CHALMERS | GÖTEBORG UNIVERSITY

MASTER'S THESIS

Jump Modelling for Financial Asset Prices

YUEMING ZHANG

Department of Mathematical Statistics
CHALMERS UNIVERSITY OF TECHNOLOGY
GÖTEBORG UNIVERSITY
Göteborg, Sweden 2005

Thesis for the Degree of Master of Science (20 credits)

Jump Modelling for Financial Asset Prices

Yueming Zhang

CHALMERS | GÖTEBORG UNIVERSITY

Department of Mathematical Statistics
Chalmers University of Technology and Göteborg University
SE – 412 96 Göteborg, Sweden

Göteborg, September 2005

Jump Modelling for Financial Asset Prices

Yueming Zhang

August 28, 2005

Abstract

In this master thesis we study the character of jumps occurring in a financial asset pricing process with IBM stock. We present a formal definition of jumps and find that based on this definition, the durations between upward jumps and the sizes of downward jumps have a heavy-tail distribution, but the durations between downward jumps and the sizes of upward jumps have light-tail distribution. By the BDS test, it is shown that the durations between jump are correlated. Here the AR(1) model is suggested to be the proper model for the logarithm of jump duration. The occurrence of jumps can be treated as a counting process. In our counting process model, we discuss the stochastic intensity, the distribution of jump duration, the distribution of arrival times and the properties of the counting process.

Acknowledgement I would like to express my great appreciation to my supervisor Patrik Albin for his comments and suggestions. And I also like to thank the following people: Oscar Hammar, Jan Lennartsson, Min Shu, Viktor Olsbo, Johan Tykesson and Alexander Herbertsson, for their help during my thesis work.

Contents

Chapter 1

Introduction

The issue to build a model beyond Black-Scholes is a classical topic in the field of mathematical finance. And in last thirty years many models have been suggested; most of them can be classified into two kinds: stochastic-volatility models and jump-diffusion models. Stochastic-volatility models use a stochastic volatility instead of the constant volatility in the Black-Scholes model with or without jumps, while Jump-diffusion models add a jump part to the Black-Scholes model (see e.g. [21]). Generally speaking, compared with stochastic-volatility models, jump-diffusion models are simpler and can qualitatively catch the financial market phenomena, like the large fluctuations in the asset prices (see e.g. [21]).

In jump-diffusion models for financial asset pricing, jumps are usually defined as all the discontinuities in a sample path of a Brownian motion; however as in the real world the stock prices are never continuous, such a definition will be quite difficult to put into practice. To solve this problem, people usually argue that when the difference between two consecutive stock prices is larger than a certain threshold, then a jump occurs. Unfortunately this threshold always varies to with different stocks. Moreover people usually assume that the arrival time of jumps can be a Poisson process. Is this assumption really true? Clearly a formal definition of jumps is needed.

In this master thesis we study the issue of defining jumps for stock prices. Further, we will study the properties of jumps.

Chapter 2 gives a formal definition of the jumps, the concepts of jump size and jump duration.

In Chapter 3 we investigate the probability distribution properties of jump size and jump duration.

Chapter 4 is in part devoted to the study of dependence structures of jump size and jump duration. Furthermore we discuss the models for jump duration.

In Chapter 5 we provide an idea about the possibility to establish a new jump model for asset pricing.

And finally in Chapter 6, we make some concluding remarks, as well as discuss the future research.

Chapter 2

Exploring Jump-Diffusions

In this work we use IBM stock prices, from 3-Jan-1984 to 22-Oct-2004, as our price process. We remove the sample mean from the corresponding log returns and then divide them with the sample standard deviation[1]. Compared with the standard normal distribution, the devolatilized log returns are more heavy-tailed, as can be seen in Figure 2.2 and 2.3.

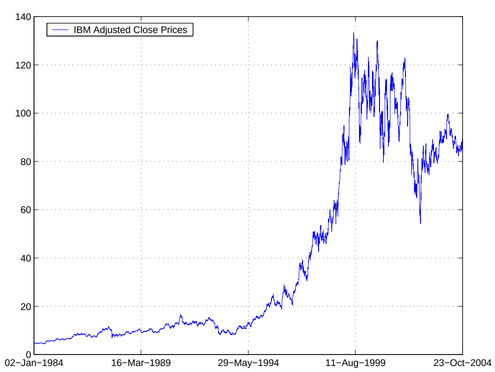


Figure 2.1: The IBM adjusted close prices from 3-Jan-1984 to 22-Oct-2004

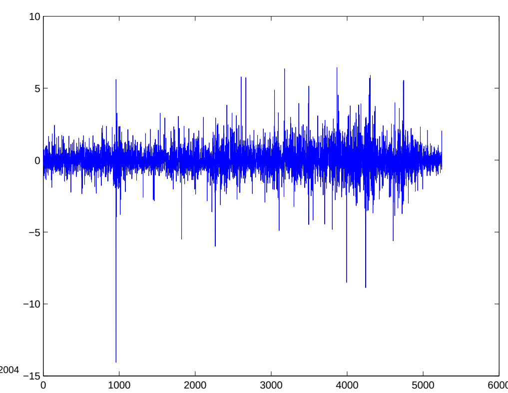


Figure 2.2: The devolatilized log returns of IBM stock prices

2.1 What are Jumps?

In jump-diffusion models, asset prices are modeled as Levy processes with a nonzero Gaussian component and a jump part (see [9], pp 103); and jumps are represented as the rare events - crashes or sudden upsurges (see [14]). Such models can explain why heavy tails will appear in the marginal distribution for some stochastic price processes. In other words, the jumps reflect the heavy tailed part on a distribution.

[1] The sample mean $\hat{\mu} = 5.47\text{e} - 4$ and the sample standard deviation $\hat{\sigma} = 0.0191$

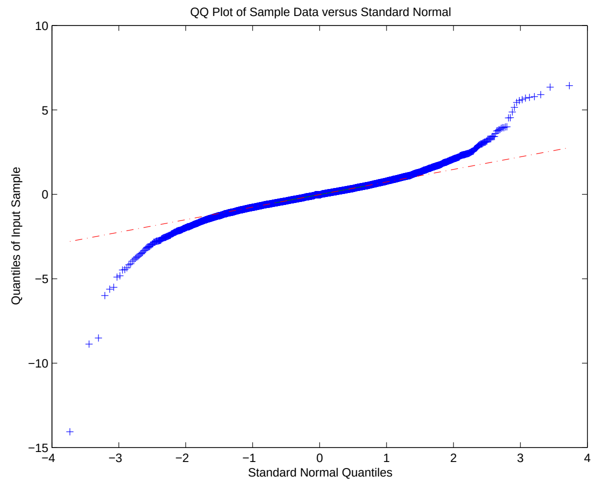


Figure 2.3: QQ plot of the devolatilized log returns on IBM stock prices

Definition 1 (Jump) *For an asset price process X_t, a threshold α ($\alpha > 0$) and $-\alpha$ are set on both the positive and the negative side of the devolatilized log returns of X_t. Once the devolatilized log return exceeds the threshold on either side at time t, i.e.*

$$\left|\frac{\log X_t - \log X_{t-1} - \hat{\mu}}{\hat{\sigma}}\right| \geq \alpha,$$

we say that a jump occurs at time t.

Following this definition, we set natural definitions of jump size and duration between jumps.

Definition 2 (Jump size) *When a jump occurs at time t, the jump size is the difference of the log prices, i.e.*

$$\{\textit{jump size}\}_t = \log X_t - \log X_{t-1}.$$

Obviously the jump size at time t equals the value of the log return at time t.

Definition 3 (Duration between jumps) *When a jump occurs at time t and the next jump occurs at time s, where $t > s$, we say that the duration between these two consecutive jumps is $t - s$.*

The following investigation of jumps are based on the above three definitions.

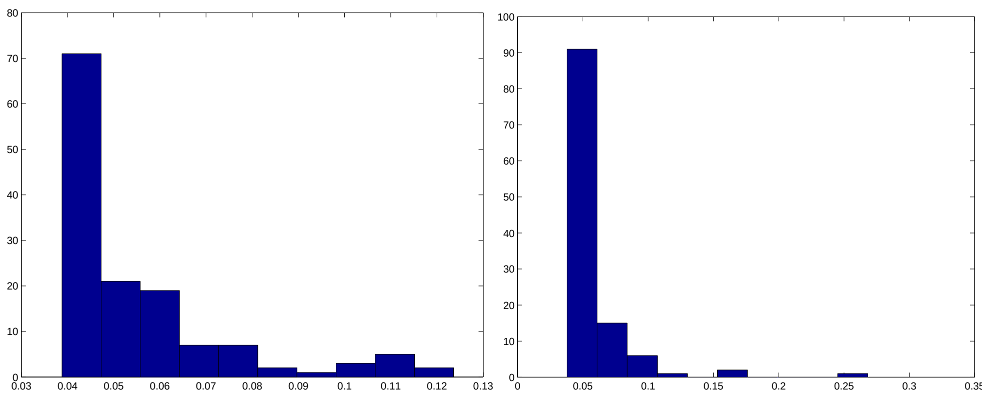
2.2 Jumps of IBM Stock Prices

Since jumps are rare events, it is reasonable to set the threshold $\alpha = 2.00$. According to the definition of jump (Definition 1), we register the time when jumps occurs. The corresponding log returns[2] are the size of jumps.

Definition 4 (Jump–up & Jump–down) *If the size of a jump > 0, then this jump is upward; and if the size < 0, then this jump is downward.*

Considering that the size of downward jumps is always negative, for simplicity, we will take the absolute value for the size of downward jumps in following parts.

It is well known that the size of downward jumps tend to be larger than that of upward jumps for many asset prices (see e.g. [14]). Hence it is motivated to investigate whether upward jumps and downward jumps really have different properties.



(a) The size of upward jumps (b) The size of downward jumps

Figure 2.4: The histograms of the jump size

	Number	Max Value	Min Value
Size of jump-up	138	0.1236	0.0388
Size of jump-down	116	0.2682	0.0379

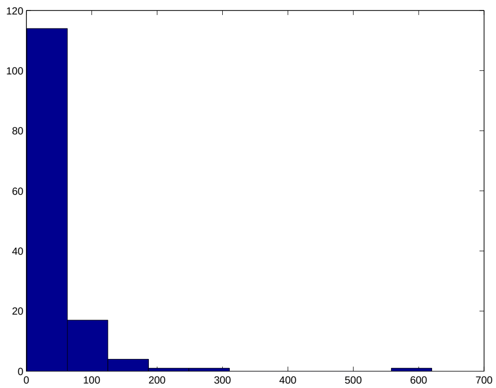
Table 2.1: Information about the jump size

In this thesis, we only consider the duration between two consecutive jumps. And as argued above, we would like to separate the durations for upward jumps and downward jumps, as well as their jump sizes.

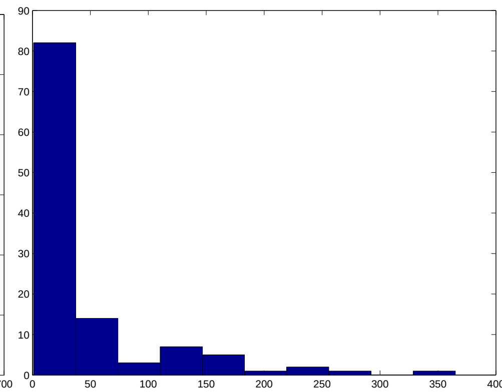
	Number	Max Value	Min Value
Duration for jump-up	138	620	1
Duration for jump-down	116	365	1

Table 2.2: Information about the duration between jumps

[2]According to the definition of height of jump (Definition 2), these log returns are non-devolatilized. In the following parts, if the log returns are not specifically indicated to be devolatilized, then they are non-devolatilized.



(a) The duration between upward jumps



(b) The duration between downward jumps

Figure 2.5: The histograms of the duration between jumps

2.3 Distribution Test

F. B. Hanson and J. J. Westman [14] consider that the sizes of jumps are uniformly distributed and the jump process is a space-time Poisson process. However, through observing the histograms of the jump size (Figure 2.4), we see that the heights of neither upward jumps nor downward jumps are uniformly distributed. And in the following part, we will investigate whether the increments of jump process, i.e. the duration between jumps, are exponentially distributed.

2.3.1 Goodness of Fit

We choose Kolmogorov-Smirnov (KS) distance as the statistical test for assumptions about distribution. The test statistic (see e.g. [16]) is given by

$$\mathrm{KS} = \max_{x \in \mathbb{R}} \left| F_{\mathrm{emp}}(x) - F_{\mathrm{fit}}(x) \right|,$$

where F_{fit} is the fitted distribution function and F_{emp} is the empirical distribution function , which is given by

$$F_{\mathrm{emp}}(x) = \frac{\#(X_i \leq x,\, i = 1, \ldots, n)}{n}, \qquad n = \text{the number of total observations.}$$

In practice, we can calculate KS distance with the following formula (see e.g. [16]):

$$\mathrm{KS} = \max_{1 \leq i \leq n} \left(\max\left\{ \left|(i - 0.5)/n - F(X_{(i)})\right|, \left|(i + 0.5)/n - F(X_{(i)})\right| \right\} \right),$$

where $X_{(1)} \leq \ldots \leq X_{(n)}$ is the ordered data set.

2.3.2 Testing Results

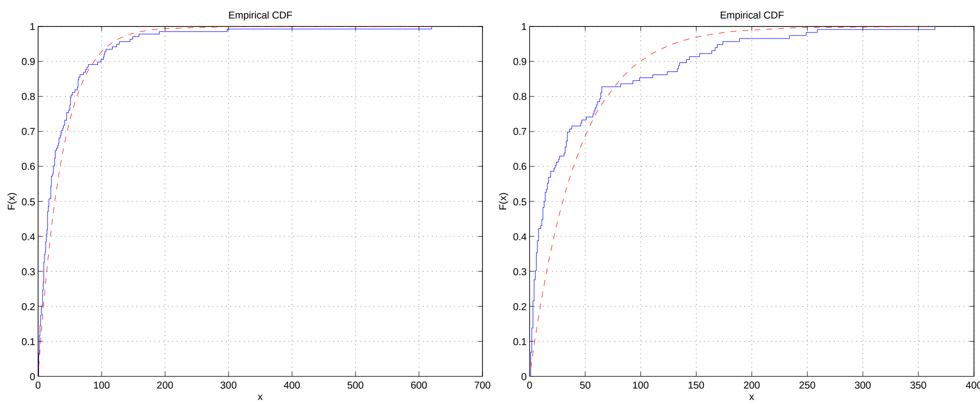
First we use the Maximum Likelihood (ML) method to estimate the parameter λ of the exponential probability density function,

$$f_{\text{Exp}}(x) = \lambda e^{-\lambda x}, \qquad x > 0,$$

which is the distribution of durations between jumps in Poisson models. Then we test goodness-of-fit with the KS-distance.

	λ	KS test
durations for jump-up	0.0263	0.1481
durations for jump-down	0.0232	0.2529

Table 2.3: The estimated parameters and KS-distance, with MLE method



(a) The duration between upward jumps (b) The duration between downward jump

Figure 2.6: The comparison between empirical distribution of duration and exponential distribution with MLE method

Observing Figure 2.6, we can notice that the empirical distribution of duration between upward jumps fits the exponential distribution quite well, though its KS-distance 0.1481 is a little too large to indicate that we really have exponential distribution; however for the durations between downward jumps, the empirical does not fit Exponential distribution well, and its KS-distance is also very large, 0.2529. Hence we conclude that Exponential distribution is not the best distribution for the jump duration.

Chapter 3

Distribution Fitting

In the previous chapter, we saw that the standard distribution assumptions about the exponential distribution and the uniform distribution can not fit the data (jump size and duration between jumps) well. In this chapter, we will try to find proper distributions for these data.

3.1 Tail Behavior

3.1.1 Mean Excess Plot

With the mean excess plot, we can test the tail properties of jump size and jump duration graphically. And this will be helpful to choose the proper distribution for the data of jump size and jump duration, as it is usually in the tails problems with fitted distributions occur.

The main idea is that with the help of mean excess plot, we choose the threshold u such that the sample mean excess function is nearly linear above u, (see e.g. [15]). We then consider the sample data, which exceed the threshold, to belong to the tail. Then we can decide whether the tail is heavy or light.

Definition 5 (Mean excess function, see [10]) *Let X be a random variable, then*

$$e(u) = E(X - u|X > u), \quad u \geq 0,$$

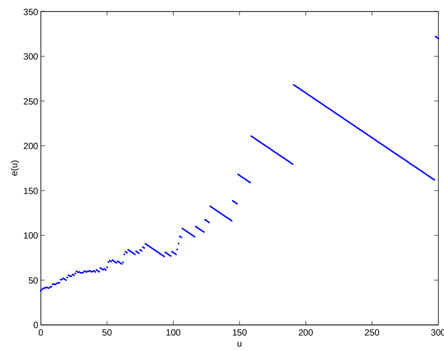
is called the mean excess function of X.

The sample mean excess function is given by

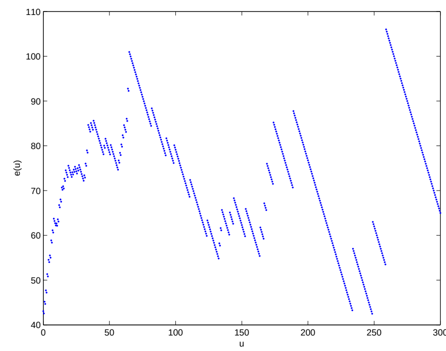
$$e_n(u) = \frac{1}{N_u}\sum_{i=1}^{n}(X_i - u)\mathbb{I}\{X_i > u\},$$

where $X_1 \dots X_n$, are iid random variables, u is the threshold, $N_u = \#\{i : 1 \leq i \leq n, X_i > u\}$ and $\mathbb{I}\{X > u\}$ is an indicator function. The mean excess plot consists of the points (see e.g. [15])

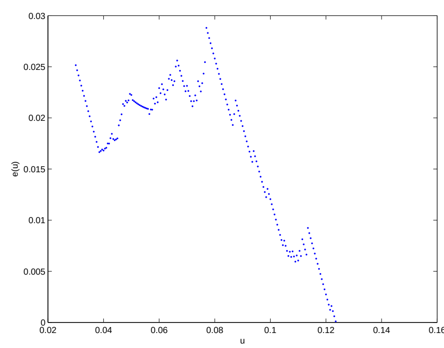
$$\{(X_{k,n}, e_n(X_{k,n})) : k = 1, \dots, n-1\}.$$



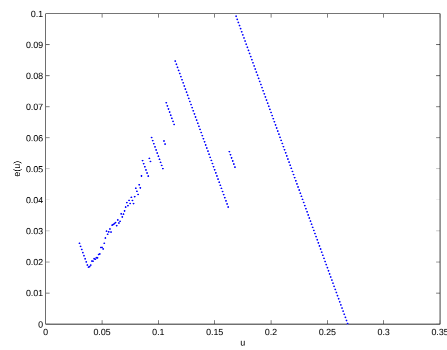
(a) Duration for upward jumps



(b) Duration for downward jumps



(c) Size of upward jumps



(d) Size of downward jumps

Figure 3.1: Mean Excess Plot

Using the mean excess plot, we find the threshold for duration to be 60 and the threshold for size to be 0.06.

3.1.2 Log Transformation

One example of a heavy-tail distribution given by the probability density function

$$f(x) = \frac{(1 + (x/\delta)^{\rho})^{-\beta}}{C(\delta, \rho, \beta)}, \quad x > 0,$$

where $\delta > 0$, $\rho > 0$ and $\beta > 0$. Heavy-tail distributions are also called polynomial distributions, since when x is large,

$$1 - F(x) \approx Cx^{-\rho},$$

Now for some $C > 0$ and $\rho > 0$, take logarithms on both sides and use the empirical distribution $F_{\text{Emp}}(X_{(i)}) = \frac{i}{n}$ instead of the distribution function $F(x)$, i.e.

$$\begin{aligned} 1 - F(x) &\approx Cx^{-\rho}, \\ \log(1 - F(x)) &\approx \log C - \rho \log x, \\ \log(1 - \frac{i}{n}) &\approx \log C - \rho \log x_i. \end{aligned} \tag{3.1}$$

Using the least square (LS) method to estimate $\log C$ and ρ, then we may observe whether the linear relation (3.1) really holds, and decide whether data are heavy-tail distributed.

One example of a light-tail distribution has a probability density function,

$$f(x) = \frac{x^{\beta-1}\mathrm{e}^{-\lambda x^{\alpha}}}{C(\beta,\lambda)}, \quad x > 0,$$

where $\alpha > 0$, $\beta > 0$ and $\lambda > 0$[1]. And light-tail distributions are also called exponential distributions, since when x is large,

$$1 - F(x) \approx C_1 x^{-\rho} \exp\{-C_2 x^{\alpha}\}.$$

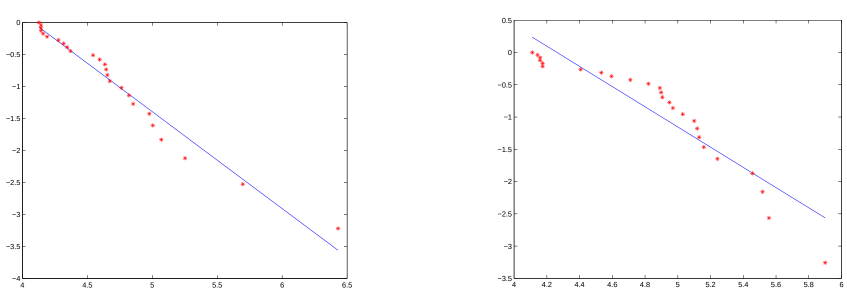
We can use the similar method as above to decide whether data are light-tail distributed; however here we need to take logarithm twice, i.e.

$$\begin{aligned}
1 - F(x) &\approx C_1 x^{-\rho} \exp\{-C_2 x^{\alpha}\}, \\
\log(1 - F(x)) &\approx \underbrace{\log C_1 - \rho \log x}_{negligible} - C_2 x^{\alpha} \\
&\approx -C_2 x^{\alpha}, \\
\log(-\log(1 - F(x))) &\approx \log C_2 + \alpha \log x, \\
\log(-\log(1 - \frac{i}{n})) &\approx \log C_2 + \alpha \log x_i. \qquad (3.2)
\end{aligned}$$

Now with the data of jump durations and jumps sizes, which exceed the thresholds, we may estimate the coefficients ρ and $\log C$ for heavy-tail distribution and check for linearity.

	ρ	$\log C$	No. of data
upward jump duration	1.5146	6.1787	25
downward jump duration	1.5650	6.6709	26

Table 3.1: coefficients for heavy-tail distribution, with the threshold $u = 60$



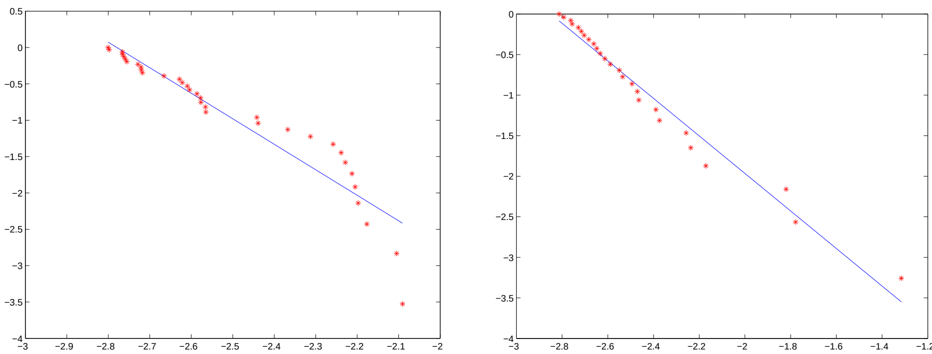
(a) Duration for upward jumps (b) Duration for downward jumps

Figure 3.2: Polynomial Curves for Jump Duration (Heavy-tail)

And similarly we estimate the coefficients α and $\log C_2$ for light-tail distribution and check the fit by looking for linearity in a plot of the transformed data (3.2).

	ρ	$\log C$	No. of data
upward jump size	3.5089	-9.7521	34
downward jump size	2.3132	-6.5922	26

Table 3.2: coefficients for heavy-tail distribution, with the threshold $u = 0.06$



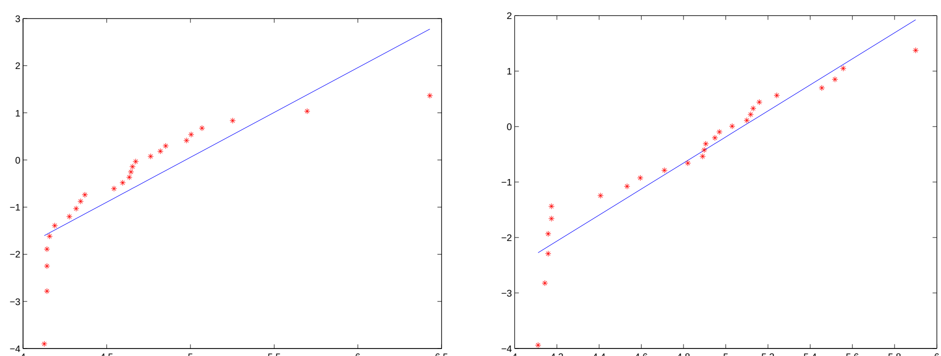
(a) Size of upward jumps

(b) Size of downward jumps

Figure 3.3: Polynomial Curves for Jump Size (Heavy-tail)

	α	$\log C_2$	No. of data
upward jump duration	1.9031	-9.4589	25
downward jump duration	2.3477	-11.9262	26

Table 3.3: coefficients for light-tail distribution, with the threshold $u = 60$



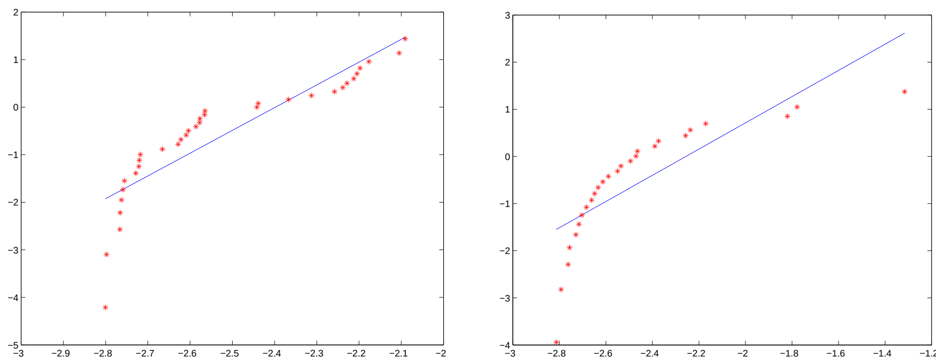
(a) Duration for upward jumps

(b) Duration for downward jumps

Figure 3.4: Exponential Curves for Jump Duration (Light-tail)

	α	$\log C_2$	No. of data
upward jump size	4.7874	11.4798	34
downward jump size	2.7809	6.2745	26

Table 3.4: coefficients for light-tail distribution, with the threshold $u = 0.06$



(a) Size of upward jumps (b) Size of downward jumps

Figure 3.5: Exponential Curves for Jump Size (Light-tail)

Observing the figures above, we conclude that the upward jump duration and the downward jump size are heavy-tail distributed, and the downward jump duration and the upward jump size are light-tail distributed.

3.2 Distributions

Here we choose four distributions to test. They are the Generalized Pareto Distribution, the Pearson VII Distribution, the Generalized Hyperbolic Distribution and the Gamma Distribution. The first two distributions have the heavy-tailed property; and the last two have the light-tailed property[2].

3.2.1 Generalized Pareto Distribution

The density function of the *Generalized Pareto* (GP) distribution is given by

$$f_{\mathrm{GP}}(x; u, \sigma, \xi) = \frac{1}{\sigma}(1 + \xi\frac{x-u}{\sigma})^{-1/\xi-1} \quad \text{for} \quad x > u,$$

where $u \in \mathbb{R}$ is a threshold, $\xi \geq 0$ is a shape parameter and $\sigma > 0$ is a scale parameter (see e.g. [16]). The distribution function is given by

$$F_{\mathrm{GP}}(x; u, \sigma, \xi) = 1 - (1 + \xi\frac{x-u}{\sigma})^{-1/\xi} \quad \text{for} \quad x > u.$$

3.2.2 Pearson VII Distribution

The *Pearson VII* distribution has the density function,

$$f_{\mathrm{Pearson}}(x; m, c) = \frac{2\Gamma(m)}{c\Gamma(m - \frac{1}{2}\sqrt{\pi})}(1 + (\frac{x}{c})^2)^{-m} \quad \text{for} \quad x > 0,$$

[1]In fact when $\alpha = 1$, this distribution is called semi-heavy-tail distribution.
[2]Actually they are semi-heavy tailed and have $\alpha = 1$.

with the distribution function,

$$F_{\text{Pearson}}(x;m,c) = \frac{2\Gamma(m)x{}_2F_1(\frac{1}{2},m,\frac{3}{2},-x^2/\sigma^2)}{\sqrt{\pi}\sigma\Gamma(m-\frac{1}{2})} \quad \text{for} \quad x>0,$$

where $m > 1/2$ is a shape parameter and $c > 0$ is a scale parameter (see e.g. [16]).

3.2.3 Generalized Hyperbolic Distribution

The probability density function for the *Generalized Hyperbolic* (GH) distribution is given by

$$\begin{aligned} f_{\text{GH}}(x;\lambda,\alpha,\beta,\delta,\mu) =& \frac{(\alpha^2-\beta^2)^{\lambda/2}(\delta^2+(x-\mu)^2)^{(\lambda-1/2)/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}\delta^{\lambda}K_{\lambda}(\delta\sqrt{\alpha^2-\beta^2})} \\ & \times K_{\lambda-1/2}(\alpha\sqrt{\delta^2+(x-\mu)^2})\mathrm{e}^{\beta(x-\mu)} \quad \text{for} \quad x\in\mathbb{R}, \end{aligned}$$

where K_λ is the modified Bessel function of the third kind

$$K_\lambda = \frac{1}{2}\int_0^\infty y^{\lambda-1}\mathrm{e}^{-x(y+1/y)/2}\mathrm{d}y \quad \text{for} \quad x\in\mathbb{R},$$

(see e.g. [16]). The permitted values of the parameters are as follows:

$$\lambda,\beta,\mu\in\mathbb{R} \quad \text{with} \quad \begin{cases} \delta\geq 0 \quad \text{and} \quad |\beta|<\alpha \quad \text{if} \quad \lambda>0, \\ \delta>0 \quad \text{and} \quad |\beta|<\alpha \quad \text{if} \quad \lambda=0, \\ \delta>0 \quad \text{and} \quad |\beta|\leq\alpha \quad \text{if} \quad \lambda<0. \end{cases}$$

3.2.4 Gamma Distribution

The probability density function for the *Gamma* distribution is given by

$$f_{\text{Gamma}} = \frac{x^{\alpha-1}\mathrm{e}^{-\lambda x}\lambda^{\alpha}}{\Gamma(\alpha)} \quad \text{for} \quad x\in[0,\infty),$$

and the corresponding distribution function is given by

$$F_{\text{Gamma}} = P(\alpha,\lambda x) \quad \text{for} \quad x\in[0,\infty),$$

where $P(a,z)$ is a regularized gamma function. (see e.g. [22])

3.3 Test Results

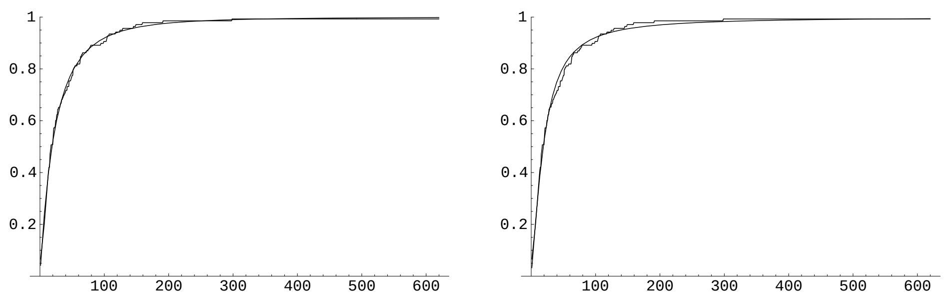
3.3.1 Heavy-tail Part

For the data of the upward jump duration and the downward jump size, which are considered as heavy-tail distributed, we use the GP distribution and the Pearson VII distribution to fit them with ML method. Notice that when estimating the parameters for the downward jump size, we shift the data set, i.e. let the data set minus the minimum value of the data. The fitting results are as follows.

	General Pareto		
	ξ	σ	KS Distance
Duration for jump-up	0.376565	23.5202	0.0398764
Size of jump-down	0.369871	0.0114926	0.0556826

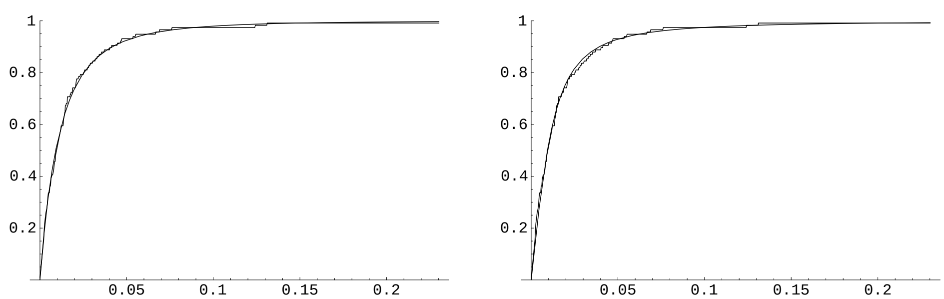
	Pearson VII		
	m	c	KS Distance
Duration for jump-up	1.20422	25.022	0.0556826
Size of jump-down	1.26035	0.0132479	0.0681976

Table 3.5: Parameters for GP and Pearson VII estimated by ML and KS distance



(a) GP distribution (b) Pearson VII distribution

Figure 3.6: Comparison of empirical distribution and fitted distribution for the data of upward jump duration



(a) GP distribution (b) Pearson VII distribution

Figure 3.7: Comparison of empirical distribution and fitted distribution for the data of downward jump size

3.3.2 Light-tail Part

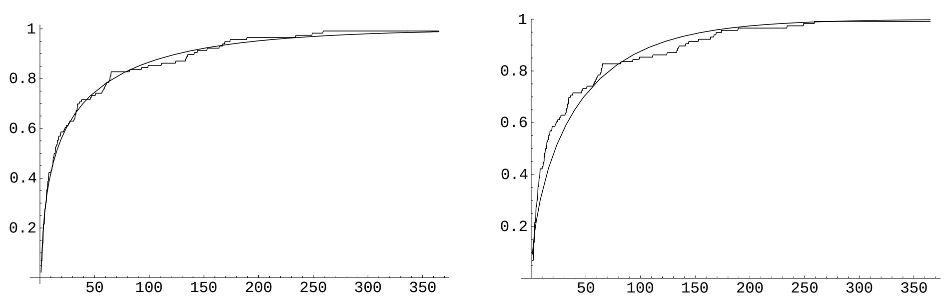
Similarly we use the GH distribution and the Gamma distribution to fit the data of the downward jump duration and the upward jump size, which were found to be light-tailed. Note that our data are all positive; however for the GH distribution, data can be both positive and negative. Hence it is necessary to set a cutoff point to keep the GH distribution just working on the positive side. Also another thing that need to be mentioned here is that when estimating the parameters of the Gamma distribution, we shift the data set; the idea is that we consider the data set minus the minimum value of the data, so that the whole data set starts out close to the zero point.[3] The fitting results are as follows.

	Generalized Hyperbolical Distribution			
	cutoff	λ	α	β
Duration for jump-down	-24.155	0.139297	4.31309	4.30681
Size of jump-up	1.94047	-4.80819	5.37966	4.55007
	δ	μ	KS Distance	
Duration for jump-down	-0.845762	0.79237	0.0545647	
Size of jump-up	0.0387968	-11.2119	0.113873	

Table 3.6: Parameters for GH distribution estimated by ML and KS distance

	Gamma distribution		
	α	λ	KS Distance
Duration for jump-down	0.576991	74.6291	0.132548
Size of jump-up	0.697845	0.0234174	0.0640025

Table 3.7: Parameters for Gamma distribution estimated by ML and KS distance



(a) GH distribution

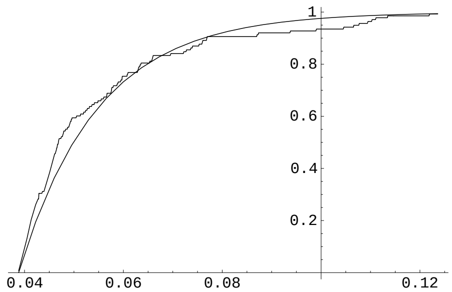
(b) Gamma distribution

Figure 3.8: Comparison of empirical distribution and fitted distribution for the data of downward jump duration

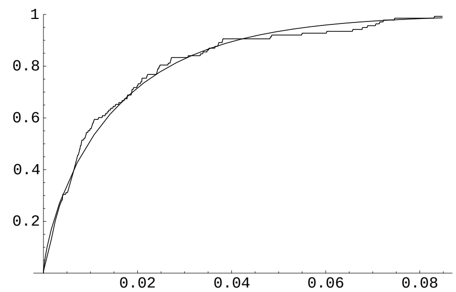
3.3.3 Conclusions

Observing the comparing plots of empirical distribution and fitted distribution, we find that for the upward jump duration and the downward jump size, both the GP distribution and the Pearson VII distribution fit data quite well; and according to

[3] In practice, we also have to plus a very small number, e.g. 0.000000001, to keep the data greater than zero such that the computer code can work correctly.



(a) GH distribution



(b) Gamma distribution

Figure 3.9: Comparison of empirical distribution and fitted distribution for the data of upward jump size

the KS distances, the GP distribution performs a little better than the Pearson VII distribution.

Further according to the comparing plots and the KS distances for the light-tail part, we consider that the GH distribution can fit the data of the downward jump duration well; and for the upward jump size, the Gamma distribution can fit well.

In conclusion, these test results show that the upward jump duration and the downward jump size are heavy-tail distributed, and the downward jump duration and the upward jump size are light-tail distributed.

Chapter 4

Dependence Structure Test

As discussed in Chapter 2, the jump process for asset pricing is usually modelled as a Poison process. This implies two assumptions. The first assumption that the duration between jumps are exponentially distributed fails, as we have shown in previous chapters. Another assumption, that the duration between jumps are independent random variables, also need to be checked. In this chapter, we will test dependence structure for both jump duration and jump size.

4.1 BDS Test

Here we use the BDS statistic (see [7]) to test dependence structure of the time series for both jump duration and jump size. The BDS statistic is defined as

$$w_{m,n}(\epsilon) = \sqrt{n-m+1}\,\frac{c_{m,n}(\epsilon) - c^m_{1,n-m+1}(\epsilon)}{\sigma_{m,n}(\epsilon)}. \tag{4.1}$$

Here n is the sample size, m is the embedding dimension, and $c_{m,n}(\epsilon)$ is defined as

$$c_{m,n}(\epsilon) = \frac{2}{(n-m+1)(n-m)} \sum_{s=m}^{n} \sum_{t=s+1}^{n} \prod_{j=0}^{m-1} I_\epsilon(X_{s-j}, X_{t-j}),$$

where

$$I_\epsilon(X_{s-j}, X_{t-j}) = \begin{cases} 1 & \text{if } |X_{s-j} - X_{t-j}| < \epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

And the consistent estimator $\sigma^2_{m,n}(\epsilon)$ is

$$\sigma^2_{m,n}(\epsilon) = 4\Big[k^m + 2\sum_{j=1}^{m-1} k^{m-j}c^{2j} + (m-1)^2c^{2m} - m^2kc^{2m-2}\Big],$$

where

$$c = c_{1,n}(\epsilon) \ ,$$

$$k = k_n(\epsilon) = \frac{6}{n(n-1)(n-2)} \sum_{t=1}^{n} \sum_{s=t+1}^{n} \sum_{r=s+1}^{n} h_\epsilon(X_t, X_s, X_r),$$

$$h_\epsilon(i,j,k) = \frac{1}{3}\big[I_\epsilon(i,j)I_\epsilon(j,k) + I_\epsilon(i,k)I_\epsilon(k,j) + I_\epsilon(j,i)I_\epsilon(i,k)\big].$$

Since the BDS statistic is asymptotically $N(0,1)$ distributed and is two-sided, the null of independence and identical distribution will be rejected at 5% level when $|w_{m,n}(\epsilon)| > 1.96$. (See [3])

4.2 Dependence Test Results

This test is carried out with the Matlab code of Ludwig Kanzler, (see [17]). Here the testing objectives are non-normally distributed and with a small sample size, therefore, besides the `bds` function, we also need the `bdssig` function. The return value of the `bdssig` function will be 0.005, 0.01, 0.025, 0.05 or 1. For example, when the return value is 0.01, the equivalent two-sided significance level will approximately be 0.02, (the detailed description for these two functions can also be found in [4]). The test results are as follows.

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
w	0.5985	0.9521	0.7380	0.3449	0.0172	−0.5839	−0.8183	−0.8076	−0.7000
return	1	1	1	1	1	1	1	1	1

Table 4.1: Size of upward jump

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
w	1.3159	1.4190	0.9864	0.8537	0.0529	−0.6276	−1.1339	−0.8389	−0.7461
return	1	1	1	1	1	1	1	1	1

Table 4.2: Size of downward jump

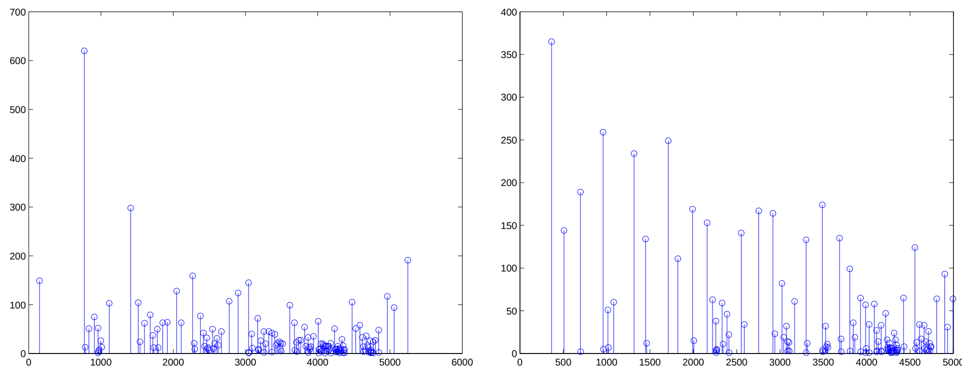
	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
w	1.3159	5.5038	6.3020	6.4904	6.3721	6.7194	7.2741	7.8510	8.3401
return	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050	0.0050

Table 4.3: Duration for upward jump

	$m=2$	$m=3$	$m=4$	$m=5$	$m=6$	$m=7$	$m=8$	$m=9$	$m=10$
w	1.2599	2.1148	2.6047	3.2856	3.2031	3.7322	4.4233	5.1866	5.8934
return	1.0000	1.0000	0.0500	0.0250	0.0250	0.0100	0.0050	0.0050	0.0050

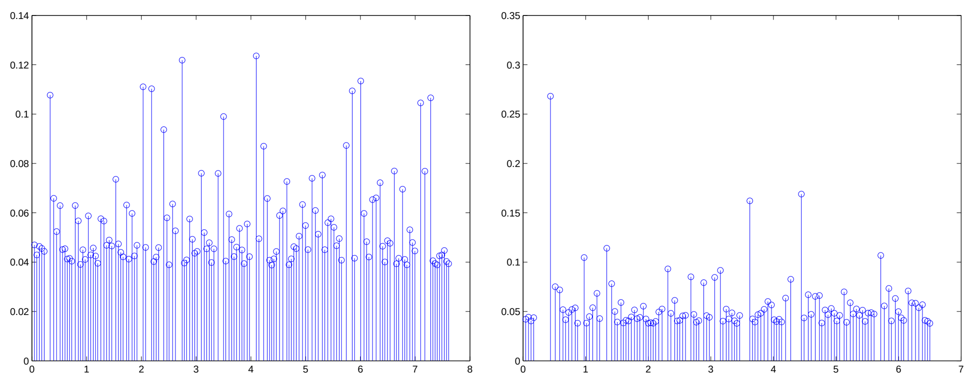
Table 4.4: Duration for downward jump

According to the test results in the above tables, we conclude that the data of jump sizes for both upward jumps and downward jumps are independent. However, for durations for upward jumps and downward jumps, the data are dependent. In the following figures, we also see that the durations of jump are clustered but the jump sizes are not.



(a) Duration for jump-up (b) Duration for jump-down

Figure 4.1: Stems of jump duration



(a) Size of jump-up (b) Size of jump-down

Figure 4.2: Stems of jump size

4.3 Modelling for Jump Duration

In the previous section, the BDS test results show that the jump durations for both jump-up and jump-down are correlated. In this section we will try to find a proper model for jump duration.

4.3.1 AR(1) Model

The AR(1) model is given by

$$y_n = a_0 + a_1 y_{n-1} + \sigma \epsilon_n, \tag{4.2}$$

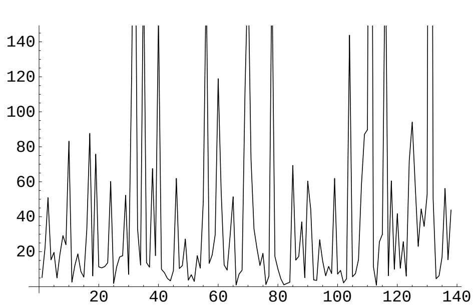
where a_0 and a_1 are constants, and $\epsilon_n \sim N(0,1)$.

Here, taking the logarithm for data of jump-up duration and jump-down duration, we estimate the parameters a_0 and a_1 by LS method, and estimate the parameter σ of the residual by ML method. The estimated parameters are listed in the following table.

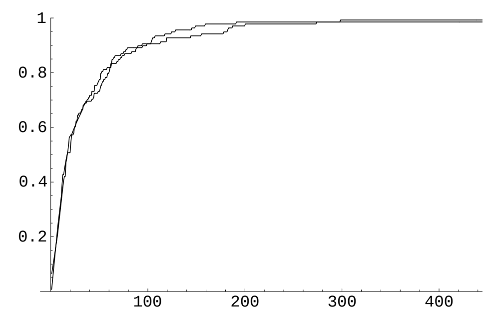
	a_0	a_1	σ
Duration for jump-up	2.17197	0.223581	1.31999
Duration for jump-down	2.34962	0.115215	1.53022

Table 4.5: Estimated parameters for jump duration

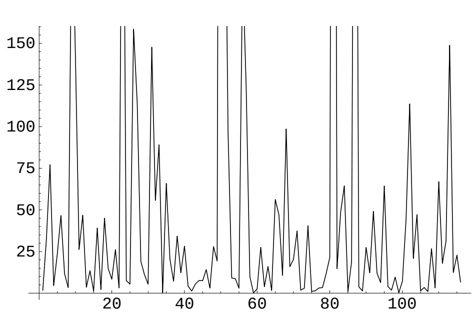
With the estimated parameters for the jump-up duration, we simulate one sample path and compare the empirical distribution of simulated sample data with the original data. And same to the jump-down duration.



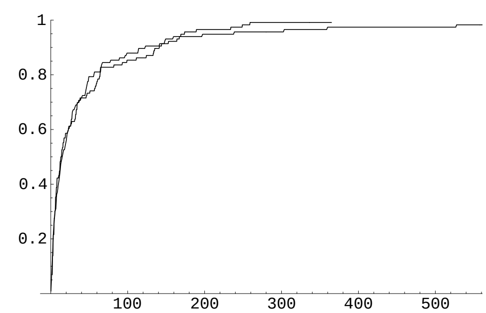
(a) One simulated sample path for jump-up duration



(b) Comparison of empirical distribution for jump-up duration



(c) One simulated sample path for jump-down duration



(d) Comparison of empirical distribution for jump-down duration

Figure 4.3: Simulated Sample Paths and Empirical Distribution Comparing

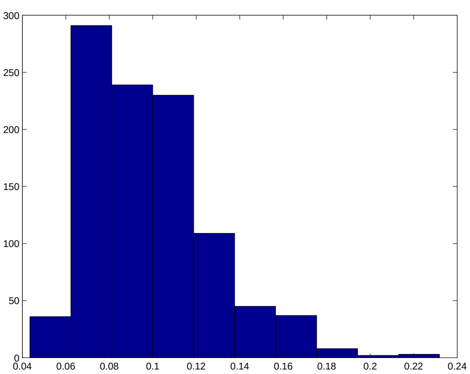
Observing the comparing plots of empirical distributions, we find that the empirical distributions of the data simulated by the AR(1) model can fit the empirical distributions of the original data very well. For further verification of the AR(1) model, we simulated 1000 sample paths for jump-up and jump-down respectively and compare their empirical distributions with the empirical distributions of the original logarithmic data with the KS distance test[1].

	mean	variance	min	max
KS distance	0.0993	$8.1053e-4$	0.0435	0.2319
p value	0.535	0.0776	$9.4238e-4$	0.9993

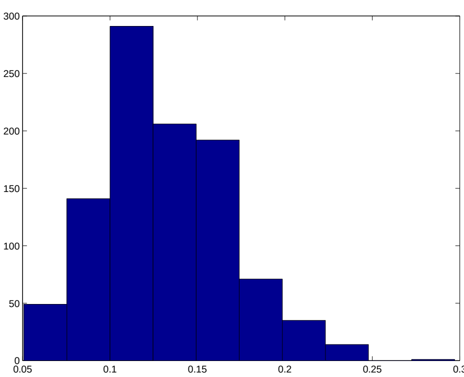
Table 4.6: KS distance and p value for 1000 simulations of jump-up duration

	mean	variance	min	max
KS distance	0.1322	0.0014	0.0507	0.2917
p value	0.2706	0.0651	$6.9142e-6$	0.9993

Table 4.7: KS distance and p value for 1000 simulated paths of jump-down duration



(a) Simulations for jump-up duration



(b) Simulation for jump-down duration

Figure 4.4: Histogram of KS distance

According to the results of KS distance test, there are 33 simulations for jump-up duration rejected by the KS test in 1000 simulations; and for jump-down duration there are 198 simulations rejected in 1000 simulations. Hence we can consider that the AR(1) model performs very well for jump-up duration. And though in the jump-down duration case, the performance of the AR(1) model is not as good as that in jump-up duration case, we still consider the AR(1) model as a reasonably proper model for jump-down duration.

[1]Here we use the Matlab built-in function, "kstest2".

Chapter 5

Counting Process

Letting $N(t)$ be a counting process for counting the jump times, we consider the jump model for asset pricing can be

$$S_t = S_0 \exp\Big\{\mu t + \sigma W_t + \sum_{i=1}^{N^1(t)} \xi_i^1 - \sum_{i=1}^{N^2(t)} \xi_i^2\Big\}, \tag{5.1}$$

where $N^1(t)$ and $N^2(t)$ count the number of upward jumps and downward jumps separately; ξ_i^1 and ξ_i^2 are the upward and downward jump size. In the last chapter, we find that duration for jumps are correlated and the logarithm of the data can be an AR(1) process. With these results, we will investigate the relation between the jump duration model and the intensity process, $\lambda(t)$, of the counting process. Moreover we will also discuss the properties of the counting process $N(t)$.

5.1 Stochastic Intensity

Definition 6 (Stochastic Intensity, (see C. G. Bowsher [6] Definition 3)**)** *Let $N(t)$ be a simple point process on $[0,\infty]$ that is adapted to some filtration $\{\mathcal{F}_t\}$, and let $\lambda(t)$ be a positive, $\mathcal{F}_t$-predictable process. If*

$$E[N(s) - N(t)|F_t] = E[\int_t^s \lambda(u)\mathrm{d}u|\mathcal{F}_t] \quad P\text{-}a.s.,$$

for all t, s such that $0 \le t \le s$, then $\lambda(t)$ is the $(P, \mathcal{F}_t)$-intensity of $N(t)$.

The relation between the stochastic intensity $\lambda(t)$ and the compensator of a counting process $A(t)$ is given by

$$A(t) = \int_0^t \lambda(s)\mathrm{d}s.$$

If $A(t)$ is differentiable, then

$$\lambda(t) = A'(t-),$$

(see [18]). Further the compensator $A(t)$ of $N(t)$ exists uniquely. Hence the stochastic intensity $\lambda(t)$ also exists uniquely. In other words, one predictable process $\lambda(t)$ can characterize one counting process $N(t)$.

Theorem 1 (F. C. Klebaner [18] Theorem 9.6**)** *Let N be a counting process generated by the sequence T_n, and denote by $U_{n+1} = T_{n+1} - T_n$ the interarrival times, $T_0 = 0$. Let $F_n(t) = P(U_{n+1} \leq t | T_1, \ldots, T_n)$ denote the regular conditional distributions, and $F_0 = P(T_1 \leq t)$. Then the compensator $A(t)$ is given by*

$$A(t) = \sum_{i=0}^{\infty} \int_0^{t \wedge T_{i+1} - t \wedge T_i} \frac{\mathrm{d}F_i(s)}{1 - F_i(s-)}. \tag{5.2}$$

For a counting process, we have

$$N(t) = \sum_{n=1}^{\infty} I(T_n \leq t), \quad N(0) = 0,$$

where $T_1, T_2, \ldots$ denotes the arrival time of the events[1]. Applying Theorem 1, we can get the compensator $A(t)$ of the counting process, and if the conditional distribution F_n are continues with $F_0 = 0$, we can simplify equation (5.2) and get

$$A(t) = -\sum_{n=0}^{\infty} \log(1 - F_n(t \wedge T_{n+1} - t \wedge T_n)), \tag{5.3}$$

(see [18]). We have

$$t \wedge T_{n+1} = \begin{cases} t & \text{if } t < T_{n+1}, \\ T_{n+1} & \text{if } t \geq T_{n+1}, \end{cases} \tag{5.4}$$

and

$$t \wedge T_n = \begin{cases} t & \text{if } t < T_n, \\ T_n & \text{if } t \geq T_n. \end{cases} \tag{5.5}$$

Since $\{T_n\}$ are the sequence of arrival times, we have $T_{n+1} > T_n$. Hence with equation (5.4) and equation (5.5), we have

$$t \wedge T_{n+1} - t \wedge T_n = \begin{cases} 0 & \text{if } t < T_n < T_{n+1}, \\ t - T_n & \text{if } T_n \leq t < T_{n+1}, \\ T_{n+1} - T_n & \text{if } T_n < T_{n+1} \leq t. \end{cases} \tag{5.6}$$

Letting $g(t, T_n, T_{n+1})$ denote $t \wedge T_{n+1} - t \wedge T_n$, we can assume that $g(t, T_n, T_{n+1})$ is a continuous function of time t with "parameters" T_n and T_{n+1}. Note that T_n and T_{n+1} are not the parameters in normal sense, they are random variables (stopping times). In other words, T_n is distributed with a certain distribution, so is T_{n+1}.

The sequence of stopping time, $T_1, T_2, \ldots, T_n, \ldots$, divide $[0, \infty)$ into many small intervals, i.e. $[T_n, T_{n+1})$, $n = 1, 2, 3, \ldots$. For any certain time t, it must locate in one of these small intervals and must locate in only one interval, (since there is

[1] In this report, the events are the jumps of the stock prices.

no intersection part in these small intervals). Now let t locate in $[T_k, T_{k+1})$, i.e. $T_k \le t < T_{k+1}$. With equation (5.6), we know that for this interval, $[T_k, T_{k+1})$,

$$t \wedge T_{n+1} - t \wedge T_n = t - T_k.$$

And for the small intervals on the left side of $[T_k, T_{k+1})$, all of $T_1, T_2, \ldots, T_{k-1}$ are less than t. Hence for these intervals, $[T_j, T_{j+1})$, $j = 0, 1, \ldots, k-1$, with equation (5.6), we know that

$$t \wedge T_{n+1} - t \wedge T_n = T_{j+1} - T_j, \quad j = 0, 1, \ldots, k-1.$$

Similarly for the intervals on the right side of $[T_k, T_{k+1})$, all the stopping time, $T_{k+1}, T_{k+2}, \ldots$, are greater than t. Hence for these intervals, $[T_i, T_{i+1})$, $i = k+1, k+2, \ldots$, according to equation (5.6), we can get that

$$t \wedge T_{n+1} - t \wedge T_n = 0, \quad i = k+1, k+2, \ldots \quad .$$

Through the analysis above, we can expand the series on right-hand side of equation (5.3) in the following form,

$$\begin{aligned} A(t) &= -\sum_{n=0}^{\infty} \log(1 - F_n(t \wedge T_{n+1} - t \wedge T_n)) \\ &= -\Big[\log(1 - F_1(T_2 - T_1)) + \log(1 - F_2(T_3 - T_2)) + \cdots + \\ &\quad + \log(1 - F_{k-1}(T_k - T_{k-1})) + \log(1 - F_k(t - T_k)) + \\ &\quad + \log(1 - F_{k+1}(0)) + \log(1 - F_{k+2}(0)) + \cdots\cdots\Big]. \end{aligned} \tag{5.7}$$

Using equation (5.7), we find that there is only one term containing the time t, that is the term $\log(1 - F_k(t - T_k))$. Hence differentiation of both sides of equation (5.7) gives

$$\begin{aligned} A'(t) &= -\frac{\mathrm{d}}{\mathrm{d}t} \log(1 - F_k(t \wedge T_{k+1} - t \wedge T_k)) \\ &= -\frac{\mathrm{d}}{\mathrm{d}t} \log(1 - F_k(t - T_k)) \\ &= \frac{F'_k(t - T_k)\frac{\mathrm{d}}{\mathrm{d}t}(t - T_k)}{1 - F_k(t - T_k)} \\ &= \frac{f_k(t - T_k)}{1 - F_k(t - T_k)}. \end{aligned} \tag{5.8}$$

Here we assume the conditional distribution function $F_k(s)$ is differentiable and let $f_k(s) = \frac{\mathrm{d}}{\mathrm{d}s} F_k(s)$ denote the crresponding conditional density function.

However we need to notice that $T_1, T_2, \ldots$ are a sequence of random variables, so for a certain time t, the small interval $[T_k, T_{k+1})$ in which t is located is also in random, i.e. k is a random variable. Hence to improve the expression of $A'(t)$ in equation (5.8), we introduce two random variables, $T_{N(t)}$ and $T_{N(t)+1}$, in which the subscripts are random. Let $T_{N(t)}$ denote the time when the last event happens just before time t (or exactly at time t) and $T_{N(t)+1}$ denote the time when the first event happens after time t. With these two random variable, we have

$$T_{N(t)} \le t < T_{N(t)+1},$$

in other words, for any certain time t, it is located in the random interval $[T_{N(t)}, T_{N(t)+1})$. Hence we use $N(t)$ instead of k in equation (5.8) and get a new expression of $A'(t)$,

$$A'(t) = \frac{f_{N(t)}(t - T_{N(t)})}{1 - F_{N(t)}(t - T_{N(t)})}. \tag{5.9}$$

If we assume that all the conditional density function $f_n(s)$, $n = 1, 2, \ldots$ are continuous functions, then equation (5.9) is also the expression of the stochastic intensity $\lambda(t) = A'(t-)$, i.e.

$$\lambda(t) = \frac{f_{N(t)}(t - T_{N(t)})}{1 - F_{N(t)}(t - T_{N(t)})}. \tag{5.10}$$

Based on our result in the previous chapter, i.e. the logarithm of the jump duration can be an AR(1) process, we can use the expression (5.10) to compute $\lambda(t)$. Recall that the AR(1) model is given by

$$y_n = a_0 + a_1 y_{n-1} + \sigma\epsilon_n,$$

where a_0 and a_1 are constant, and $\epsilon_n \sim N(0, 1)$. And in Theorem 1, we let U_n denote the interarrival time between the nth event and the $n-1$th event[2]. Then we have

$$y_n = \log U_n$$

and

$$U_n = T_n - T_{n-1}.$$

Since

$$U_{n+1} \le t \Longleftrightarrow \mathrm{e}^{y_{n+1}} \le t,$$

we can get

$$\begin{aligned} F_n(t) &= P(U_{n+1} \le t | T_1, \ldots, T_n) \\ &= P(\mathrm{e}^{y_{n+1}} \le t | T_1, \ldots, T_n) \\ &= P(\mathrm{e}^{a_0 + a_1 y_n + \sigma\epsilon_{n+1}} \le t | y_n) \\ &= P(\mathrm{e}^{\sigma\epsilon_{n+1}} \le t\mathrm{e}^{-a_0 - a_1 y_n} | y_n) \\ &= \psi(t\mathrm{e}^{-a_0 - a_1 y_n}), \end{aligned} \tag{5.11}$$

here $\psi(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_0^x \frac{1}{u} \mathrm{e}^{-\frac{(\ln u)^2}{2\sigma^2}} \mathrm{d}u$ is the lognormal distribution function with parameters $(0, \sigma)$. This is result quite obvious, since that in $\mathrm{e}^{a_0 + a_1 y_n + \sigma\epsilon_{n+1}}$, y_n just depends

[2]In this report, U_n is also called as the duration between the jumps.

on $T_n - T_{n-1}$ and ϵ_{n+1} is independent from $T_1, \ldots, T_n$; hence $\mathrm{e}^{a_0+a_1y_n+\sigma\epsilon_{n+1}}$ just depends on $T_n - T_{n-1}$, i.e. just depends on U_n; or we can say it just depends on y_n.

From (5.11), we have

$$\begin{aligned} F_n(t - T_{N(t)}) &= \psi((t - T_{N(t)})\mathrm{e}^{-(a_0+a_1y_n)}) \\ &= \psi((t - T_{N(t)})\mathrm{e}^{-a_0}U_n^{-a_1}) \\ &= \psi((t - T_{N(t)})\mathrm{e}^{-a_0}(T_n - T_{n-1})^{-a_1}). \end{aligned} \tag{5.12}$$

Since $F_{N(t)}(s)$ must be one of $F_0(s), F_1(s), \ldots, F_n(s)$, we therefore get

$$F_{N_{(t)}}(t - T_{N(t)}) = \psi((t - T_{N(t)})\mathrm{e}^{-a_0}(T_{N_{(t)}} - T_{N_{(t)}-1})^{-a_1}), \tag{5.13}$$

and

$$\begin{aligned} f_{N(t)}(t - T_{N(t)}) &= F'_{N_{(t)}}(t - T_{N(t)}) \\ &= \frac{1}{\sqrt{2\pi}\sigma}\mathrm{e}^{-(a_0+a_1y_n)}\frac{1}{(t - T_{N(t)})\mathrm{e}^{-(a_0+a_1y_n)}} \cdot \mathrm{e}^{-\frac{[\log((t-T_{N(t)}))\mathrm{e}^{-(a_0+a_1y_n)}]^2}{2\sigma^2}} \\ &= \frac{\exp\Big\{-\frac{1}{2\sigma^2}\big[-a_0 + \log(t - T_{N(t)}) - a_1\log(T_{N(t)} - T_{N(t)-1})\big]^2\Big\}}{\sqrt{2\pi}\sigma(t - T_{N(t)})}. \end{aligned} \tag{5.14}$$

Now the stochastic intensity $\lambda(t)$ of the counting process $N(t)$ in equation (5.1) is given by

$$\lambda(t) = \frac{\exp\Big\{-\frac{1}{2\sigma^2}\big[-a_0 - a_1\log(T_{N(t)} - T_{N(t)-1}) + \log(t - T_{N(t)})\big]^2\Big\}}{\sqrt{2\pi}\sigma(t - T_{N(t)})\Big(1 - \psi\big(\mathrm{e}^{-a_0}(t - T_{N(t)})(T_{N(t)} - T_{N(t)-1})^{-a_1}\big)\Big)}, \tag{5.15}$$

where $\psi(x)$ is the lognormal distribution function with parameter $(0, \sigma)$.

5.2 Properties of the Counting Process

In the previous section, we discussed the counting process $N(t)$ with the stochastic intensity given by equation (5.15). Now we will further discuss the properties of this counting process model.

5.2.1 Distribution Function for the Interarrival Time

Here we will compute $F_{U_n}(t)$, the distribution function for the interarrival time $U_n, (n = 1, 2, \ldots)$.

We have $y_1 = a_0 + a_1y_0 + \sigma\epsilon_1$, $y_0 = \frac{a_0}{1-a_1}$ and get

$$\begin{aligned} y_1 &= a_0 + \frac{a_1a_0}{1 - a_1} + \sigma\epsilon_1 \\ &= \frac{a_0}{1 - a_1} + \sigma\epsilon_1, \end{aligned}$$

where $\epsilon_1 \sim N(0,1)$ and $y_1 \sim N(\frac{a_0}{1-a_1}, \sigma^2)$. Further $U_1 = \mathrm{e}^{y_1}$, so that U_1 is lognormal distributed.

For $F_{U_1}(t)$, we have

$$\begin{aligned} F_{U_1}(t) &= P(U_1 \le t) = F(\mathrm{e}^{y_1} \le t) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_0^t \frac{1}{u} \mathrm{e}^{-\frac{(\log u - \frac{a_0}{1-a_1})^2}{2\sigma^2}} \mathrm{d}u \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_0^t \frac{1}{u} \mathrm{e}^{-\frac{(\log(\mathrm{e}^{-\frac{a_0}{1-a_1}} u))^2}{2\sigma^2}} \mathrm{d}u. \end{aligned} \tag{5.16}$$

Let $\mathrm{e}^{-\frac{a_0}{1-a_1}} u = \tau$, then $\frac{1}{u}\mathrm{d}u = \mathrm{e}^{-\frac{a_0}{1-a_1}} \frac{1}{\tau} \cdot \mathrm{e}^{\frac{a_0}{1-a_1}} \mathrm{d}\tau = \frac{1}{\tau}\mathrm{d}\tau$. And equation (5.16) can be rewritten as

$$\begin{aligned} F_{U_1}(t) &= \frac{1}{\sqrt{2\pi}\sigma} \int_0^{t\mathrm{e}^{-\frac{a_0}{1-a_1}}} \frac{1}{\tau} \mathrm{e}^{-\frac{\log(\tau)^2}{2\sigma^2}} \mathrm{d}\tau \\ &= \psi(t\mathrm{e}^{-\frac{a_0}{1-a_1}}), \end{aligned} \tag{5.17}$$

where

$$\psi(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_0^x \frac{1}{u} \mathrm{e}^{-\frac{(\log u)^2}{2\sigma^2}} \mathrm{d}u$$

is the lognormal distribution function with parameters $(0, \sigma)$.

For $F_{U_n}(t)$, $n = 2, 3, \ldots$, we have

$$F_{U_n(t)} = P(U_n \le t) = \int_0^\infty P(U_n \le t | U_{n-1} = \tau) f_{U_{n-1}} \mathrm{d}\tau, \tag{5.18}$$

where $f_{U_{n-1}}$ is the density function of the interarrival time U_{n-1}. And since $y_n = a_0 + a_1 y_{n-1} + \sigma\epsilon_n$ and $y_n = \log U_n$, we get

$$\begin{aligned} P(U_n \le t | U_{n-1} = \tau) &= P(\mathrm{e}^{y_n \le t} | \mathrm{e}^{y_{n-1}} = \tau) \\ &= P(\mathrm{e}^{a_0 + a_1 y_{n-1} + \sigma\epsilon_n} \le t | y_{n-1} = \log\tau) \\ &= P(\mathrm{e}^{\sigma\epsilon_n} \le t\mathrm{e}^{-a_0 - a_1 \log\tau} | y_{n-1} = \log\tau) \\ &= \psi(t\mathrm{e}^{-a_0 - a_1 \log\tau}). \end{aligned} \tag{5.19}$$

Letting $F(t,\tau) = \psi(t\mathrm{e}^{-a_0 - a_1 \log\tau})$, with equation (5.18), we get the recursive formula for $F_{U_n}(t)$,

$$F_{U_n}(t) = \int_0^\infty F(t,\tau) f_{U_{n-1}}(\tau) \mathrm{d}\tau. \tag{5.20}$$

Putting equation (5.16) and (5.20) together, we have the distribution function for the interarrival time

$$F_{U_n}(t) = \begin{cases} \psi(t\mathrm{e}^{-\frac{a_0}{1-a_1}}) & \text{if } n = 1, \\ \int_0^\infty F(t,\tau) f_{U_{n-1}}(\tau) \mathrm{d}\tau & \text{if } n > 1. \end{cases} \tag{5.21}$$

5.2.2 Distribution Function for the Arrival Time

Let $F_{T_n}(t)$ denote the distribution function for the arrival time $T_n, (n = 1, 2, \ldots)$. With the result of equation (5.16), we get

$$F_{T_1}(t) = P(T_1 \leq t) = P(U_1 \leq t) = \psi(t\mathrm{e}^{-\frac{a_0}{1-a_1}}). \tag{5.22}$$

For $F_{T_n}(t)$, $n = 2, 3, \ldots$, we have

$$\begin{aligned} F_{T_n}(t) = P(T_n \leq t) &= \int_0^t P(T_n \leq t | T_{n-1} = \tau) f_{T_{n-1}}(\tau)\mathrm{d}\tau \\ &= \int_0^t P(\sum_{j=1}^{n} U_j \leq t | \sum_{j=1}^{n-1} U_j = \tau) f_{T_{n-1}}(\tau)\mathrm{d}\tau \\ &= \int_0^t P(U_n \leq t - \tau | \sum_{j=1}^{n-1} U_j = \tau) f_{T_{n-1}}(\tau)\mathrm{d}\tau. \end{aligned} \tag{5.23}$$

Let $R_n(t, \tau) = P(U_n \leq t | \sum_{j=1}^{n-1} U_j = \tau)$. Then equation (5.23) can be written as

$$F_{T_n}(t) = \int_0^t R_n(t - \tau, \tau) f_{T_{n-1}}(\tau)\mathrm{d}\tau. \tag{5.24}$$

Now the function $R_n(t, \tau), (n \geq 2)$ can be computed recursively. Because $R_2(t, \tau) = P(U_2 \leq t | U_1 = \tau)$, with the definition of $F(t, \tau)$ in equation (5.19), we get

$$R_2(t, \tau) = F(t, \tau). \tag{5.25}$$

For $R_n(t, \tau), (n > 2)$, we have

$$R_n(t, \tau) = \int_0^\tau F(t, s) R'_{n-1}(s, \tau - s)\mathrm{d}s, \quad n > 2, \tag{5.26}$$

where $R'_{n-1}(s, u) = \frac{\partial}{\partial s} R_{n-1}(s, u)$. We prove this by mathematical induction:

First when $n = 3$,

$$\begin{aligned} R_3(t, \tau) &= P(U_3 \leq t | U_1 + U_2 = \tau) \\ &= \int_0^\tau P(U_3 \leq t | U_2 = s, U_1 + U_2 = \tau) f_{U_2}(s | U_1 + U_2 = \tau)\mathrm{d}s \\ &= \int_0^\tau P(U_3 \leq t | U_2 = s) f_{U_2}(s | U_1 = \tau - s)\mathrm{d}s \qquad \text{(since } U_3 \text{ just depends on } U_2\text{)} \\ &= \int_0^\tau F(t, s) R'_2(s, \tau - s)\mathrm{d}s. \end{aligned} \tag{5.27}$$

If the formula for $R_n(t,\tau)$ holds, then

$$\begin{aligned}
R_{n+1}(t,\tau) &= P(U_{n+1} \le t | \sum_{j=1}^{n} U_j = \tau) \\
&= \int_0^\tau P(U_{n+1} \le t | U_n = s, \sum_{j=1}^{n} U_j = \tau) f_{U_n}(s | \sum_{j=1}^{n} U_j = \tau)\mathrm{d}s \\
&= \int_0^\tau P(U_{n+1} \le t | U_n = s) f_{U_n}(s | \sum_{j=1}^{n-1} U_j = \tau - s)\mathrm{d}s \\
&= \int_0^\tau F(t,s) R_n'(s, \tau - s)\mathrm{d}s. \qquad (5.28)
\end{aligned}$$

Hence equation (5.26) holds by induction.

Now we can write the distribution function for arrival time

$$F_{T_n}(t) = \begin{cases} \psi(t\mathrm{e}^{-\frac{a_0}{1-a_1}}) & \text{if } n = 1, \\ \int_0^t R_n(t-\tau,\tau) f_{T_{n-1}}(\tau)\mathrm{d}\tau & \text{if } n \ge 2, \end{cases} \qquad (5.29)$$

where

$$R_n(t,\tau) = \begin{cases} F(t,\tau) & \text{if } n = 2, \\ \int_0^\tau F(t,s) R_{n-1}'(s,\tau - s)\mathrm{d}s & \text{if } n \ge 3, \end{cases} \qquad (5.30)$$

and $F(t,\tau) = \psi(t\mathrm{e}^{-a_0 - a_1 \log \tau})$.

5.2.3 Probability Distribution of N(t)

The probability of n jumps occurring up till time t is

$$\begin{aligned}
P\{N(t) = n\} &= P\{N(t) \ge n\} - P\{N(t) \ge n+1\} \\
&= P\{T_n \le t\} - P\{T_{n+1} \le t\} \\
&= F_{T_n}(t) - F_{T_{n+1}}(t) \\
&= \int_0^t \mathrm{d}F_{T_n}(t) - \int_0^t R_{n+1}(t-\tau,\tau) f_{T_n}(\tau)\mathrm{d}\tau \\
&= \int_0^t (1 - R_{n+1}(t-\tau,\tau))\mathrm{d}F_{T_n}(\tau). \qquad (5.31)
\end{aligned}$$

Since $N(t) = \sum_{n=1}^{\infty} I(T_n \le t)$, the mean value of the jump times in $[0,t]$ is

$$\begin{aligned}
\mathrm{E}[N(t)] &= \sum_{n=1}^{\infty} \mathrm{E}[I(T_n \le t)] \\
&= \sum_{n=1}^{\infty} P(T_n \le t) \\
&= \sum_{n=1}^{\infty} F_{T_n}(t). \qquad (5.32)
\end{aligned}$$

5.3 Conclusions

Using the assumption that the logarithm of the jump duration is an AR(1) process, we derive the analytical expression of the stochastic intensity $\lambda(t)$. Moreover through analysis of the properties of the counting process, we find the distributions for inter-arrival times or arrival times.

Chapter 6

Conclusions

In this project, we focus on jump characteristics for financial asset pricing. We give a formal definition of jumps, based on which we analyze the properties of jump sizes and jump durations for upward and downward jumps.

We have shown that the durations for jump-up and the sizes of jump-down are heavy-tailed distributed, while the durations for jump-down and the sizes of jump-up are light-tailed distributed. We also investigate the dependence structure for both jump size sequence and jump duration sequence. Moreover one very interesting result is that the duration between the jumps are correlated and the logarithm of the data can be modelled as an AR(1) process. Though our test result is based on one stock, we notice that it is a common phenomenon that in financial market, jumps are clustered. Hence we think that for most stocks, jump durations are dependent and can not simply be modelled as a Poisson process as is assumed in the literatures (see e.g. [14]).

Finally we discuss some properties of the new counting process model which we can derive form our empirical findings.

In the future work, we need to test more stocks, and verify whether other stocks have the similar tail properties of jumps as IBM stock and whether the jumps are also correlated. Further in the previous chapters, we saw that the AR(1) model for jump-down duration still can be improved: to find a better model for jump duration is also one important part in future research.

Bibliography

[1] P. K. ANDERSEN, Ø. BORGAN, R. D. GILL AND N. KEIDING *Statistical Models Based on Counting Processes* Springer-Verlag, 1993.

[2] OLE E. BARNDORFF-NIELSEI AND NEIL SHEPHARD *Financial volatility and Levy based models*
Unpublished manuscript, 2002.

[3] JORGE BELAIR-FRANCH AND DULCE CONTRERAS-BAYARRI *How to compute the BDS test: a software comparison.*
Journal of Applied Econometrics, Volume 17, Issue 6, $691 - 699$(2002).

[4] JORGE BELAIR-FRANCH AND DULCE CONTRERAS-BAYARRI *The BDS Test: A Practitioner's Guide.*
Unpublished, 2002.

[5] MATTIAS BENGTSSON AND VIKTOR OLSBO *Value at Risk Using Stochastic Volatility Models, Master Thesis.*
Department of Mathematical Statistics, Chalmers University of Technology and Gothenburg University, 2003.

[6] CLIVE G. BOWSHER *Modelling Security Market Events in Continuous Time: Intensity Based, Multivariate Point Process Models.*
`http://www.nuff.ox.ac.uk/economics/papers/2003/W3/Hawk_ECTA_finald.pdf`, 2003

[7] W. A. BROCK, W. D. DECHERT, J. A. SCHEINKMAN AND B. LEBARON
A Test for Independence Based on the Correlation Dimension.
Econometric Reviews, 15(3), $197 - 235$(1996).

[8] K. C. CHAN, G. A. KORALYI, F. A. LONGSTAFF AND A. B. SANDERS *An Empirical Comparison of Alternative Models of the Short-Term Interest Rate*
The Journal of Finance, Vol XLVII, No. 3, July 1992.

[9] RAMA CONT AND PETER TANKOV *Financial Modelling with Jump Processes*
Chapman & Hall/CRC, 2004.

[10] P. EMBRECHTS, C. KLÜPPELBERG AND T. MIKOSCH *Modelling Extremal Events for Insurance and Finance*
Springer, 1997.

[11] Walter Enders *Applied econometric time series*
Wiley New York, 2004.

[12] Rajna Gibson *Model Risk: Concepts, Calibration and Pricing.*
Risk Books, 2000.

[13] Christian Gouriéroux *ARCH Models and Financial Applications*
Springer, 1997.

[14] Floyd B. Hanson and J. J. Westman *Jump-Diffusion Stock Return Models in Finance: Stochastic Process Density with Uniform-Jump Amplitude*
ftp://www.math.uic.edu/pub/Hanson/MTNS2002/mtns02fmt.pdf, 2002.

[15] Henrik Hult and Filip Lindskog *Mathematical Methods in Risk Management*
Lecture Notes, October 23, 2004.

[16] Fredrik Jonsson *On the Fitting of Generalized Hyperbolical Distributions to Financial Data, Master Thesis.*
Department of Mathematical Statistics, Chalmers University of Technology and Gothenburg University, 2004.

[17] Ludwig Kanzler *MATLAB /Octave Code for BDS test.*
http://www2.gol.com/users/kanzler/index.htm

[18] Fima C. Klebaner *Introduction to Stochastic Calculus with Applications*
Imperial College Press, 1998.

[19] Anders Muszta *Contributions to Numerical Solution of Stochastic Differential Equations, Phd thesis.*
Department of Mathematical Statistics, Chalmers University of Technology and Gothenburg University, 2005.

[20] Steven E. Shreve *Stochastic Calculus for Finance II: Continuous-Time Models*
Springer, 2004.

[21] Zongwu Zhu and Floyd B. Hanson *Log-Double-Uniform Jump-diffusion Model.*
www.math.uic.edu/~hanson/pub/ZZhu/dbuniformfree2.pdf, 2004.

[22] Website of mathworld.
http://mathworld.wolfram.com

[23] Yahoo Finance
http://finance.yahoo.com/