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Explicit Martingale Representations

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Abstract

In this thesis we derive a formula which can be used to give the so called martingale representation theorem an explicit form with arbitrarily good approximation.

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Contents

1 Introduction

In this thesis we derive a formula which can be used to give the so called martingale representation theorem an explicit form with arbitrarily good approximation.

2 Martingale representation theorem

Let $\{B(t)\}_{t\geq 0}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and $\{\mathcal{F}_t\}_{t\geq 0}$ the filtration generated by this Brownian motion, which is to say that $\mathcal{F}_t = \sigma(B(r) : 0 \leq r \leq t)$ for $t \geq 0$. If $T > 0$ is a constant and $\{M(t)\}_{t\in[0,T]}$ is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, then the so called martingale representation theorem (see e.g., Klebaner, 2005, Section 8.12) states that there exists an adapted and measurable stochastic process $\{Y(t)\}_{t\in[0,T]}$ such that

$$\mathbf{P}\biggl\{\int_0^T Y(r)^2\,dr < \infty\biggr\} = 1 \quad \text{and} \quad \{M(t)\}_{t\in[0,T]} = \biggl\{M(0) + \int_0^t Y\,dB\biggr\}_{t\in[0,T]}. \tag{1}$$

Moreover, if Z is an integrable $\mathcal{F}_T$-measurable random variable, then there exists an adapted and measurable stochastic process $\{Y(t)\}_{t\in[0,T]}$ such that

$$\mathbf{P}\biggl\{\int_0^T Y(r)^2\,dr < \infty\biggr\} = 1 \quad \text{and} \quad Z = \mathbf{E}[Z] + \int_0^T Y\,dB. \tag{2}$$

Note that (1) implies (2), as taking $\{M(t)\}_{t\in[0,T]} = \{\mathbf{E}[Z\,|\,\mathcal{F}_t]\}_{t\in[0,T]}$ the adaptedness of Z to $\mathcal{F}_T$ together with (1) and the fact that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ give

$$Z = \mathbf{E}[Z\,|\,\mathcal{F}_T] = M(T) = M(0) + \int_0^T Y\,dB = \mathbf{E}[Z\,|\,\mathcal{F}_0] + \int_0^T Y\,dB = \mathbf{E}[Z] + \int_0^T Y\,dB.$$

It turns out that it is of great importance in mathematical finance to find the explicit form of the process $\{Y(t)\}_{t\in[0,T]}$ in the representation (2) for a given $\mathcal{F}_T$-measurable random variable Z. We refer to Shiryaev and Yor (2004, 2007) for more information on that importance. Also, Shiryaev and Yor explains that although there exist more or less elaborate theoretical schemes to calculate the process Y in (2), see, e.g., Øksendal (1996), those schemes do not give explicit closed form expressions for Y except in a few special cases.

In this thesis we will derive a formula which can be used to find the process Y in (2) in explicit closed form with arbitrarily good approximation in the sense of convergence in probability.

3 Approximate explicit martingale representation

A random variable Z that is $\mathcal{F}_T$-measurable is determined by the values of $\{B(t)\}_{t\in[0,T]}$. As B is continuous the values of $\{B(t)\}_{t\in[0,T]}$ are determined by the values of B on all finite subsets $\{t_1,\ldots,t_n\}\subseteq(0,T]$. It follows that if we can give the representation (2) for any bounded and continuous function $f(B(t_1),\ldots,B(t_n))$ of the Brownian motion values $B(t_1),\ldots,B(t_n)$, then we can get an arbitrarily good approximation of a random variable Z that is $\mathcal{F}_T$-measurable in the sense of convergence in probability (see also Albin, 2008). For this in turn, by the Weirstrass approximation theorem, it is sufficient to find the representation (2) of any monomial of the Brownian motion values $B(t_1),\ldots,B(t_n)$, that is, to find the representation (2) for random variables Z of the type

$$Z=\prod_{i=1}^{n}B(t_i)^{k_i}\quad\text{for any }\ 0<t_1<\ldots<t_n\le T,\ k_1,\ldots,k_n\in\mathbb{N}\ \text{ and }\ n\in\mathbb{N}.\tag{3}$$

By elementary algebra, to find the representation (2) for random variables Z of the type (3) it is sufficient to find the representation (2) for random variables Z of the type

$$Z=\prod_{i=1}^{n}(B(t_i)-B(t_{i-1}))^{k_i}\quad\text{for any }\ 0=t_0<\ldots<t_n\le T,\ k_1,\ldots,k_n\in\mathbb{N}\ \text{ and }\ n\in\mathbb{N}.\tag{4}$$

4 The case when $n = 1$

Here we treat the case when Z in (2) is given by (3) with $n=1$.

Denoting

$$I(i,j;t) = \int_0^t (t-r)^i\, B(r)^j\, dB(r) \quad \text{and} \quad J(i,j;t) = \int_0^t (t-r)^i\, B(r)^j\, dr$$

for $t>0$ and $i,j\geq 0$ with $J(i,-1;t)=0$, Itô's formula shows that

$$\begin{aligned} J(i,j{+}1;t) &= \int_0^t (t-r)^i \bigg((j{+}1) \int_0^r B(\xi)^j\, dB(\xi) + \frac{(j{+}1)\, j}{2} \int_0^r B(\xi)^{j-1}\, d\xi \bigg)\, dr \\ &= (j{+}1) \int_0^t \frac{(t-r)^{i+1}}{i+1}\, B(r)^j\, dB(r) + \frac{(j{+}1)\, j}{2} \int_0^t \frac{(t-r)^{i+1}}{i+1}\, B(r)^{j-1}\, dr \\ &= \frac{j+1}{i+1}\, I(i{+}1,j;t) + \frac{(j{+}1)\, j}{2\,(i+1)}\, J(i{+}1,j{-}1;t) \end{aligned}$$

for $t>0$ and $i,j\geq 0$. Moreover, another application of Itô's formula gives

$$\begin{aligned} B(t)^{k+1} &= (k{+}1) \int_0^t B(r)^k\, dB(r) + \frac{(k{+}1)\, k}{2} \int_0^t B(r)^{k-1}\, dr \\ &= (k{+}1)\, I(0,k;t) + \frac{(k{+}1)\, k}{2}\, J(0,k{-}1;t) \end{aligned}$$

for $t>0$ and $k\geq 0$. Putting these identities together we obtain by recursion

$$\begin{aligned} & B(t)^{2k+1} \\ &= (2k{+}1)\, I(0,2k;t) + \frac{(2k{+}1)\, 2k}{2}\, J(0,2k{-}1;t) \\ &= (2k{+}1)\, I(0,2k;t) + \frac{(2k{+}1)\, 2k\, (2k{-}1)}{2}\, I(1,2k{-}2;t) + \frac{(2k{+}1)\, 2k\, (2k{-}1)\, (2k{-}2)}{4}\, J(1,2k{-}3;t) \\ &\qquad \vdots \\ &= \sum_{\ell=0}^{k} \frac{(2k{+}1)!}{\ell!\, 2^\ell\, (2k-2\ell)!}\, I(\ell,2k-2\ell;t) \\ &= \int_0^t \bigg(\sum_{\ell=0}^{k} \frac{(2k{+}1)!\, (t-r)^\ell\, B(r)^{2k-2\ell}}{\ell!\, 2^\ell\, (2k-2\ell)!} \bigg)\, dB(r) \end{aligned}$$

for $t>0$ and $k\geq 0$. (Of course, a rigorous verification of this formula is done by means of induction after a few iterations have shown that the above formula really is what to prove by induction.) As $\mathbf{E}[B(t)^{2k+1}]=0$ we have obtained the representation (2) of Z given by (3) when $n=1$ and $k_1 = 2k{+}1 \geq 1$ is odd.

Moving over to the case when $k_1=2k$ is even we note that by recursion

$$
\begin{aligned}
&B(t)^{2k}\\
&= 2k\, I(0,2k-1;t) + \frac{2k\,(2k-1)}{2}\, J(0,2k-2;t)\\
&= 2k\, I(0,2k-1;t) + \frac{2k\,(2k-1)\,(2k-2)}{2}\, I(1,2k-3;t) + \frac{2k\,(2k-1)\,(2k-2)\,(2k-3)}{4}\, J(1,2k-4;t)\\
&\qquad\vdots\\
&= \sum_{\ell=0}^{k-1} \frac{(2k)!}{\ell!\,2^\ell\,(2k-1-2\ell)!}\, I(\ell,2k-1-2\ell;t) + \frac{(2k)!}{(k-1)!\,2^k}\, J(k-1,0;t)\\
&= \int_0^t \bigg(\sum_{\ell=0}^{k-1} \frac{(2k)!\,(t-r)^\ell\, B(r)^{2k-1-2\ell}}{\ell!\,2^\ell\,(2k-1-2\ell)!}\bigg)\, dB(r) + \frac{(2k)!\,t^k}{k!\,2^k}
\end{aligned}
$$

for $t>0$ and $k\geq 1$. (Again, a rigorous verification of this formula is done by means of induction.) As $\mathbf{E}[B(t)^{2k}] = (2k)!\,t^k/(k!\,2^k)$ by elementary calculations, we have obtained the representation (2) of Z given by (3) when $n=1$ and $k_1=2k\geq 2$ is even.

We may put the odd and the even case together by the following formula

$$
B(t)^k = \int_0^t \bigg(\sum_{\ell=0}^{\lfloor (k+1)/2\rfloor -1} \frac{k!\,(t-r)^\ell\, B(r)^{k-1-2\ell}}{\ell!\,2^\ell\,(k-1-2\ell)!}\bigg)\, dB(r) + \frac{k!\,\mathbf{1}_{\{k \text{ even}\}}\, t^{k/2}}{(k/2)!\,2^{k/2}}
$$

for $t>0$ and $k\geq 1$, where $\lfloor x\rfloor$ denotes the integer part of $x\in\mathbb{R}$. From this in turn we readily get the following representation (2) for random variables Z of the type (4) when $n=1$:

$$
\begin{aligned}
&(B(t)-B(s))^k\\
&= \int_0^t \bigg(\sum_{\ell=0}^{\lfloor (k+1)/2\rfloor -1} \frac{k!\,(t-r)^\ell\, B(r)^{k-1-2\ell}}{\ell!\,2^\ell\,(k-1-2\ell)!}\bigg)\, \mathbf{1}_{(s,\infty)}(r)\, dB(r) + \frac{k!\,\mathbf{1}_{\{k \text{ even}\}}\,(t-s)^{k/2}}{(k/2)!\,2^{k/2}}\\
&= k!\,\bigg[\int_0^T \bigg(\sum_{\ell=0}^{\lfloor (k+1)/2\rfloor -1} \frac{(t-r)^\ell\, B(r)^{k-1-2\ell}}{\ell!\,2^\ell\,(k-1-2\ell)!}\bigg)\, \mathbf{1}_{(s,t]}(r)\, dB(r) + \frac{\mathbf{1}_{\{k \text{ even}\}}\,(t-s)^{k/2}}{(k/2)!\,2^{k/2}}\bigg]
\end{aligned}
\tag{5}
$$

for $0\leq s<t\leq T$ and $k\geq 1$.

5 The general case when $n \geq 1$

Here we treat the case when Z in (2) is given by (4) with a general $n \geq 1$.

With the convention that the empty product is 1 and that the empty sum is 0, we have by (5) together with Itô's formula

$$
\begin{aligned}
&\prod_{i=1}^{n}(B(t_i)-B(t_{i-1}))^{k_i} \\
&= \prod_{i=1}^{n} k_i! \left[\int_0^t \left(\sum_{\ell=0}^{\lfloor (k_i+1)/2\rfloor -1} \frac{(t_i-r)^\ell\, B(r)^{k_i-1-2\ell}}{\ell!\, 2^\ell\, (k_i-1-2\ell)!}\right) \mathbf{1}_{(t_{i-1},t_i]}(r)\, dB(r) + \frac{\mathbf{1}_{\{k_i \text{ even}\}}\,(t_i-t_{i-1})^{k_i/2}}{(k_i/2)!\, 2^{k_i/2}}\right] \\
&= \prod_{i=1}^{n} k_i! \int_0^t \left(\sum_{\ell=0}^{\lfloor (k_i+1)/2\rfloor -1} \frac{(t_i-r)^\ell\, B(r)^{k_i-1-2\ell}}{\ell!\, 2^\ell\, (k_i-1-2\ell)!}\right) \mathbf{1}_{(t_{i-1},t_i]}(r)\, dB(r) \\
&\quad + k_n! \frac{\mathbf{1}_{\{k_n \text{ even}\}}\,(t_n-t_{n-1})^{k_n/2}}{(k_n/2)!\, 2^{k_n/2}} \\
&\qquad \times \prod_{i=1}^{n-1} k_i! \left[\int_0^t \left(\sum_{\ell=0}^{\lfloor (k_i+1)/2\rfloor -1} \frac{(t_i-r)^\ell\, B(r)^{k_i-1-2\ell}}{\ell!\, 2^\ell\, (k_i-1-2\ell)!}\right) \mathbf{1}_{(t_{i-1},t_i]}(r)\, dB(r) + \frac{\mathbf{1}_{\{k_i \text{ even}\}}\,(t_i-t_{i-1})^{k_i/2}}{(k_i/2)!\, 2^{k_i/2}}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \vdots \\
&= \sum_{j=0}^{n} \left[\prod_{i=j+1}^{n} \frac{k_i!\, \mathbf{1}_{\{k_i \text{ even}\}}\,(t_i-t_{i-1})^{k_i/2}}{(k_i/2)!\, 2^{k_i/2}}\right] \\
&\qquad \times \left[\prod_{i=1}^{j} \int_0^t \left(\sum_{\ell=0}^{\lfloor (k_i+1)/2\rfloor -1} \frac{k_i!\,(t_i-r)^\ell\, B(r)^{k_i-1-2\ell}}{\ell!\, 2^\ell\, (k_i-1-2\ell)!}\right) \mathbf{1}_{(t_{i-1},t_i]}(r)\, dB(r)\right] \\
&= \sum_{j=0}^{n} \left[\prod_{i=j+1}^{n} \frac{k_i!\, \mathbf{1}_{\{k_i \text{ even}\}}\,(t_i-t_{i-1})^{k_i/2}}{(k_i/2)!\, 2^{k_i/2}}\right] \\
&\qquad \times \left[\sum_{i=1}^{j} \int_0^t \left\{ \int_0^s \prod_{m\in\{1,\dots,j\},\, m\neq i} \left(\sum_{\ell=0}^{\lfloor (k_m+1)/2\rfloor -1} \frac{k_m!\,(t_m-r)^\ell\, B(r)^{k_m-1-2\ell}}{\ell!\, 2^\ell\, (k_m-1-2\ell)!}\right) \mathbf{1}_{(t_{m-1},t_m]}(r)\, dB(r)\right]\right\} \\
&\qquad\qquad \times \left(\sum_{\ell=0}^{\lfloor (k_i+1)/2\rfloor -1} \frac{k_i!\,(t_i-s)^\ell\, B(s)^{k_i-1-2\ell}}{\ell!\, 2^\ell\, (k_i-1-2\ell)!}\right) \mathbf{1}_{(t_{i-1},t_i]}(s)\, dB(s) \\
&= \prod_{i=1}^{n} \frac{k_i!\, \mathbf{1}_{\{k_i \text{ even}\}}\,(t_i-t_{i-1})^{k_i/2}}{(k_i/2)!\, 2^{k_i/2}} \\
&\quad + \int_0^T \sum_{j=1}^{n} \left[\prod_{i=j+1}^{n} \frac{k_i!\, \mathbf{1}_{\{k_i \text{ even}\}}\,(t_i-t_{i-1})^{k_i/2}}{(k_i/2)!\, 2^{k_i/2}}\right] \\
&\qquad \times \left[\sum_{i=1}^{j} \left\{ \int_0^s \prod_{m\in\{1,\dots,j\},\, m\neq i} \left(\sum_{\ell=0}^{\lfloor (k_m+1)/2\rfloor -1} \frac{k_m!\,(t_m-r)^\ell\, B(r)^{k_m-1-2\ell}}{\ell!\, 2^\ell\, (k_m-1-2\ell)!}\right) \mathbf{1}_{(t_{m-1},t_m]}(r)\, dB(r)\right]\right\}
\end{aligned}
$$

$$\times\Bigg(\sum_{\ell=0}^{\lfloor (k_i+1)/2\rfloor -1} \frac{k_i!\,(t_i-s)^\ell\,B(s)^{k_i-1-2\ell}}{\ell!\,2^\ell\,(k_i-1-2\ell)!}\Bigg)\,\mathbf{1}_{(t_{i-1},t_i]}(s)\,dB(s)$$

for $0 \le t_0 < \ldots < t_n \le t \le T$, $k_1,\ldots,k_n \ge 1$ and $n \ge 1$. In view of the elementary fact that

$$\mathbf{E}\Bigg[\prod_{i=1}^n (B(t_i)-B(t_{i-1}))^{k_i}\Bigg] = \prod_{i=1}^n \frac{k_i!\,\mathbf{1}_{\{k_i\text{ even}\}}\,(t_i-t_{i-1})^{k_i/2}}{(k_i/2)!\,2^{k_i/2}}$$

for $0 \le t_0 < \ldots < t_n \le t \le T$, $k_1,\ldots,k_n \ge 1$ and $n \ge 1$, this means that we have obtained the representation (2) when Z is given by (4). And that is our contribution to math folks!

6 Conclusion

In this thesis we derived a formula which can be used to give the so called martingale representation theorem an explicit form with arbitrarily good approximation.

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