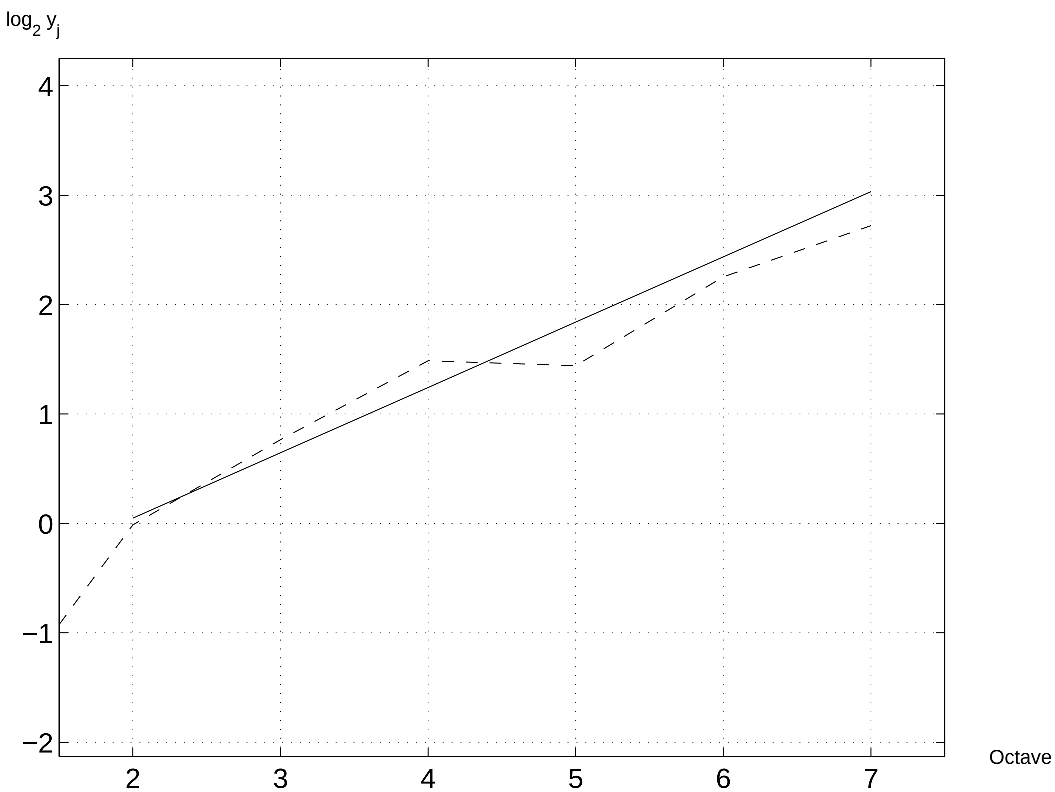

Figure 5.2: The wavelet method for FGN with Hurst parameter $H = 0.8$. The slope of the regression line is 0.597, i.e. $\hat{H} = \frac{0.597+1}{2} = 0.798$. Here $j_1 = 2$, $j_2 = 7$ and $N = 10$.

total 1200 simulations. The accuracy of the estimation methods is evaluated by calculating the empirical mean and standard deviation of the estimated H from the 40 independent simulations.

For the wavelet method the Daubechies wavelet with $N = 10$ vanishing moments was chosen, and with $j_1 = 2$ and $j_2 = 8$.

Both the $\mathcal{R}/\mathcal{S}$-analysis and Wavelet method as described above assumes processes with finite variance. However, in [TTW] it is suggested that most estimators of long range dependence is based on $H = \omega + \frac{1}{2}$ which in the stable case generalizes to $H = \omega + 1/\alpha$. This adjustion is made below when $\alpha \neq 2$. The Matlab code used for the wavelet estimation can be found at [CL].

		$\alpha = 2.0$		
		$H = 1/\alpha = 0.5$	$H = 0.6$	$H = 0.7$
	$\hat{H}$	0.545	0.628	0.703
	$\hat{\sigma}$	0.028	0.027	0.027
		$\alpha = 1.9$		
		$H = 1/\alpha = 0.526$	$H = 0.6$	$H = 0.7$
$\beta = -1$	$\hat{H}$	0.570	0.572	0.532
	$\hat{\sigma}$	0.031	0.036	0.035
$\beta = -0.5$	$\hat{H}$	0.570	0.614	0.628
	$\hat{\sigma}$	0.026	0.039	0.038
$\beta = 0$	$\hat{H}$	0.570	0.630	0.706
	$\hat{\sigma}$	0.026	0.026	0.035
		$\alpha = 1.6$		
		$H = 1/\alpha = 0.625$	$H = 0.7$	$H = 0.8$
$\beta = -1$	$\hat{H}$	0.662	0.545	0.465
	$\hat{\sigma}$	0.027	0.041	0.045
$\beta = -0.5$	$\hat{H}$	0.650	0.646	0.604
	$\hat{\sigma}$	0.026	0.035	0.041
$\beta = 0$	$\hat{H}$	0.660	0.717	0.795
	$\hat{\sigma}$	0.022	0.025	0.027
		$\alpha = 1.3$		
		$H = 1/\alpha = 0.769$	$H = 0.8$	$H = 0.9$
$\beta = -1$	$\hat{H}$	0.794	0.720	0.679
	$\hat{\sigma}$	0.016	0.038	0.052
$\beta = -0.5$	$\hat{H}$	0.781	0.781	0.773
	$\hat{\sigma}$	0.024	0.030	0.053
$\beta = 0$	$\hat{H}$	0.789	0.811	0.893
	$\hat{\sigma}$	0.021	0.022	0.031

Table 5.1: $\mathcal{R}/\mathcal{S}$ estimators of the Hurst parameter.

$\alpha = 2.0$				
		$H = 1/\alpha = 0.5$	$H = 0.6$	$H = 0.7$
	$\hat{H}$	0.496	0.611	0.717
	$\hat{\sigma}$	0.021	0.018	0.018
$\alpha = 1.9$				
		$H = 1/\alpha = 0.526$	$H = 0.6$	$H = 0.7$
$\beta = -1$	$\hat{H}$	0.526	0.585	0.690
	$\hat{\sigma}$	0.030	0.024	0.026
$\beta = -0.5$	$\hat{H}$	0.524	0.606	0.729
	$\hat{\sigma}$	0.028	0.030	0.043
$\beta = 0$	$\hat{H}$	0.524	0.617	0.713
	$\hat{\sigma}$	0.028	0.027	0.030
$\alpha = 1.6$				
		$H = 1/\alpha = 0.625$	$H = 0.7$	$H = 0.8$
$\beta = -1$	$\hat{H}$	0.624	0.700	0.803
	$\hat{\sigma}$	0.046	0.051	0.056
$\beta = -0.5$	$\hat{H}$	0.621	0.717	0.833
	$\hat{\sigma}$	0.039	0.060	0.075
$\beta = 0$	$\hat{H}$	0.624	0.710	0.804
	$\hat{\sigma}$	0.040	0.055	0.082
$\alpha = 1.3$				
		$H = 1/\alpha = 0.769$	$H = 0.8$	$H = 0.9$
$\beta = -1$	$\hat{H}$	0.767	0.794	0.885
	$\hat{\sigma}$	0.062	0.062	0.107
$\beta = -0.5$	$\hat{H}$	0.760	0.794	0.903
	$\hat{\sigma}$	0.056	0.078	0.078
$\beta = 0$	$\hat{H}$	0.762	0.786	0.897
	$\hat{\sigma}$	0.060	0.077	0.107

Table 5.2: Wavelet estimators of the Hurst parameter.

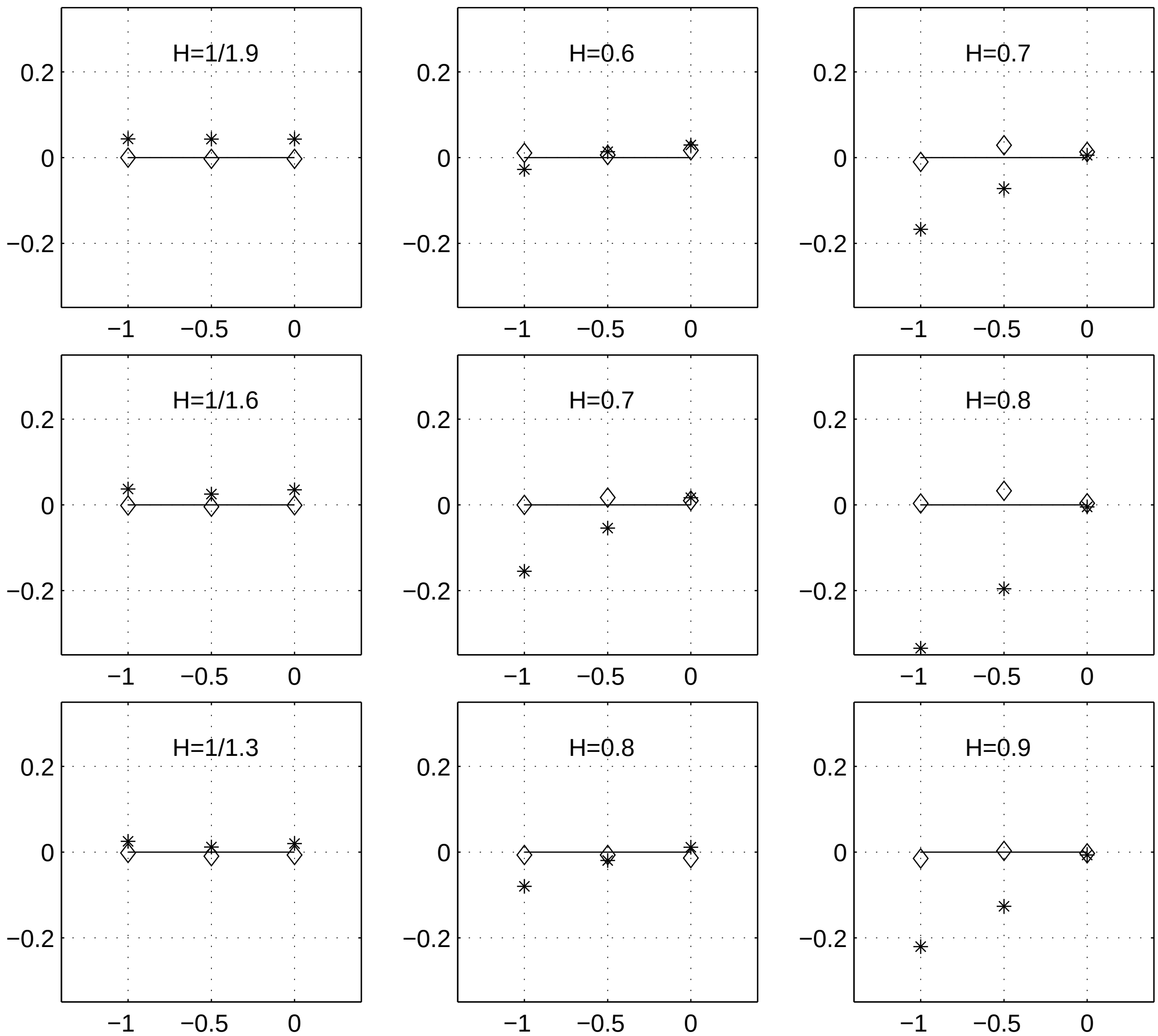

Figure 5.3: $(\hat{H} - H)$ plotted against β for $\alpha = 1.9$ (top), $\alpha = 1.6$ (middle) and $\alpha = 1.3$ (bottom). $\diamond$ = Wavelet estimator $\quad *$ = $\mathcal{R}/\mathcal{S}$ estimator.

5.5 Conclusions

It is clear that β is a critical parameter for the $\mathcal{R}/\mathcal{S}$ estimator. As the LFSN process becomes more negatively skewed, the $\mathcal{R}/\mathcal{S}$ analysis seems to systematically underestimate the Hurst parameter. This error also seems to increase with H. The wavelet estimator displays unbiasedness for all parameter values of the LFSN process but also a high variance.

Chapter 6

Fitting marginal distributions to dependent data

When fitting the marginal distribution to some observed stochastic process, the assumption that the observed data are independent simplifies the estimation of parameters. The usual 'casual' usage of the maximum likelihood method, for example, usually assumes independent observations. As does the theory that lies behind the maximum likelihood method.

Since the increments from fractional Brownian motion and linear fractional stable motion are dependent one makes a false assumption when treating them as independent. The intent of this section is to somewhat investigate how large the error of the estimated parameters become when employing methods that assume independence.

6.1 Estimating parameters for simulated data

The parameters for the normal and NIG distribution are estimated by the maximum likelihood method, i.e. maximizing the log likelihood function

$$\ell(\Theta) = \sum_{i=1}^{n} \log f(X_i|\Theta),$$

where $\Theta = (\mu, \sigma)$ and $(\alpha, \beta, \delta, \mu)$ respectively.

The lack of explicit expressions for probability density functions for stable distributions makes the parameter estimation in that case more problematic. However, Nolan [N] describes a numerical maximum likelihood method for stable distributions. The Mathematica package can be found at [W].

Our investigation is carried out by performing maximum likelihood estimation on 10 different simulated sample paths of FGN and LFSN of size $n = 5,000$. The error of the estimation of some parameter θ is calculated as the mean of $|\hat{\theta} - \theta|$. The investigation is restricted to FGN with Hurst parameter $H = 0.6$ and $H = 0.7$, symmetric ($\beta = 0$) LFSN with $\alpha = 1.6$, $H = 0.7$ and $H = 0.8$ and symmetric LFSN with $\alpha = 1.3$, $H = 0.8$ and $H = 0.9$. The estimated parameters are compared with the parameters estimated from their independent ($H = 1/\alpha$) counterparts.

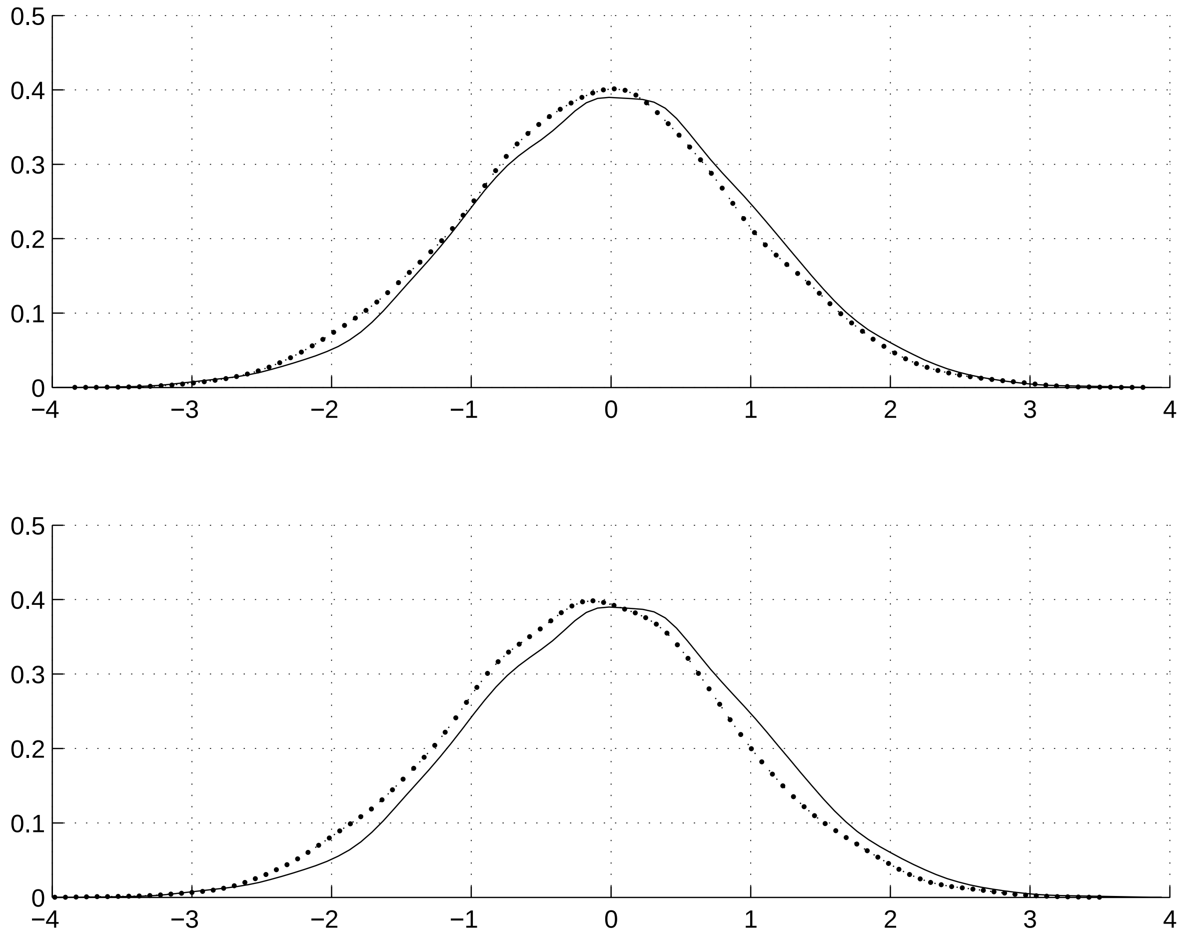

Figure 6.1: Empirical histogram from $5,000$ independent $S_{1.6}(1,0,0)$-distributed random variables (solid line), and simulated linear fractional stable noise $\{Y_{1.6,H}(j)\}_{j=1}^{5000}$ (dotted line) with Hurst parameter $H = 0.7$ (top) and $H = 0.8$ (bottom).

Parameter	$H = 0.5$	$H = 0.6$	$H = 0.7$
$\lvert\hat{\mu} - \mu\rvert$	0.0090	0.0323	0.0709
$\lvert\hat{\sigma} - \sigma\rvert$	0.0052	0.0096	0.0102

Table 6.1: Estimated parameters for fractional Gaussian noise

	$\alpha = 1.6$		
Parameter	$H = 1/\alpha = 0.625$	$H = 0.7$	$H = 0.8$
$\lvert\hat{\alpha} - \alpha\rvert$	0.0100	0.0180	0.0296
$\lvert\hat{\beta} - \beta\rvert$	0.0471	0.0497	0.0450
$\lvert\hat{\sigma} - \sigma\rvert$	0.0101	0.0110	0.0199
$\lvert\hat{\mu} - \mu\rvert$	0.0359	0.0562	0.0902
	$\alpha = 1.3$		
Parameter	$H = 1/\alpha = 0.769$	$H = 0.8$	$H = 0.9$
$\lvert\hat{\alpha} - \alpha\rvert$	0.0196	0.0167	0.0432
$\lvert\hat{\beta} - \beta\rvert$	0.0502	0.0227	0.0388
$\lvert\hat{\sigma} - \sigma\rvert$	0.0106	0.0168	0.0680
$\lvert\hat{\mu} - \mu\rvert$	0.0295	0.0637	0.1817

Table 6.2: Estimated parameters for linear fractional stable noise

6.2 Conclusions

For FGN, it seems that a large value of H may lead to an over- or underestimation of μ. Also, the dependence (i.e. $H > \frac{1}{2}$) does seem to produce larger error on the estimation of σ.

For LFSN all parameters except β display an increase of estimation errors when H increases. The errors also increas with heavier tails. As for FGN it is the estimation error $|\hat{\mu} - \mu|$ that displays the most pronounced increase with H, and for large H the error is quite large.

Notice in Figure 6.1 that the implied probability density function is shifted to the right compared with the empirically observed density when $H > 1/\alpha$.

Chapter 7

Empirical financial data

In this chapter the models that have been described in previous chapters, i.e. FBM (Wick-Itô solution), a stochastic process with copulas and NIG distributed marginals, and LFSM are tested against empirical data. The data sets consist of the closing prices $\{S_k\}_{0\leq k\leq n}$ of three financial indeces and three individual stock prices, that can all be found at [Y]. The data sets studied are:

- Standard & Poor 500 Index (S&P), 6 February 1950 to 22 April 2004 (13639 data);
- Nasdaq Bank Index (NBI), 5 February 1982 to 15 July 2004 (5655 data);
- Nasdaq Composite Index (NCI), 12 October 1984 to 2 August 2004 (4996 data);
- Coca Cola Co. (CC), 11 April 1988 to 16 July 2004 (4422 data);
- IBM Corp. (IBM), 21 September 1984 to 15 July 2004 (5000 data);
- Ford Motor Co. (Ford), 9 February 1987 to 16 July 2004 (4400 data).

We study the log returns $\{X_k\}_{1\leq k\leq n} = \{\log(S_k/S_{k-1})\}_{1\leq k\leq n}$. The returns are manupilated so that two exact equal values never occur. This is done by adding a small random number. This is for technical reasons, and of no practical importance.

7.1 Long range dependence

Since $\omega = H - 1/\alpha$ it quantifies the long range dependence in both the Gaussian and stable case. Thus, instead of the Hurst parameter H, we estimate ω for the log returns. We use the Wavelet method with Daubechies wavelet with $N = 10$ vanishing moments, $j_1 = 2$ and $j_2 = 8$.

To estimate the LRD modeled by the Gaussian copulas, the log returns are transformed to be Gaussian. Note that by the discussion above, this should not affect the value of ω. The confidence intervals are calculated using the usual methods for linear regression. Note that the they are calculated under the assumtion of lognormal returns [AV2]. In Chapter 5 we noticed that the variance of $\hat{H}$ increased with lower α and hence the confidence intervals should perhaps be larger under the assumption of stable marginals.

Asset	$\hat{\omega}$	P	$\hat{\omega}_{\text{Copula}}$	P
S&P	-0.009 $[-0.029, 0.012]$	0.05	-0.002 $[-0.023, 0.019]$	0.09
S&P July 1964 - June 1984	0.073 $[0.033, 0.112]$	0.28	0.069 $[0.030, 0.108]$	0.79
S&P July 1984 - June 2004	-0.074 $[-0.114, -0.035]$	0.63	-0.079 $[-0.118, -0.039]$	0.41
NBI	0.105 $[0.069, 0.141]$	0.64	0.115 $[0.079, 0.151]$	0.73
NCI	-0.028 $[-0.067, 0.012]$	0.54	-0.015 $[-0.054, 0.025]$	0.16
CC	0.023 $[-0.024, 0.069]$	0.27	0.009 $[-0.038, 0.055]$	0.25
IBM	-0.008 $[-0.048, 0.031]$	0.03	-0.008 $[-0.047, 0.032]$	0.04
Ford	0.001 $[-0.043, 0.044]$	0.04	-0.058 $[-0.102, -0.015]$	0.04

Table 7.1: Estimated ω for empirical financial data and for financial data that are transformed to be Gaussian. Also included are 95%- confidence intervals and P-values for the linear fit.

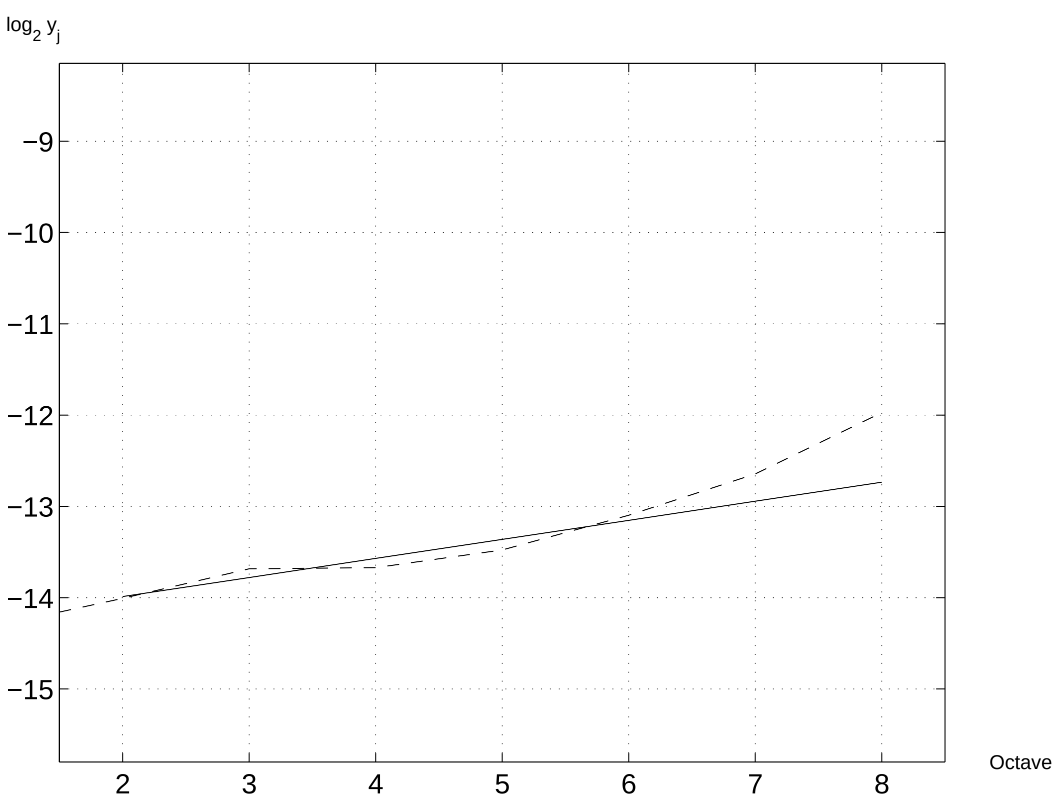

Figure 7.1: Wavelet based estimator of ω for Nasdaq Bank Index, $\hat{\omega} = 0.105$.

Asset	μ	σ
S&P	0.000402	0.01090
NBI	0.000530	0.00790
NCI	0.000409	0.0144
CC	0.000432	0.0220
IBM	0.000539	0.0194
Ford	0.000722	0.0226

Table 7.2: Estimated normal distribution parameters for financial data.

Asset	α	β	δ	μ
S&P 500	75.680	-3.80609	0.008362	0.00082278
NBI	76.562	-8.33385	0.0046431	0.00103864
NCI	37.2043	-6.44085	0.00764422	0.00175328
CC	38.1001	1.38101	0.0171161	-0.000188096
IBM	45.3864	2.30732	0.0161315	-0.00028208
Ford	48.9789	5.30753	0.0226033	-0.00174172

Table 7.3: Estimated NIG distribution parameters for financial data.

7.2 Marginal distributions

According to Chapter 6 we can treat the observations as independent under the assumption that $\omega = H - 1/\alpha$ is not too large. In such a case the estimated parameters should not deviate much from the true value. The parameters for Normal and NIG distributions are estimated by maximum likelihood. The parameters for the stable marginals are estimated by the maximum likelihood method described in [N].

For computational reasons the marginal distribution for the log returns of Standard & Poor Index is estimated for only the last $5,000$ data values.

From (2.4) the theoretical log returns for the Wick-Itô solution can be derived as

$$\log\left(\frac{S(k)}{S(k-1)}\right) = \sigma X_H(k) + \mu - \frac{\sigma^2}{2}\left(k^{2H} - (k-1)^{2H}\right).$$

Here $X_H(k)$ denotes fractional Gaussian noise. Thus, the increments of $\log S(t)$ are not stationary in this model. In fact, this process is not appropriate for larger time intervals since the log returns on average then becomes smaller and smaller. However, the data sets has been modified to be stationary by a recursive method and it turns out that the parameter estimations are not affected.

To estimate the fit of the estimated distribution function to the empirical we use the *Anderson & Darling* statistic

$$\mathrm{AD} = \max_{x\in\mathbb{R}} \frac{|\hat{F}(x) - F_{\mathrm{est}}(x)|}{\sqrt{F_{\mathrm{est}}(x)(1 - F_{\mathrm{est}}(x))}}.$$

Here F_{est} is the estimated cumulative distribution and $\hat{F}$ the empirical distribution function of the data set.

Asset	α	β	σ	μ
S&P 500	1.6546	-0.072565	0.00575222	0.00040659
NBI	1.487	-0.110	0.00364	0.000489
NCI	1.388	-0.196	0.00614	0.000279
CC	1.682	0.101	0.0116	0.000764
IBM	1.697	0.160	0.0102	0.000854
Ford	1.780	0.196	0.0129	0.000785

Table 7.4: Estimated stable distribution parameters for financial data.

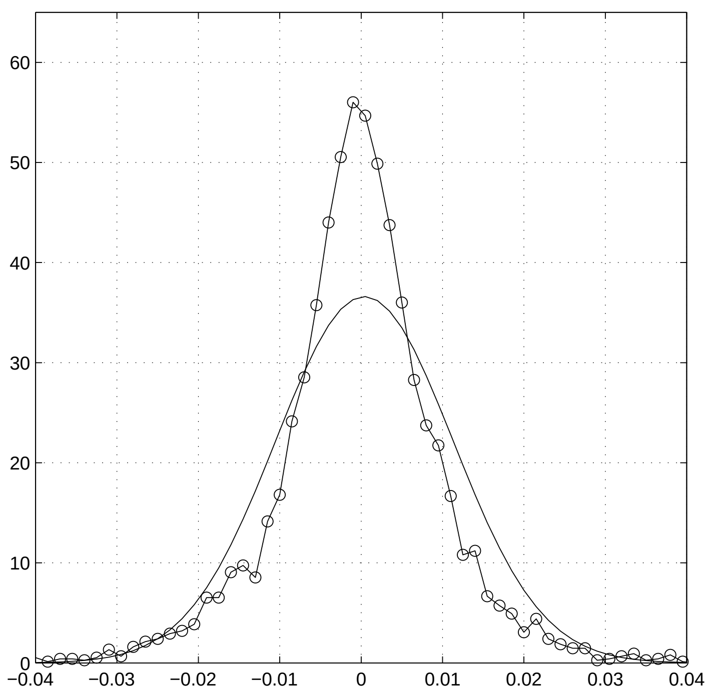

Figure 7.2: Fitted Normal distribution (solid line) and Standard & Poor Index (dotted line).

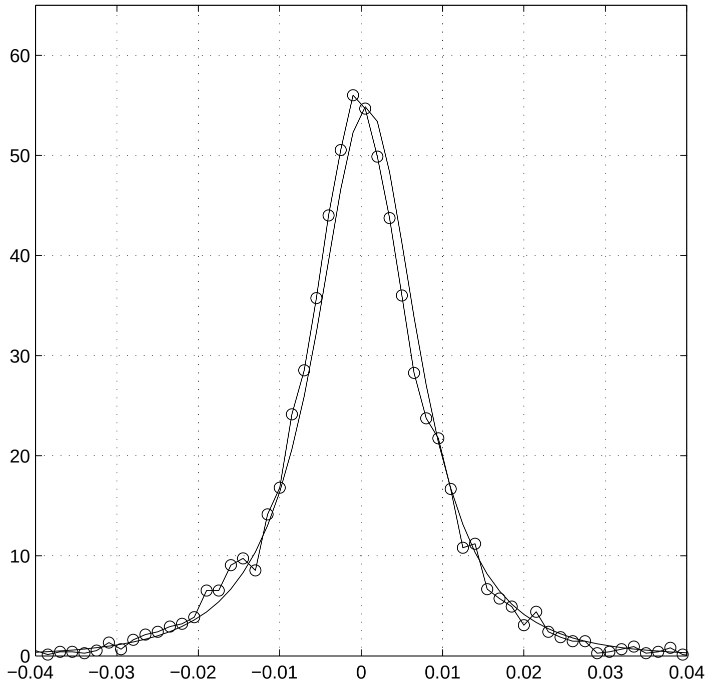

Figure 7.3: Fitted NIG distribution (solid line) and Standard & Poor Index (dotted line).

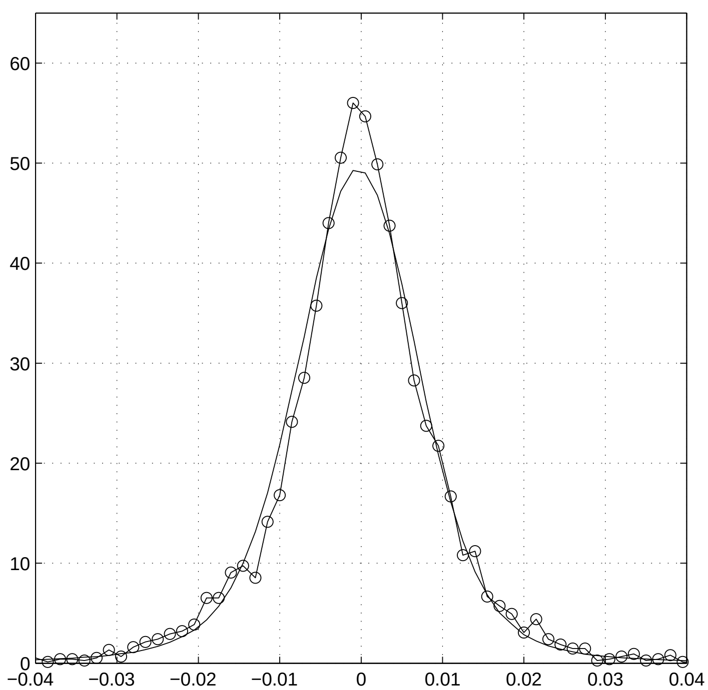

Figure 7.4: Fitted stable distribution (solid line) and Standard & Poor Index (dotted line).

Asset	Normal	NIG	Stable
S&P	9.8716×10^{44}	9.6582	0.0561
NBI	8.4890×10^{11}	0.0480945	0.0906
NCI	4.5432×10^{4}	0.0402288	0.1843
CC	3.5799×10^{29}	4.51911	0.18378
IBM	9.6808×10^{17}	1.09333	0.0735
Ford	7.8695×10^{5}	0.298411	0.1599

Table 7.5: The Anderson & Darling statistic.

For the case of stable marginal distribution, F_{est} is approximated by simulating $1,500,000$ independent stable random variables with the estimated parameters. The Anderson & Darling distance is a measure of fit that gives special importance to the tails of the distribution [BFO] [T].

7.3 Conclusions

The wavelet analysis shows that most assets considered here most likely lack long range dependence in their log returns. All estimators of ω but one includes 0 in their confidence intervals.

The exception is the Nasdaq Bank Index for which the estimated ω has a confidence interval significantly far from 0, although this interval should be considered with caution. The estimated ω in this case has a high probability of the linear fit, which also can be seen from Figure 7.1.

In addition, the Standard & Poor 500 Index from 1950 to 2004 displays LRD over one 20-year period, but short range dependence ($\omega < 0$) over the next 20-year period. Over the full period it has an ω very close to 0. This may be due to changing properties of the underlying dynamics over time, but might as well be due to the significant variance of the wavelet estimator. To transforme the log returns to be Gaussian should not change the value of ω and it is also noticed that the estimators does not change dramatically.

For the marginal distributions the Anderson & Darling statistic illustrates the normal distributions incapability of capturing the extreme events on the financial markets. The difference between the stable and NIG distributions is quite small, eventhough the stable distribution performs more evenly. The stable distribution also has the economically appealing feature as the limit distribution of many small effects on the price of an asset.

If one wants to model risky assets where the returns have LRD, the LFSM process has the nice property that it can be expressed explicitly, which is not case for the copula process presented here.

It is obvious that the assumption of totally negatively skewed stable log returns by Carr and Wu is much too restrictive: For our data sets, as many assets display negative skewness, as display positive. In addition, the skewness parameters is closer to 0 than to 1 for all assets.

Appendix A

Simulation

A.1 Hosking method

By the Hosking method a stationary Gaussian time series $\{X_i\}_{i=1,\dots,n}$ with covariance function ρ[5] is generated by the following scheme [TTW]:

- generate n independent random variables $\epsilon_i \sim N(0,1)$ for $i = 1,\dots,n$;
- set $X_1 = \sqrt{\rho(0)}\epsilon_1$;
- set $X_{i+1} = \phi_{i,1}X_i + \cdots + \phi_{i,i}X_1 + \sigma_i\epsilon_{i+1}$ for $i = 1,\dots,n-1$.

The variances σ_i^2 and the coefficients $\{\phi_{i,j}\}_{i=1,\dots,n-1,j=1,\dots,i}$ are computed recursively by:

- set $\sigma_0 = \sqrt{\rho(0)}$ and compute $\sigma_i^2 = \sigma_{i-1}^2(1-\phi_{i,i})^2$ for $i = 1,\dots,n-1$;
- compute $\phi_{i,i} = \big(\rho(i) - \sum_{k=1}^{i-1}\phi_{i-1,k}\rho(i-k)\big)\sigma_{i-1}^{-2}$ for $i = 1,\dots,n-1$;
- compute $\phi_{i,j} = \phi_{i-1,j} - \phi_{i,i}\phi_{i-1,i-j}$ for $j < i$.

A.2 Davies-Harte algorithm

The Davies-Harte algorithm uses the discrete Fourier transform (DFT) of the covariance function to generate a stationary Gaussian process [CR]. With $i = \sqrt{-1}$ denoting the imaginary unit, a time series $\{X_t\}_{t=0,\dots,n-1}$ with covariance function ρ is generated in the following way:

- generate $2n$ independent random variables $\epsilon_j \sim N(0,1)$, $j = 0,\dots,2n-1$;
- compute

$$A_{k,n}(\rho) = \sum_{j=0}^{n}\rho(j)\mathrm{e}^{-i\pi kj/n} + \sum_{j=n+1}^{2n-1}\rho(2n-j)\mathrm{e}^{-i\pi kj/n} \quad \text{for } k = 0,\dots,2n-1;$$

- check that $A_{k,n}(\rho) \geq 0 \;\; \forall\, k$;

[5]When simulating FGN ρ is given by (2.1).

- compute

$$Y_k = \begin{cases} \sqrt{2nA_{0,n}(\rho)}\epsilon_0 & \text{when} \quad k=0 \\ \sqrt{nA_{k,n}(\rho)}(\epsilon_{2k-1}+i\epsilon_{2k}) & \text{when} \quad 1 \le k \le n-1 \\ \sqrt{2nA_{n,n}(\rho)}\epsilon_{2n-1} & \text{when} \quad k=n \\ \sqrt{nA_{k,n}(\rho)}(\epsilon_{4n-1-2k}-i\epsilon_{4n-2k}) & \text{when} \quad n+1 \le k \le 2n-1 \end{cases};$$

- the simulated Gaussian is obtained as

$$X_t = \frac{1}{2n}\sum_{k=0}^{2n-1} Y_k e^{i\pi kt/n} \quad \text{for } t=0,\dots,n-1.$$

A.3 Stoev and Taqqu algorithm for LFSM

In the stable infinite variance case exact simulation techniques for the LFSM process $L_{\alpha,H}$ are not known. Stoev and Taqqu [SST] suggests an algorithm based on, as Davies-Harte, the DFT. The LFSN process in (4.6) is approximated by a Riemann sum

$$Y_{m,M}(k) = \sum_{j=1}^{mM}\Big(\big((j/m)^+\big)^{H-1/\alpha} - \big((j/m-1)^+\big)^{H-1/\alpha}\Big) Z_{\alpha,\beta,m}(mk-j) \quad \text{for } m, M \in \mathbb{N}, \tag{A.1}$$

where

$$Z_{\alpha,\beta,m}(j) = L((j+1)/m) - L(j/m) \quad \text{for } j \in \mathbb{Z}.$$

Here L is an α-stable Lévy motion with skewness intensity β. The random variables $Z_{\alpha,\beta,m}(j)$ are iid. with $Z_{\alpha,m}(j) \sim S_\alpha(m^{-1/\alpha},\beta,0)$.

Let

$$W(n) = \sum_{j=1}^{mM} a_{H,m}(j) Z_{\alpha,\beta}(n-j) \quad \text{for } n \in \mathbb{N}, \tag{A.2}$$

where

$$a_{H,m}(j) = \Big(\big((j/m)^+\big)^{H-1/\alpha} - \big((j/m-1)^+\big)^{H-1/\alpha}\Big) m^{-1/\alpha} \quad \text{for } j \in \mathbb{N},$$

and where the random variables $\{Z_{\alpha,\beta}(j)\}_{j\in\mathbb{Z}}$, are iid. with $Z_{\alpha,\beta}(j) \sim S_\alpha(1,\beta,0)$. Since

$$\{Z_{\alpha,\beta,m}(j)\}_{j\in\mathbb{Z}} =_d \{m^{-1/\alpha} Z_{\alpha,\beta}(j)\}_{j\in\mathbb{Z}},$$

(A.1) and (A.2) imply that

$$\{Y_{m,M}(k)\}_{k=1,\dots,N} =_d \{W(mk)\}_{k=1,\dots,N}.$$

The process $\{W(n)\}_{n=1,\dots,nM}$, can be computed efficiently using the DFT: Let

$$\tilde{a}_{H,m}(j) = \begin{cases} a_{H,m}(j) & \text{when} \quad j=1,\dots,mM \\ 0 & \text{when} \quad j=mM+1,\dots,m(M+N) \end{cases}.$$

Then

$$\{W(n)\}_{n=1}^{mM} =_d \left\{ \sum_{j=1}^{m(M+N)} \tilde{a}_{H,m}(j) Z_{\alpha,\beta}(n-j) \right\}_{n=1}^{mM}.$$

The discrete Fourier transform is defined as

$$\hat{a}(k) = \mathcal{D}_R(a)(k) = \sum_{j=0}^{R-1} \mathrm{e}^{2\pi ijk/R} a(j) \quad \text{for } \in \mathbb{Z}.$$

Thus the sequence $\hat{a}$ is R-periodic and it satisfies the inversion formula

$$a(j) = \mathcal{D}_R^{-1}(\hat{a})(j) = \frac{1}{R} \sum_{k=0}^{R-1} \mathrm{e}^{-2\pi ijk/R} \hat{a}(k) \quad \text{for } j \in \mathbb{Z}.$$

Furthermore, for any two R-periodic sequences a and b, we have the following convolution theorem

$$\mathcal{D}_R(a)(k)\mathcal{D}_R(b)(k) = \mathcal{D}_R(a * b)(k) \quad \text{for } k \in \mathbb{Z},$$

where

$$(a * b)(n) = \sum_{j=0}^{R-1} a(n-j)b(j) \quad \text{for } n \in \mathbb{Z}.$$

If R is an integer power of two, then the DFT can be computed efficiently by using the Fast Fourier Transform (FFT) algorithm (Matlab built-in function 'fft.m').

The algorithm:

- pick large enough integers m and M , so that $m(M+N)$ is an integer power of two;
- using the FFT algorithm, compute the DFT

  $$\hat{a}(k) = \mathcal{D}_{m(M+N)}(\tilde{a}_{H,m})(k) \quad \text{for } k = 0, \ldots, m(M+N)-1$$

  of the $m(M+N)$-periodic sequence $\tilde{a}_{H,m}$;
- generate $m(M+N)$ iid. $S_\alpha(1,\beta,0)$ distributed random variables $Z(j),\ j = 1, \ldots, m(M+N)$ and, by using the FFT algorithm, compute

  $$\hat{Z}(k) = \mathcal{D}_{m(M+N)}(Z)(k) \quad \text{for } k = 0, \ldots, m(M+N)-1;$$

- using the FFT algorithm, compute the inverse DFT of the sequence $\hat{a}(k)\hat{Z}(k),\ k = 0, \ldots, m(N+M)-1$, and keep only

  $$W(n+1) = \mathcal{D}_{m(M+N)}^{-1}(\hat{a}\hat{Z})(n) \quad \text{for } n = 0, \ldots, mN-1$$

  and where $(\hat{a}\hat{Z})(k) = \hat{a}(k)\hat{Z}(k)$ for $k = 0, 1, \ldots, m(M+N)-1$;
- set

  $$Y_{m,M}(k) = W(mk) \quad \text{for } k = 1, \ldots, N$$

  and let

  $$X(n) = \sum_{k=1}^{n} Y_{m,M}(k) \quad \text{for } n = 1, \ldots, N.$$

Here $Y_{m,M}(k),\ k = 1, \ldots, N$ is the approximation of a LFSN process $Y_{\alpha,H}$, and the sequence $X(n),\ n = 1, \ldots, N$ is the desired approximate path of the LFSM process $L_{\alpha,H}$.

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