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On Simulating SDEs by Transformation

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Abstract

In this thesis we briefly discuss simulation techniques for stochastic differential equations. The method of transforming stochastic differential equations with non-Lipschitz coefficients onto a new stochastic differential equation which is easier to simulate will be discussed and its scope will be analyzed.

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Contents

1 Introduction

This thesis is mainly concerned with exploring the scope of the Euler-Maruyama method and the so-called stochastic theta method. For an autonomous stochastic differential equation (SDE)

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$$

where $\{B_t\}$ represents Brownian motions with globally Lipschitz drift and diffusion coefficients the Euler-Maruyama scheme is strongly convergent (see e.g. [8]).

A strategy for simulating a solution to a SDE with non-Lipschitz coefficients which is explored in this thesis is to apply the Itô lemma to it with a bijective transformation $f \in C^2$ to obtain a new transformed SDE $Y_t = f(X_t)$. Which may be easier to simulate with the Euler-Maruyama scheme or some of its variants and then transform the simulated solution back onto the original SDE using the inverse transformation f^{-1}.

2 Numerical solutions to SDEs

A (one dimensional autonomous) stochastic differential equation (SDE) is an equation on the form

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \qquad X_0 = \zeta,$$

for $t \in [0,T]$, where μ and σ are the measurable coefficient functions, $\{B_t\}_{t\geq 0}$ is Brownian motion and ζ is a random variable that is independent of the Brownian motions B. A solution to the preceding SDE is any process $\{X_t\}_{t\in[0,T]}$ that satisfies

$$X_t = \zeta + \int_0^t \mu(X_s)ds + \int_0^t \sigma(X_s)dB_s,$$

for $t \in [0,T]$. The theory of SDEs guarantees the existence of solutions to a specific class of SDEs. However, although we may know that such a process exists it might not be trivial to find an analytical expression for it. This is where numerical methods for solving SDEs come in. If we know that a certain SDE has a solution we can try to use numerical methods to simulate the solution. All equations that feature in this thesis have well-defined and unique so called-strong solutions, since the conditions that ensure convergence of simulation schemes will also ensure the existence of unique strong solutions. For more information see e.g. [7] and [8]. Before considering a family of Euler schemes we introduce and discuss the Lipschitz condition.

2.1 The Lipschitz condition

A function $f : I \to \mathbb{R}$, where $I \subset \mathbb{R}$, is *globally Lipschitz* if there exists some $C > 0$ such that

$$|f(x) - f(y)| \leq C|x - y|$$

for all $x, y \in I$. If the Lipschitz condition holds for all compact subsets of I then f is *locally Lipschitz*. Finally if $f : I \to \mathbb{R}$ satisfies

$$(x - y)(f(x) - f(y)) \leq C|x - y|^2$$

then it called *one-sided Lipschitz*. The following lemma is presented as an exercise in [3], and its proof is given here for the convenience of the reader:

Lemma 2.1. *A function $f : \mathbb{R} \to \mathbb{R}$ is globally Lipschitz with Lipschitz constant M if and only if f is absolutely continuous and $|f'| \leq M$ almost everywhere.*

Proof. Suppose that f is Lipschitz with Lipschitz constant $M > 0$, i.e. that $|f(x) - f(y)| \leq M|x - y|$ for all $x, y \in \mathbb{R}$. Then f is absolutely continuous since if $\varepsilon > 0$ and $\{(a_i, b_i)\}_{i\geq 1}$ is any sequence of disjoint intervals that satisfies $\sum_{i\geq 1} |b_i - a_i| < \varepsilon/M$ then

$$\sum_{i\geq 1} |f(b_i) - f(a_i)| \leq \sum_{i\geq 1} M|b_i - a_i| < \varepsilon.$$

Since f is absolutely continuous its derivative f' is defined almost everywhere. But the Lipschitz condition gives us that

$$\frac{|f(x)-f(y)|}{|x-y|} \leq M \qquad \text{for all } x, y \in \mathbb{R},$$

so taking limits yields $|f'| \leq M$ almost everywhere. Now suppose conversely that f is absolutely continuous and that $|f'| \leq M$ almost everywhere. Then by absolute continuity $f(x) = f(x_0) + \int_{x_0}^{x} f'(t)dt$, so

$$|f(x)-f(y)| = \left| \int_{x \wedge y}^{x \vee y} f'(t)dt \right| \leq \int_{x \wedge y}^{x \vee y} |f'(t)|dt \leq M|x-y|.$$

□

Now as we are also going to consider one-sided Lipschitz functions, it would nice have some sort of a result that describes the character of one-sided Lipschitz functions. Notice first that a function can be one-sided Lipschitz without being absolutely continuous. This can be seen by noting that any decreasing function is one-sided Lipschitz, which is due the left hand side of the one-sided Lipschitz inequality being negative in that case. So a decreasing function that is not absolutely continuous is one-sided Lipschitz.

Lemma 2.2. *Suppose that $f : \mathbb{R} \to \mathbb{R}$ is an absolutely continuous function, then f is one-sided Lipschitz with the constant M if and only if $f' \leq M$ almost everywhere.*

Proof. If f is absolutely continuous and one-sided Lipschitz with the Lipschitz constant $M > 0$ then by absolute continuity f' exists almost everywhere. If $x \neq y$ we may divide through the one-sided Lipschitz continuity inequality for f with $|x-y|^2$ to obtain the inequality

$$\frac{f(x)-f(y)}{x-y} \leq M.$$

So taking limits yields $f' \leq M$ almost everywhere. If on the other hand f is absolutely continuous and $f' \leq M$ almost everywhere then by absolute continuity $f(x) = f(x_0) + \int_{x_0}^{x} f'(t)dt$, so if sgn denotes the sign function

$$\begin{aligned}
(x-y)(f(x)-f(y)) &= (x-y)\,\mathrm{sgn}(x-y)\left(\int_{x \wedge y}^{x \vee y} f'(t)dt\right) \\
&\leq (x-y)\,\mathrm{sgn}(x-y) M|x-y| \\
&= M|x-y|^2.
\end{aligned}$$

□

2.2 A family of Euler schemes

Suppose we have an autonomous SDE:

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t, \qquad X_0 = \zeta,$$

where $\{B_t\}_{t\geq 0}$ is the Brownian motion. The stochastic theta method computes approximates $Z_k \approx X_{\Delta k}$ starting at $Z_0 = \zeta$:

$$Z_{n+1} = Z_n + (1-\theta)\Delta\mu(Z_n) + \theta\Delta\mu(Z_{n+1}) + \sigma(Z_n)\Delta B_n, \tag{1}$$

for $n = 1, \ldots, N$ where $0 = t_0 < t_1 < \cdots < t_N$ are equidistant with a constant step size $\Delta > 0$, $\Delta B_n = B_{t_{n+1}} - B_{t_n}$ and $\theta \in [0, 1]$. If $\theta = 0$ we get the *Euler-Maruyama* method and if $\theta = 1$ we get the so-called *backward Euler* scheme. Another variant of the Euler method is the *split step backward Euler* scheme which is given by the rule

$$\begin{aligned} Z_n^* &= Z_n + \Delta\mu(Z_n^*) \\ Z_{n+1} &= Z_n^* + \sigma(Z_n^*)\Delta B_n. \end{aligned}$$

For the sake of completeness we present the following result, which is taken from the literature, the reader is referred to Theorem 10.2.2 in [8] for the proof of the result.

Theorem 2.3. *Consider the Euler-Maruyama ($\theta = 0$) approximation scheme (1). Suppose that*

$$E(|X_0|^2) < \infty,$$

$$E\left(|X_0 - Z_0|^2\right)^{1/2} \leq K\Delta^{1/2}$$

and that μ and σ are globally Lipschitz, then there exists a constant C such that

$$E\left(|X_T - Z_T|\right) \leq C\Delta^{1/2}.$$

A version of this result for the split step backward Euler method applied on a SDE with one-sided Lipschitz drift coefficient and a globally Lipschitz diffusion coefficient exists, the reader is referred to Theorem 3.3. in [5] for the statement and proof of that result.

3 Transformed processes

SDEs with ill behaved (non-Lipschitz) coefficients are more challenging to simulate than SDEs with well behaved (Lipschitz) coefficients. For such ill behaved SDEs the Euler scheme may diverge. One way to get around this problem is to transform the ill behaved SDE to obtain a new SDE which in turn is hopefully easier to simulate, simulate the new SDE and then transform it back using the inverse transformation. By an SDE that is easier to simulate the most usual criterion is that it has globally Lipschitz coefficients, or if that fails then a SDE that has a globally Lipschitz diffusion coefficient and a one-sided Lipschitz drift coefficient.

3.1 The Itô lemma

Consider a SDE, which is a stochastic process $\{X_t\}$ which can be written as the sum of a Lebesgue integral and an integral with respect to Brownian motion

$$X_t = \zeta + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s.$$

For such an SDE the Itô lemma (in differential form) states that for a function f that is twice continuously differentiable (which will be denoted by $f \in C^2$ henceforth)

$$df(X_t) = \left(f'(X_t)\mu_t + \frac{1}{2} f''(X_t)\sigma_t^2 \right) dt + f'(X_t)\sigma_t dB_t.$$

In what follows we will apply the Itô lemma on a SDE and investigate whether or not we can get Lipschitz coefficients.

3.2 Transforming SDEs

Let's consider transforming a SDE

$$dX_t = \mu(X_t)dt + \sigma(X_t)dB_t \qquad X_0 = \zeta.$$

By transforming we mean that we consider the process $Y_t = f(X_t)$ where $f \in C^2$ is a bijective function. By the Itô lemma

$$dY_t = \left(g(Y_t)\mu_f(Y_t) + \frac{1}{2} g(Y_t)g'(Y_t)\sigma_f^2(Y_t) \right) dt + g(Y_t)\sigma_f(Y_t)dB_t \qquad Y_0 = f(\zeta),$$

where $g(y) = f'(f^{-1}(y))$, $\mu_f(y) = \mu(f^{-1}(y))$ and $\sigma_f(y) = \sigma(f^{-1}(y))$. Now what do we want to achieve by this transformation? First of all it would be interesting to see when we can find a transformation f that gives us globally Lipschitz drift and diffusion coefficients, so that the transformed process may be simulated using the Euler-Maruyama method. Secondly it is interesting to see when we can obtain a one-sided Lipschitz drift coefficient and a globally Lipschitz diffusion coefficient which may in turn make the split step backward Euler method appealing for simulating the transformed process.

3.2.1 The diffusion coefficient

Now let's suppose that this new SDE has a globally Lipschitz diffusion coefficient $b(y) = g(y)\sigma_f(y)$ and try to infer what that means in terms of the transformation f. Since b is Lipschitz it must by Lemma 2.1 have an essentially bounded derivative, in other words we can equate the derivative of b with an essentially bounded function:

$$b'(y) = \frac{f''(f^{-1}(y))}{f'(f^{-1}(y))}\sigma(f^{-1}(y)) + \sigma'(f^{-1}(y)) = \varphi(f^{-1}(y)),$$

where $\varphi \circ f^{-1}$ is an essentially bounded function. Assuming that σ is non-zero almost everywhere we may rewrite this as

$$\frac{f''(x)}{f'(x)} = \frac{\varphi(x) - \sigma'(x)}{\sigma(x)},$$

and since f' is either strictly positive or strictly negative f''/f' equals either the derivative of $\log f'$ if $f' > 0$ or the derivative of $\log(-f')$ if $f' < 0$. Using this fact and assuming that σ is non-zero almost everywhere and solving for f yields

$$f(x) = C \int_{x_0}^{x} \exp\left(\int_{t_0}^{t} \frac{\varphi(s) - \sigma'(s)}{\sigma(s)} ds\right) dt \tag{2}$$

almost everywhere, where $C \in \mathbb{R}$ is a non-zero constant. If σ is strictly positive then f can be written as

$$f(x) = C \int_{x_0}^{x} \frac{k(t)}{\sigma(t)} dt \tag{3}$$

almost everywhere, where

$$k(x) = \exp\left(\int_{x_0}^{x} \varphi(t)/\sigma(t) dt\right).$$

Thus our choice of φ affects the slope of f, that is if it grows fast or slow.

3.2.2 The drift coefficient

Assuming that f has the form (2) which makes the diffusion coefficient globally Lipschitz, the drift coefficient $a(y) = g(y)\mu_f(y) + \frac{1}{2}g(y)g'(y)\sigma_f^2(y)$ takes the form

$$\begin{aligned} a(y) &= g(y)\mu_f(y) + \frac{1}{2}f''(f^{-1}(y))\sigma^2(f^{-1}(y)) \\ &= C\exp\left(\int_{x_0}^{f^{-1}(y)} \frac{\varphi(s) - \sigma'(s)}{\sigma(s)} ds\right)\left(\mu_f(y) + \frac{1}{2}\left(\varphi(f^{-1}(y)) - \sigma'(f^{-1}(y))\right)\sigma(f^{-1}(y))\right). \end{aligned}$$

Now if furthermore we assume that σ is strictly positive then we may use (3) to obtain

$$\begin{aligned} a(y) &= g(y)\mu_f(y) + \frac{1}{2}f''(f^{-1}(y))\sigma^2(f^{-1}(y)) \\ &= Ck(f^{-1}(y))\left(\frac{\mu_f(y)}{\sigma_f(y)} + \frac{1}{2}\left(\varphi(f^{-1}(y)) - \sigma'(f^{-1}(y))\right)\right). \end{aligned}$$

Now the question is when we can select φ that makes the drift coefficient a either globally Lipschitz or one-sided Lipschitz. Differentiating a yields

$$\begin{aligned} a'(y) &= C\Bigg(\varphi(f^{-1}(y))\left(\frac{\mu_f(y)}{\sigma_f(y)} + \frac{1}{2}\left(\varphi(f^{-1}(y)) - \sigma'(f^{-1}(y))\right)\right) \\ &\quad + \mu'(f^{-1}(y)) - \frac{\mu_f(y)\sigma'(f^{-1}(y))}{\sigma_f(y)} \\ &\quad + \frac{\sigma_f(y)}{2}\left(\varphi'(f^{-1}(y)) - \sigma''(f^{-1}(y))\right)\Bigg). \end{aligned}$$

Now if this derivative is bounded and a is absolutely continuous then by Lemma 2.1 a is globally Lipschitz. Likewise if this derivative is bounded from above then a is one-sided Lipschitz by Lemma 2.2. In what remains of of this section we will present our results, regarding when there exists a transformation that gives us globally Lipschitz drift and diffusion coefficients on the one hand and one-sided Lipschitz drift and globally Lipschitz diffusion on the other hand.

3.3 General processes

The following result is the most general result of this thesis.

Proposition 3.1. *A SDE $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$ where $\mu \in C^1$ is absolutely continuous and $\sigma \in C^2$ is absolutely continuous with an absolutely continuous derivative σ' can be transformed onto a new SDE $dY_t = a(Y_t)dt + b(Y_t)dB_t$ with globally Lipschitz drift and diffusion coefficients if and only if there exists some essentially bounded ψ such that the function*

$$x \mapsto \exp\left(-\int_{x_0}^{x} \frac{2\mu(t)}{\sigma^2(t)} - \frac{\sigma'(t)}{\sigma(t)}dt\right)\int_{x_0}^{x} q(t)\exp\left(\int_{t_0}^{t} \frac{2\mu(s)}{\sigma^2(s)} - \frac{\sigma'(s)}{\sigma(s)}ds\right)dt$$

is essentially bounded, where

$$q(x) = \frac{2\mu(x)\sigma'(x)}{\sigma^2(x)} + \frac{2(\psi(x) - \mu'(x))}{\sigma(x)} + \sigma''(x).$$

If furthermore $\sigma > 0$ then we may rewrite this function in the following way

$$x \mapsto \sigma(x)\exp\left(-\int_{x_0}^{x} \frac{2\mu(t)}{\sigma^2(t)}dt\right)\int_{x_0}^{x} \frac{q(t)}{\sigma(t)}\exp\left(\int_{t_0}^{t} \frac{2\mu(s)}{\sigma^2(s)}ds\right)dt,$$

where we have the same q as before.

Proof. Let us assume that such a transformation $f \in C^2$ exists, i.e. that there exists some essentially bounded φ such that f is like in equation (2) that transforms onto $dY_t = a(Y_t)dt + b(Y_t)dB_t$, where as before

$$a(y) = C\exp\left(\int_{x_0}^{f^{-1}(y)} \frac{\varphi(s)-\sigma'(s)}{\sigma(s)}ds\right)\left(\mu_f(y) + \frac{1}{2}\left(\varphi(f^{-1}(y)) - \sigma'(f^{-1}(y))\right)\sigma(f^{-1}(y))\right)$$

and

$$b(y) = C\exp\left(\int_{x_0}^{f^{-1}(y)} \frac{\varphi(s)-\sigma'(s)}{\sigma(s)}ds\right)\sigma(f^{-1}(y)).$$

Now b should be absolutely continuous and $b'(y) = \varphi(f^{-1}(y))$ must be essentially bounded by Lemma 2.1. By that same Lemma we must have that a is absolutely continuous and that a' is essentially bounded, but

$$\begin{aligned} a'(y) \quad = \quad & C\left(\varphi(f^{-1}(y))\left(\frac{\mu_f(y)}{\sigma_f(y)} + \frac{1}{2}\left(\varphi(f^{-1}(y)) - \sigma'(f^{-1}(y))\right)\right)\right. \\ & + \mu'(f^{-1}(y)) - \frac{\mu_f(y)\sigma'(f^{-1}(y))}{\sigma_f(y)} \\ & \left. + \frac{\sigma_f(y)}{2}\left(\varphi'(f^{-1}(y)) - \sigma''(f^{-1}(y))\right)\right). \end{aligned}$$

Which in turn means that the function

$$\varphi\left(\frac{\mu}{\sigma} + \frac{1}{2}(\varphi - \sigma')\right) + \mu' - \frac{\mu\sigma'}{\sigma} + \frac{\sigma}{2}(\varphi' - \sigma'')$$

is essentially bounded. Now since we know that φ is essentially bounded we may exclude the term $\frac{1}{2}\varphi^2$ from the previous equation and it will still have to be essentially bounded. So let us write

$$\varphi\left(\frac{\mu}{\sigma} - \frac{1}{2}\sigma'\right) + \mu' - \frac{\mu\sigma'}{\sigma} + \frac{\sigma}{2}(\varphi' - \sigma'') = \psi$$

where ψ is some essentially bounded function. But this can rewritten as

$$\varphi' + \left(\frac{2\mu}{\sigma^2} - \frac{\sigma'}{\sigma}\right)\varphi = \left(\frac{2\mu\sigma'}{\sigma^2} + \frac{2(\psi - \mu')}{\sigma} + \sigma''\right).$$

This first degree ordinary differential equation can be solved using an integrating factor: It has the solution

$$\varphi(x) = \exp\left(-\int_{x_0}^{x} \frac{2\mu(t)}{\sigma^2(t)} - \frac{\sigma'(t)}{\sigma(t)}dt\right)\int_{x_0}^{x} q(t)\exp\left(\int_{t_0}^{t} \frac{2\mu(s)}{\sigma^2(s)} - \frac{\sigma'(s)}{\sigma(s)}ds\right)dt \qquad (4)$$

where

$$q(x) = \frac{2\mu(x)\sigma'(x)}{\sigma^2(x)} + \frac{2(\psi(x) - \mu'(x))}{\sigma(x)} + \sigma''(x).$$

So if the right hand side of equation (4) were unbounded for all essentially bounded ψ it would contradict our selection of a essentially bounded φ. Therefore we may conclude that it is essentially bounded. If on the other hand there exists some essentially bounded ψ such that the right hand side of equation (4) is essentially bounded then for an f as in equation (2) with φ equal to the right hand side of equation (4) we get globally Lipschitz a and b. □

If in stead of having an essentially bounded ψ in the preceding proposition, we have a ψ that is bounded above, i.e. $\psi < M$ for some $M \in \mathbb{R}$. Then the drift coefficient a would be one-sided Lipschitz, so we get the following result.

Proposition 3.2. *A SDE $dX_t = \mu(X_t)dt + \sigma(X_t)dB_t$ where $\mu \in C^1$ is absolutely continuous and $\sigma \in C^2$ is absolutely continuous with an absolutely continuous derivative σ' can be transformed onto a new SDE $dY_t = a(Y_t)dt + b(Y_t)dB_t$ with a one-sided Lipschitz drift coefficient and a globally Lipschitz diffusion coefficients if and only if there exists some ψ that is bounded above, i.e. $\psi < M$ for some constant $M \in \mathbb{R}$ such that the function*

$$x \mapsto \exp\left(-\int_{x_0}^{x} \frac{2\mu(t)}{\sigma^2(t)} - \frac{\sigma'(t)}{\sigma(t)} dt\right) \int_{x_0}^{x} q(t) \exp\left(\int_{t_0}^{t} \frac{2\mu(s)}{\sigma^2(s)} - \frac{\sigma'(s)}{\sigma(s)} ds\right) dt$$

is essentially bounded, where

$$q(x) = \frac{2\mu(x)\sigma'(x)}{\sigma^2(x)} + \frac{2(\psi(x) - \mu'(x))}{\sigma(x)} + \sigma''(x).$$

If furthermore $\sigma > 0$ then we may rewrite this function in the following way

$$x \mapsto \sigma(x) \exp\left(-\int_{x_0}^{x} \frac{2\mu(t)}{\sigma^2(t)} dt\right) \int_{x_0}^{x} \frac{q(t)}{\sigma(t)} \exp\left(\int_{t_0}^{t} \frac{2\mu(s)}{\sigma^2(s)} ds\right) dt,$$

where we have the same q as before.

Proof. This is done in the same manner as the proof of the previous proposition apart from the fact that in the preceding proof ψ is essentially bounded, whereas here it is bounded above: $\psi < M$ for some constant M, yielding a one-sided Lipschitz drift coefficient by Lemma 2.2. Thus giving us an essentially bounded derivative for the diffusion coefficient and a derivative that is bounded from above for the drift coefficient. If conversely there exists some ψ that is bounded above such that the right hand side of equation (4) is essentially bounded then for an f as in equation (3) with φ equal to the right hand side of equation (4) we get one-sided Lipschitz a and globally Lipschitz b. □

3.4 Diffusions in natural scale

For the special case of a SDEs with zero drift coefficients, i.e. $\mu = 0$ we get the following conditions.

Corollary 3.3. *A diffusion in natural scale $dX_t = \sigma(X_t)dB_t$ where $\sigma \in C^2$ is absolutely continuous with an absolutely continuous derivative σ' can be transformed onto a new SDE $dY_t = a(Y_t)dt + b(Y_t)dB_t$ with globally Lipschitz coefficients if and only if there exists some essentially bounded function ψ such that the function*

$$\varphi(x) = \exp\left(\int_{x_0}^{x} \frac{\sigma'(t)}{\sigma(t)}dt\right) \int_{x_0}^{x} \left(\frac{2\psi(t)}{\sigma(t)} + \sigma''(t)\exp\left(\int_{t_0}^{t} \frac{\sigma'(s)}{\sigma(s)}ds\right)\right) dt$$

is essentially bounded. If furthermore $\sigma > 0$ we may rewrite φ as

$$\varphi(x) = \sigma(x) \int_{x_0}^{x} \left(\frac{2\psi(t)}{\sigma^2(t)} + \frac{\sigma''(t)}{\sigma(t)}\right) dt.$$

It is also worth stating the corresponding corollary for the case of transforming to a one-sided drift coefficient and globally Lipschitz diffusion coefficient. For the special case of a diffusions in natural scale process, i.e. $\mu = 0$ we get the following conditions.

Corollary 3.4. *A diffusion in natural scale $dX_t = \sigma(X_t)dB_t$ where $\sigma \in C^2$ is absolutely continuous with an absolutely continuous derivative σ' can be transformed onto a new SDE $dY_t = a(Y_t)dt + b(Y_t)dB_t$ with globally Lipschitz coefficients if and only if there exists some function ψ that is bounded from above, i.e. $\psi < M$ such that the function*

$$\varphi(x) = \exp\left(\int_{x_0}^{x} \frac{\sigma'(t)}{\sigma(t)}dt\right) \int_{x_0}^{x} \left(\frac{2\psi(t)}{\sigma(t)} + \sigma''(t)\exp\left(\int_{t_0}^{t} \frac{\sigma'(s)}{\sigma(s)}ds\right)\right) dt$$

is essentially bounded. If furthermore $\sigma > 0$ we may rewrite φ as

$$\varphi(x) = \sigma(x) \int_{x_0}^{x} \left(\frac{2\psi(t)}{\sigma^2(t)} + \frac{\sigma''(t)}{\sigma(t)}\right) dt.$$

3.5 Constant diffusion processes

In this section we turn our attention to yet another special case of SDEs, namely SDEs with constant non-zero diffusion coefficients. Our results on this class of equations are the following.

Proposition 3.5. *A SDE $dX_t = \mu(X_t)dt + \sigma dB_t$ where $\mu \in C^1$ is absolutely continuous and σ is a non-zero constant can be transformed onto a new SDE $dY_t = a(Y_t)dt + b(Y_t)dB_t$ with globally Lipschitz coefficients if and only if $|\mu - \mu'|$ is bounded on the set $\{|\mu| < |\mu'|\}$.*

Proof. Suppose that $f \in C^2$ is bijective and write $Y_t = f(X_t)$, then by the Itô lemma

$$dY_t = \left(g(Y_t)\mu_f(Y_t) + \frac{1}{2} g(Y_t) g'(Y_t)\sigma^2 \right) dt + g(Y_t)\sigma dB_t,$$

where $g(y) = f'(f^{-1}(y)$ and $\mu_f(y) = \mu(f^{-1}(y)$. Now in order for the diffusion coefficient $b(y) = g(y)\sigma$ to be globally Lipschitz by Lemma 2.1 g should be absolutely continuous and g' needs to be essentially bounded. Writing $g'(y) = f''(f^{-1}(y))/f'(f^{-1}(y) = \varphi(f^{-1}(y))$ where $\varphi \circ f^{-1}$ is some essentially bounded function yields the following expression for the transformation f:

$$f(x) = C \int_{x_0}^{x} \exp\left(\int_{t_0}^{t} \varphi(s) ds \right) dt,$$

where C is some constant. Plugging this into the drift coefficient $a(y) = (g(y)\mu_f(y) + \frac{1}{2} g(y) g'(y)\sigma^2)$ yields

$$a(y) = C \exp\left(\int_{x_0}^{f^{-1}(y)} \varphi(s) ds \right) \left(\mu_f(y) + \frac{\sigma^2}{2} \varphi(f^{-1}(y)) \right),$$

and differentiating a to check if a' is essentially bounded yields:

$$a'(y) = \varphi(f^{-1}(y)) \left(\mu_f(y) + \frac{\sigma^2}{2} \varphi(f^{-1}(y)) \right) + \mu'(f^{-1}(y)) + \frac{\sigma^2}{2} \varphi'(f^{-1}(y)). \tag{5}$$

Now if $|\mu - \mu'|$ is bounded on the set $\{|\mu| < |\mu'|\}$ then we can let

$$\varphi(x) = \begin{cases} -\mu'(x)/\mu(x) & \text{if } |\mu(x)| \geq |\mu'(x)| \\ \psi(x) & \text{if } |\mu(x)| < |\mu'(x)| \end{cases}$$

where ψ is some non-zero globally Lipschitz function which is equal to $-\mu'/\mu$ on the set $\{|\mu| = |\mu'|\}$, and a' will be bounded. If on the other hand $|\mu - \mu'|$ is unbounded on the set $\{|\mu| < |\mu'|\}$ we can conclude that a' will be unbounded for any bounded φ. □

Now that we have established that not all SDE with a constant diffusion coefficient can be transformed onto a new SDE with globally Lipschitz drift and diffusion coefficients a new question arises. Can we transform a SDE with a constant diffusion coefficient onto a new SDE with a one-sided Lipschitz drift coefficient and a globally Lipschitz diffusion coefficient?

Proposition 3.6. *A SDE $dX_t = \mu(X_t)dt + \sigma dB_t$ where $\mu \in C^1$ is absolutely continuous and σ is a non-zero constant can be transformed onto a new SDE $dY_t = a(Y_t)dt + b(Y_t)dB_t$ with a one-sided Lipschitz drift coefficient and a globally Lipschitz diffusion coefficient if and only if $|\mu - \mu'|$ is bounded on the set $\{\mu < \mu', \mu' > 0\}$.*

Proof. By arguing in the same manner as in the previous proposition we see that a' in equation (5) will be bounded from above if and only if $|\mu - \mu'|$ is bounded on the set $\{\mu < \mu', \mu' > 0\}$. So by Lemma 2.2 the result follows.

□

As an immediate corollary to Propositions 3.1 and 3.5, we get the following result for the case of a non-zero constant diffusion coefficient.

Corollary 3.7. *For an absolutely continuous $\mu \in C^1$ and a non-zero constant σ the following two conditions are equivalent and imply that for an SDE $dX_t = \mu(X_t)dt + \sigma dB_t$ we may find a transformation f such that $Y_t = f(X_t)$, $Y_t = a(Y_t)dt + b(Y_t)dB_t$ and a and b are globally Lipschitz.*

1. *$|\mu - \mu'|$ is bounded on the set $\{|\mu| < |\mu'|\}$.*
2. *There exists some essentially bounded function ψ such that the function*

$$\varphi(x) = \exp\left(-\int_{x_0}^{x} \frac{2\mu(t)}{\sigma^2}dt\right) \int_{x_0}^{x} \frac{2(\psi(t) - \mu'(t))}{\sigma} \exp\left(\int_{t_0}^{t} \frac{2\mu(s)}{\sigma^2}ds\right) dt$$

   is essentially bounded.

We get a corresponding corollary from Propositions 3.2 and 3.6.

Corollary 3.8. *For an absolutely continuous $\mu \in C^1$ and a non-zero constant σ the following two conditions are equivalent and imply that for an SDE $dX_t = \mu(X_t)dt + \sigma dB_t$ we may find a transformation f such that $Y_t = f(X_t)$, $Y_t = a(Y_t)dt + b(Y_t)dB_t$ and a is one-sided Lipschitz and b is globally Lipschitz.*

1. *$|\mu - \mu'|$ is bounded on the set $\{|\mu| < |\mu'|, \mu' > 0\}$.*
2. *There exists some function ψ that is bounded above such that the function*

$$\varphi(x) = \exp\left(-\int_{x_0}^{x} \frac{2\mu(t)}{\sigma^2}dt\right) \int_{x_0}^{x} \frac{2(\psi(t) - \mu'(t))}{\sigma} \exp\left(\int_{t_0}^{t} \frac{2\mu(s)}{\sigma^2}ds\right) dt$$

   is essentially bounded.

4 Examples

In this section we will look at examples of SDE models that have non-Lipschitz coefficients and explore how the method of transforming can be applied on them.

4.1 A double-well potential

The following model has its name from [6], it satisfies

$$dX_t = (X_t - X_t^3)dt + dB_t, \qquad X_0 = \zeta.$$

Notice that the drift coefficient $\mu(x) = x - x^3$ is non-Lipschitz. However $|\mu'(x)| = |1 - 3x^2|$ is bounded on the bounded set $\{x : |x - x^3| < |1 - 3x^2|\}$, so by Proposition 3.5 there exists a bijective transformation $f \in C^2$ such that $Y_t = f(X_t)$, $dY_t = a(Y_t)dt + b(Y_t)dB_t$ with a and b globally Lipschitz. Indeed, by taking

$$f(x) = C \int_{x_0}^{x} \exp\left(\int_{t_0}^{t} \varphi(s)ds\right) dt,$$

where

$$\varphi(x) = \begin{cases} \psi(x) & \text{if } -2 < x < 2 \\ -(1 - 3x^2)/(x - x^3) & \text{otherwise} \end{cases}$$

and ψ is an interpolation function between the points $(-2, 11/6)$ and $(2, -11/6)$ we get globally Lipschitz drift and diffusion coefficients a and b:

$$a(y) = C \exp\left(\int_{x_0}^{f^{-1}(y)} \varphi(s)ds\right)\left(\mu_f(y) + \frac{\sigma^2}{2}\varphi(f^{-1}(y))\right),$$

and

$$b(y) = C \exp\left(\int_{t_0}^{f^{-1}(y)} \varphi(s)ds\right),$$

since their derivatives

$$a'(y) = C\left(\varphi(f^{-1}(y))\left(\mu_f(y) + \frac{\sigma^2}{2}\varphi(f^{-1}(y))\right) + \mu'(f^{-1}(y)) + \frac{\sigma^2}{2}\varphi'(f^{-1}(y))\right)$$

and $b'(y) = C\varphi(f^{-1}(y))$ are bounded.

Finally we remark that the drift coefficient $\mu(x) = x - x^3$ is actually one-sided Lipschitz since its derivative is bounded from above and it is absolutely continuous. Thus an alternative to transforming the double-well potential for simulating with the Euler-Maruyama method would be to simulate it with the split step backward Euler method.

4.2 Diffusion with constant diffusion coefficient

Consider the SDE

$$dX_t = (1 + X_t^2)dt + dB_t \qquad X_0 = 1.$$

The drift coefficient for this SDE is neither globally Lipschitz nor one-sided Lipschitz, since its derivative is unbounded in both directions. We can however transform it onto the SDE

$$dY_t = \left(1 - \frac{\tan Y_t}{(1 + \tan^2 Y_t)^2}\right) dt + \frac{1}{1 + \tan^2 Y_t} dB_t \qquad Y_0 = \arctan(1)$$

using the transformation $f(x) = \arctan(x)$. But the coefficient functions of this SDE are both globally Lipschitz since

$$a(y) = 1 - \frac{\tan y}{(1 + \tan^2 y)^2} = 1 - \sin(y)\cos^3(y)$$

and

$$b(y) = \frac{1}{1 + \tan^2 y} = \cos^2(y)$$

are absolutely continuous functions with bounded derivatives

$$a'(y) = 3\sin^2(y)\cos^2(y) - \cos^4(y)$$

and

$$b'(y) = -2\sin(y)\cos(y).$$

So this is an example of a SDE with non-Lipschitz coefficients that can be transformed onto a SDE with Lipschitz coefficients.

4.3 A hyperbolic diffusion

The following SDE is taken from [1] and it has the form

$$dX_t = \tau \exp\left(\frac{1}{2}\left(\alpha\sqrt{\delta^2 + (X_t - \mu)^2} - \beta(X_t - \mu)\right)\right) dB_t, \quad X_0 = \zeta,$$

where the the parameters satisfy $\alpha > |\beta| \geq 0$, $\delta, \tau > 0$. Since this hyperbolic diffusion is a diffusion in natural scale we may use Corollary 3.3 to deduce whether or not it may be transformed onto a process with globally Lipschitz coefficients. Notice that the test function from the corollary

$$x \mapsto \sigma(x) \int_{x_0}^{x} \left(\frac{2\psi(t)}{\sigma^2(t)} + \frac{\sigma''(t)}{\sigma(t)}\right) dt. \tag{6}$$

is unbounded for each choice of a bounded ψ, since

$$\lim_{x \to +\infty} \tau \exp\left(\frac{1}{2}\left(\alpha\sqrt{\delta^2 + (x - \mu)^2} - \beta(x - \mu)\right)\right) = +\infty$$

so $\psi(x)/\sigma^2(x)$ will tend to zero as $x \to \infty$, and

$$\begin{aligned}\frac{\sigma''(x)}{\sigma(x)} &= \frac{1}{4}\left(\frac{\alpha(x-\mu)}{\sqrt{\delta^2+(x-\mu)^2}} - \beta\right)^2 \\ &\quad + \frac{1}{2}\left(\frac{\alpha}{\sqrt{\delta^2+(x-\mu)^2}} - \frac{\alpha(x-\mu)^2}{(\delta^2+(x-\mu)^2)^{3/2}}\right)\end{aligned}$$

tends to $(\alpha-\beta)^2/4$ as $x \to \infty$. From all this we can conclude that the integrand $2\psi/\sigma^2 + \sigma''/\sigma$ is certainly not zero at infinity so the function (6) is unbounded for all bounded ψ. Thus we conclude from Corollary 3.3 that there exists no transformation f that transforms the hyperbolic diffusion onto a SDE with globally Lipschitz coefficients.

On the other hand there exists a transformation f that transforms the hyperbolic diffusion onto a SDE with one-sided Lipschitz drift and globally Lipschitz diffusion. Indeed we can take

$$\psi = -\frac{\sigma^2}{2}\frac{\sigma''}{\sigma}$$

so that the integrand $2\psi/\sigma^2 + \sigma''/\sigma = 0$. But ψ is bounded from above and thus we may use Corollary 3.4 to conclude that there exists some transformation f that transforms the hyperbolic diffusion onto a SDE with one-sided Lipschitz drift and globally Lipschitz diffusion.

4.4 A family of heavy tailed SDEs

For a given constant $c < 0$ this model is taken from [9] and it has the form

$$dX_t = 3X_t^c dt + 3X_t^{2/3} dB_t, \qquad X_0 = \zeta > 0.$$

The Euler-Maruyama scheme turns out to be unstable for this scheme. Transforming this model with a bijective $f \in C^2$ yields

$$dY_t = \left(3g(Y_t)(f^{-1}(Y_t))^c + \frac{9}{2}g(Y_t)g'(Y_t)(f^{-1}(Y_t))^{4/3}\right)dt + 3g(Y_t)(f^{-1}(Y_t))^{2/3}dB_t,$$

where $g(y) = f'(f^{-1}(y))$. If we take our $\varphi = 0$ so that $k = 1$ and thus $f(x) = \int_{x_0}^x 1/\sigma = x^{1/3}$ this yields

$$dY_t = (Y_t^{3c-2} - Y_t^{-1})dt + dB_t.$$

The drift coefficient is not globally Lipschitz but it is one-sided Lipschitz so we can simulate $\{Y_t\}$ with the backward Euler method, by simulating $\tilde{Y}_n = Y_n + \Delta B_n$ and then solving $Y_{n+1} - (Y_{n+1}^{3c-2} - Y_{n+1}^{-1})\Delta = \tilde{Y}_n$ for Y_{n+1}. A trajectory for $\{X_t\}$ can then be obtained by transforming the $\{Y_t\}$ trajectory back using the inverse transformation $f^{-1}(y) = y^3$.

Now it would be interesting to see if we could get any further than this, that is if there exists an transformation f such that the resulting SDE has globally Lipschitz drift and diffusion coefficients. Supposing that φ is non-negative and essentially bounded we obtain a drift coefficient on the form

$$a(y) = k(f^{-1}(y))\left((f^{-1}(y))^{c-2/3} - \frac{1}{2}\varphi(f^{-1}(y)) - (f^{-1}(y))^{-1/3}\right),$$

where as before $k(x) = \exp\left(\int_{x_0}^{x} \varphi(t)/(3t^{2/3})dt\right)$ and its derivative has the form

$$\begin{aligned} a'(y) &= \varphi(f^{-1}(y))\left((f^{-1}(y))^{c-2/3} - \frac{1}{2}\varphi(f^{-1}(y)) - (f^{-1}(y))^{-1/3}\right) \\ &\quad + \frac{3}{2}(f^{-1}(y))^{2/3}\varphi'(f^{-1}(y)) + (3c-2)(f^{-1}(y))^{c-1} + (f^{-1}(y))^{-2/3}. \end{aligned}$$

For y's such that $f^{-1}(y)$ is close to zero a' tends to infinity regardless of our choice of φ, since $\varphi \circ f^{-1}$ is essentially bounded. Similarly for y's such that $f^{-1}(y)$ tends to infinity a' will tend to infinity regardless of which essentially bounded φ we choose. Our conclusion is that we can not transform this SDE onto a new SDE with globally Lipschitz drift and diffusion coefficients.

4.5 The CKLS model

The CKLS model, which was first introduced in [2] is a model on the form

$$dX_t = (\alpha + \beta X_t)dt + \sigma X_t^{\gamma} dB_t \qquad X_0 = \zeta > 0,$$

where $\alpha, \beta \in \mathbb{R}$ and $\sigma, \gamma > 0$. Transforming the CKLS model with a bijective $f \in C^2$ yields

$$dY_t = \left(g(Y_t)(\alpha + \beta f^{-1}(Y_t)) + \frac{1}{2}g(Y_t)g'(Y_t)\sigma^2(f^{-1}(Y_t))^{2\gamma}\right)dt + g(Y_t)\sigma(f^{-1}(Y_t))^{\gamma}dB_t,$$

where $g(y) = f'(f^{-1}(y))$. If we allow only positive values for the original CKLS process $\{X_t\}$ then the diffusion coefficient $x \mapsto \sigma x^{\gamma}$ is strictly positive. From section 3.2.1 we can deduce that in order for the transformed diffusion coefficient $b(y) = g(y)\sigma(f^{-1}(y))^{\gamma}$ to be globally Lipschitz, f needs to have the form

$$f(x) = \int_{x_0}^{x} \frac{k(t)}{\sigma t^{\gamma}} dt,$$

almost everywhere, where $k(x) = \exp\left(\int_{x_0}^{x} \varphi(t)/\sigma t^{\gamma} dt\right)$ and φ is some essentially bounded function. Taking $\varphi = 0$ yields the transformation $f(x) = x^{1-\gamma}/\sigma(1-\gamma)$, $\gamma \neq 1$ and gives us the following form for $\{Y_t\}$.

$$dY_t = \alpha\sigma^{1/(\gamma-1)}((1-\gamma)Y_t)^{\gamma/(\gamma-1)} + \beta(1-\gamma)Y_t - \frac{\gamma}{2(1-\gamma)}Y_t^{-1}dt + dB_t.$$

Let's take a closer look at the drift coefficient

$$a(y) = \alpha\sigma^{1/(\gamma-1)}((1-\gamma)y)^{\gamma/(\gamma-1)} + \beta(1-\gamma)y - \frac{\gamma}{2(1-\gamma)}y^{-1},$$

its derivative is

$$a'(y) = \alpha\gamma(\sigma(1-\gamma)y)^{1/(\gamma-1)} + \beta(1-\gamma) + \frac{\gamma}{2(1-\gamma)}\gamma\sigma y^{-2}$$

which is clearly not bounded, the y^{-2} term tends to infinity for low values of y and $y^{1/(\gamma-1)}$ tends to infinity for high values of y. We can conclude that choosing $\varphi = 0$ as our essentially bounded function does not yield a global Lipschitz drift coefficient. What can however be done is to notice that a is one-sided Lipschitz and simulate the transformed equation with the split step backward method. But it would be interesting to see if we can find some essentially bounded φ that makes the resulting a globally Lipschitz. If we consider non-zero φ the drift coefficient takes the following form

$$a(y) = k(f^{-1}(y))\left(\frac{\alpha+\beta f^{-1}(y)}{\sigma(f^{-1}(y))^{\gamma}} + \frac{1}{2}\left(\varphi(f^{-1}(y)) - \gamma\sigma(f^{-1}(y))^{\gamma-1}\right)\right)$$

and its derivative has the following form

$$\begin{aligned}
a'(y) &= \varphi(f^{-1}(y))\left(\frac{\alpha+\beta f^{-1}(y)}{\sigma(f^{-1}(y))^{\gamma}} + \frac{1}{2}\left(\varphi(f^{-1}(y)) - \gamma\sigma(f^{-1}(y))^{\gamma-1}\right)\right) \\
&\quad + k(f^{-1}(y))\left(\frac{\beta}{g(y)\sigma(f^{-1}(y))^{\gamma}} - \gamma\frac{\alpha+\beta f^{-1}(y)}{g(y)\sigma(f^{-1}(y))^{\gamma+1}}\right) \\
&\quad + \frac{k(f^{-1}(y))}{2}\left(\frac{\varphi'(f^{-1}(y))}{g(y)} - \gamma(\gamma-1)\sigma\frac{(f^{-1}(y))^{\gamma-2}}{g(y)}\right) \\
&= \varphi(f^{-1}(y))\left(\frac{\alpha+\beta f^{-1}(y)}{\sigma(f^{-1}(y))^{\gamma}} + \frac{1}{2}\left(\varphi(f^{-1}(y)) - \gamma\sigma(f^{-1}(y))^{\gamma-1}\right)\right) \\
&\quad + \beta - \frac{\gamma(\alpha+\beta f^{-1}(y))}{f^{-1}(y)} + \frac{1}{2}\left(\sigma(f^{-1}(y))^{\gamma}\varphi'(f^{-1}(y)) - \gamma(\gamma-1)\sigma^2(f^{-1}(y))^{2\gamma-2}\right).
\end{aligned}$$

If $\gamma < 1$ then the $(\alpha+\beta f^{-1}(y))/\sigma(f^{-1}(y))^{\gamma}$ and $\gamma(\alpha+\beta f^{-1}(y))/f^{-1}(y)$ terms push a' toward infinity for y's such that $f^{-1}(y)$ is close to zero. The former one of those can be handled by making φ tend to zero as $f^{-1}(y)$ tends to zero, but that does not really improve the situation as the latter still tends to infinity for as $f^{-1}(y)$ tends to zero. In the case of $\gamma > 1$ these two terms along with the term $\gamma(\gamma-1)\sigma^2(f^{-1}(y))^{2\gamma-2}$ pushes to zero as $f^{-1}(y)$ tends to zero, so the the situation is no better here. Finally we have assumed that $\gamma \neq 1$, in that case the original CKLS equation is globally Lipschitz in both coefficients. We conclude that there does not exist a transformation f that transforms the CKLS model onto a new SDE with globally Lipschitz drift and diffusion coefficients.

5 Conclusion

In this thesis we have explored the scope of the method of transforming a SDE with non-Lipschitz coefficients onto another SDE with either globally Lipschitz coefficients or a one-sided Lipschitz diffusion coefficient and a globally Lipschitz diffusion coefficient using a bijective transformation. We have seen that such a transformation does not necessarily exist. We have proven general results on when such a transformation exists, as well as some results on more specific cases.

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