Exact Simulation of Weak Solutions of Stochastic Differential Equations

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Abstract

In this thesis we investigate the Exact Algorithm that simulates weak, though not approximate, solutions of stochastic differential equations. The method is based on retrospective rejection sampling on measures and outputs a skeleton of a path ω.

To a start, we only consider stochastic differential equations with bounded drift and unit diffusion. Later, as the algorithm is extended, these constraints will be somewhat relaxed.

Solutions simulated with the Exact Algorithm are compared to ones approximated by Euler-Maruyama. By analyzing the behavior of the paths in several time instances, we conclude that the Exact Algorithm is not only accurate but also efficient.

Acknowledgements

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Contents

1 Preliminaries

In this chapter we will define some of the standard processes and methods to be used in this thesis. We assume that the reader has knowledge about probability spaces, stochastic processes and their corresponding filtrations.

1.1 The Brownian bridge

Definition 1. *Let W_t, $0 \leq t \leq T$, be a Brownian motion. Fix $a, b \in \mathbb{R}$ and define the* Brownian bridge *from a to b on $[0, T]$ as*

$$W_t^{a\to b} = a + \frac{(b-a)t}{T} + W_t - \frac{t}{T}W_T.$$

Then

$$\mathbb{E}[W_t^{a\to b}] = a + (b-a)\frac{t}{T}, \quad Var[W_t^{a\to b}] = \frac{t(T-t)}{T}.$$

For simplicity we denote the Brownian bridge $W_t^{0\to 0}$ by $\overline{W}_t$.

1.2 The Bessel bridge

A Bessel bridge of dimension n in time t is the distance to the origin of an n-dimensional Brownian bridge. Define the 3-dimensional Bessel bridge starting at 0 and ending at δ for $t \in [0, 1]$ as

$$B_t^{0\to\delta} \overset{\mathcal{D}}{=} \sqrt{\Big(\frac{\delta t}{\sqrt{3}} + W_{1,t}^{0\to 0}\Big)^2 + \Big(\frac{\delta t}{\sqrt{3}} + W_{2,t}^{0\to 0}\Big)^2 + \Big(\frac{\delta t}{\sqrt{3}} + W_{3,t}^{0\to 0}\Big)^2},$$

where $W_{i,t}^{0\to 0}$, $i = 1, 2, 3$ are three independent Brownian bridges starting and ending at 0 for $t \in [0, 1]$. Note that this process is positive, which will be crucial for the algorithms presented in Chapter 5.

1.3 Inverse Gaussian simulation

The inverse Gaussian distribution with parameters $\mu > 0$ and $\lambda > 0$ has probability density function (pdf)

$$f(x) = \sqrt{\frac{\lambda}{2\pi x^3}} \exp\Big\{\frac{-\lambda(x-\mu)^2}{2\mu^2 x}\Big\}, \quad for\ x > 0.$$

which we denote $X \sim$InvG(μ, λ). Drawing points from this distribution is not as straightforward as with the normal- or uniform distribution. John, William and Roy [9] suggest the following method that will result in a random number simulated from InvG(μ, λ);

First, let $X \sim$ InvG(μ, λ) and define Y as

$$Y = \lambda\frac{(X-\mu)^2}{X\mu^2}.$$

Next, solving this equality for X will yield the two roots

$$X_1 = \mu + \frac{\mu}{2\lambda}\big(\mu Y - \sqrt{4\mu\lambda Y + \mu^2 Y^2}\,\big)$$

and

$$X_2 = \frac{\mu^2}{X_1}.$$

Jonathan Shuster [12] showed that Y is a chi-square random variable with one degree of freedom. Using this, it is shown in John, William and Roy [9] that X can be simulated by choosing the root X_1 with probability $p_1 = \mu/(\mu + X_1)$ and X_2 with probability $p_2 = 1 - p_1$.

1.4 Rejection sampling

When dealing with none-standard distributions, rejection sampling will be used to simplify the simulations.

Algorithm 1 (Rejection sampling). *Let f be the pdf from which we want to simulate and g a pdf for which there are standard ways to draw from. Further, let $M > 1$ be a constant such that*

$$\frac{f(x)}{g(x)} \leq M \quad \forall x \in \mathbb{R}.$$

Perform the following steps:

1. *Sample x from g.*
2. *Sample u uniformly on $[0, 1]$.*
3. *If $u \leq f(x)/Mg(x)$, set $X = x$.*
4. *Otherwise, restart from 1.*

Now X is a random number with pdf f.

For an efficient algorithm it is crucial that the envelope function g is as similar to f as possible. As a matter of fact, a point x is accepted with probability

$$\begin{aligned}
\mathbb{P}\Big(\text{x accepted}\Big) &= \mathbb{P}\Big(u \leq \frac{f(x)}{Mg(x)}\Big) \\
&= \int_{\mathbb{R}} \mathbb{P}\Big(u \leq \frac{f(x)}{Mg(x)}\Big) g(x)dx \\
&\overset{u\sim Uni(0,1)}{=} \int_{\mathbb{R}} \frac{f(x)}{Mg(x)} g(x)dx \\
&= \frac{1}{M}.
\end{aligned}$$

2 Properties of stochastic differential equations

In this chapter we will present definitions and theorems involving stochastic integrals and probability transition densities for stochastic processes. We assume that a stochastic process Y_t on $[0,T]$ is $\mathcal{F}_t$-adapted. Furthermore, if

$$\mathbb{Q}(A) = \mathbb{P}\Big(\omega : Y(\omega) \in A\Big)$$

for Borel sets A of $C[0,T]$ we call $\mathbb{Q}$ *the distribution* of Y.

We consider stochastic differential equations (SDEs) of the type

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \tag{1}$$

where μ is the *drift coefficient* and σ the *diffusion coefficient*, satisfying the linear growth condition

$$|\mu(x)| + |\sigma(x)| \leq K(1+|x|), \;\; x \in \mathbb{R}, t \in [0,T]$$

for some constant K and the Lipschitz conditions

$$|\mu(x) - \mu(y)| + |\sigma(x) - \sigma(y)| \leq D|x-y|, \;\; x, y \in \mathbb{R}, t \in [0,T]$$

for some constant D.

2.1 Girsanov's theorem and other useful theory

Theorem 1 (Girsanov, see for instance [14], Chapter 7).
Let

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t, \;\; Y_0 = y_0, \;\; t \in [0,T]. \tag{2}$$

where $\mu, \sigma : \mathbb{R} \to \mathbb{R}$ and define

$$d\widetilde{W}_t = \gamma(Y_t)dt + dW_t, \;\; t \in [0,T] \tag{3}$$

where $\gamma(y) : \mathbb{R} \to \mathbb{R}$ is a bounded Borel function. Further, define

$$\widetilde{\mathbb{P}}(A) = \int_A Z d\mathbb{P}, \;\; A \in \widetilde{\mathcal{F}},$$

where $\widetilde{\mathcal{F}}$ is the σ-algebra generated by $\widetilde{W}$. Z is the Radon-Nikodym derivative for $\widetilde{\mathbb{P}}$ wrt $\mathbb{P}$ defined by

$$Z = Z_T,$$

where

$$Z_t = \exp\Big\{-\frac{1}{2}\int_0^t \gamma(Y_s)^2 ds - \int_0^t \gamma(Y_s)dW_s\Big\}.$$

Then $\widetilde{W}_t$ is a Brownian motion wrt $\widetilde{\mathbb{P}}$, $\mathbb{E}[Z] = 1$ and we have

$$dY_t = (\mu(Y_t) - \gamma(Y_t))dt + d\widetilde{W}. \tag{4}$$

This implies that $(Y_t, \widetilde{W}_t, \widetilde{\mathbb{P}})$ is a weak solution to (4) if $(Y_t, W_t, \mathbb{P})$ is a (weak or strong) solution of (2).

Theorem 2 (See for instance [6], p. 91). *Let $\mathbb{Q}$, $\mathbb{W}$ and $\mathbb{Z}$ be probability measures on $(\Omega, \mathcal{F})$ such that $\mathbb{Q}$ is absolutely continuous wrt $\mathbb{W}$ and $\mathbb{W}$ is absolutely continuous wrt $\mathbb{Z}$. Then*

$$\frac{d\mathbb{Q}}{d\mathbb{Z}} = \frac{d\mathbb{Q}}{d\mathbb{W}}\frac{d\mathbb{W}}{d\mathbb{Z}}$$

2.2 Transformation

The Exact Algorithm requires unit diffusion SDEs. For σ integrable, this requirement is achieved through the following transformation. Let X_t be a process defined by

$$X_t := g(Y_t) := \int_y^{Y_t} \frac{du}{\sigma(u)}, \quad y < Y_t, \tag{5}$$

where Y is given by (1). By applying Itôs formula, we derive the *SDE*

$$dX_t = g'(Y_t)dY_t + \frac{1}{2}g''(Y_t)d[Y,Y]_t{}^1.$$

Calculating the derivatives and inserting dY_t from (1) yields

$$dX_t = \Big(\frac{\mu(g^{-1}(X_t))}{\sigma(g^{-1}(X_t))} - \frac{1}{2}\sigma'(g^{-1}(X_t))\Big)dt + dW_t.$$

2.3 Transition densities

Let $\{Y_t\}_{t\geq 0}$ be a Markov stochastic process. A discrete set of observations $Y_{t_0}, \ldots, Y_{t_n}$ at time instances $t_0 < t_1 < \ldots < t_n$ has joint density function:

$$f_{Y_{t_0},Y_{t_1},\ldots,Y_{t_n}}(y_0, y_1, ..., y_n) = \Big(\prod_{i=1}^{n} f_{Y_{t_i}|Y_{t_{i-1}}}(y_i|y_{i-1})\Big) f_{Y_{t_0}}(y_0),$$

due to the Markov property of Y. Now, since $f_{Y_{t_i}|Y_{t_{i-1}}}(y_i|y_{i-1})$ is the transition density function for the stochastic process Y_t, we denote it, as in more common manners, with

$$L = \prod_{i=1}^{n} p(y_i, y_{i-1}, t_i, t_{i-1}). \tag{6}$$

When evaluating different numerical methods for solving stochastic differential equations, (6) will be used to perform the statistical tests. The following theorem provides a method to approximate the product (6), even if the analytical transition density function is unknown.

Theorem 3 ([13] page 97). *Consider the one-dimensional time homogeneous diffusion type SDE*

$$dY_t = \mu(Y_t)dt + \sigma(Y_t)dW_t \ \ for\ t \geq 0,$$

with $\mu \in \mathbb{C}^1(\mathbb{R})$ satisfying a global Lipschitz condition, and $\sigma : \mathbb{R} \to \mathbb{R}^+$ twice continuously differentiable (slightly weaker conditions work). Let

$$G(y) = \Big(\frac{\mu(g^{-1}(y))}{\sigma(g^{-1}(y))} - \frac{\sigma'(g^{-1}(y))}{2}\Big)^2 + \frac{d}{dy}\Big(\frac{\mu(g^{-1}(y))}{\sigma(g^{-1}(y))} - \frac{\sigma'(g^{-1}(y))}{2}\Big),$$

where $g(y) = \int_0^y 1/\sigma(z)dz$. Then the diffusion has transition density function

$$\begin{aligned} p(y,x,t,s) = \frac{\sqrt{\sigma(x)}}{\sqrt{2\pi(t-s)\sigma(y)^3}} \exp\Big\{-\frac{(g(y)-g(x))^2}{2(t-s)} + \int_x^y \frac{\mu(z)}{\sigma(z)^2}dz\Big\} \\ \times \mathbb{E}\Big[\exp\Big\{\frac{s-t}{2}\int_0^1 G\big(r(g(y)-g(x)) + g(x) + \sqrt{t-s}\ \overline{W}_r\big)\Big\}\Big], \end{aligned} \tag{7}$$

where $\overline{W}_t$ is a Brownian bridge on $[0,1]$.

[1] $d[Y,Y]_t$ denotes the quadratic variation of Y

For the unit diffusion case, (7) simplifies a lot. Since $g(y) = y$ we will end up with just

$$p(y,x,t,s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\Big\{-\frac{(y-x)^2}{2(t-s)} + \int_x^y \mu(z)dz\Big\} \\ \times \mathbb{E}\Big[\exp\Big\{\frac{s-t}{2}\int_0^1 G\big(r(y-x)+x+\sqrt{t-s}\,\overline{W}_r\big)\Big\}\Big],$$

where $G(y) = \mu(y)^2 + \mu'(y)$.

2.4 Euler-Maruyama

The approximative Euler-Maruyama (EM) scheme will be used as an aid when evaluating the Exact Algorithm (EA). EM will create a discretized realization of Y, using a somewhat naive method. Let t_i, $i = 0, 1, ...n$ be a partition of $[0, T]$. Then

$$Y_{t_{i+1}} = Y_{t_i} + \mu(Y_{t_i})\Delta t + \sigma(Y_{t_i})\Delta W,\ Y_{t_0} = y_0,$$

is an approximative strong solution to (1).

3 Exact Algorithm

The aim of this chapter is to present the theory behind the Exact Algorithm (EA). Later, this algorithm will be applied to a selection of examples, and we will investigate issues such as accuracy, efficiency considering time and also compare it to the EM scheme. But first, we formulate the problem to be solved.

The SDE (1) is often analytically unsolvable. If the functions μ and σ satisfy a number of conditions, usage of EA might be considered to simulate paths from the distribution of Y. Denote these paths as ω. To do this, first transform (1) to a SDE with unit diffusion, using (5):

$$dX_t = \alpha(X_t)dt + dW_t, t \in [0,T], \;\; X_0 = x_0 \in \mathbb{R} \tag{8}$$

Denote the distribution of the solution of this latter SDE by $\mathbb{Q}$. Note that after having simulated ω from $\mathbb{Q}$, we also need to invert the transformation in order for the path to be a solution of (1).

3.1 The first Exact Algorithm

3.1.1 Retrospective rejection sampling

The main idea behind exact simulation is to do a rejection sampling on two probability measures for random paths retrospectively. That is, first we generate the time instances at which we evaluate the rejection event. Then we simulate the path from the proposal measure $\mathbb{Z}$ at these times only, and call it a skeleton. Skeletons are accepted as an realization from $\mathbb{Q}$ if an event has occurred at each of the time instances. This is the major difference compared to approximation methods such as Euler-Maruyama. Instead of discretized the path and approximate it at each time step, we are actually able to produce a path from the sought distribution by a modified rejection sampling.

Theorem 4 (Rejection Sampling on Probability Measures, [5] p. 2425). *Let $\mathbb{Q}$ be as above and $\mathbb{Z}$ a probability measures on $C[0,T]$ such that $\mathbb{Q}$ is absolutely continuous wrt $\mathbb{Z}$. Assume there is an $\epsilon > 0$ such that*

$$f := \epsilon \frac{d\mathbb{Q}}{d\mathbb{Z}} \leq 1, \;\; \mathbb{Z}\text{-}a.s.$$

Let $(X_n, I_n)_{n>0}$ be i.i.d. random elements that belong to $C[0,T] \otimes \{0,1\}$ such that $X_1 \sim \mathbb{Z}$ and $\mathbb{P}(I_1 = 1 | X_1 = x) = f(x)$. Define $\tau = \min\{i > 0 : I_i = 0\}$. Then $X_\tau \sim \mathbb{Q}$.

Now, all we need are the two measures $\mathbb{Q}$ and $\mathbb{Z}$ and a set of conditions, ensuring that Theorem 4 can be applied. The theorem suggests that rejection sampling for weak solutions of (8) should be done in three major steps.

Algorithm 2 (Outline of the Exact Algorithm).
1: Draw a path ω from $\mathbb{Z}$
2: Simulate the rejection event I
3: If I occurred, accept ω as a realization from $\mathbb{Q}$. Otherwise, go to 1.

3.1.2 Finding $d\mathbb{Q}/d\mathbb{Z}$

Let $\mathbb{W}$ denote the measure induced by the Brownian motion. Theorem 1 provides what we need in order to derive the Radon-Nikodym derivative $Z = d\mathbb{Q}/d\mathbb{W}$ knowing only the SDE (8). Regarding $\mathbb{W}$, consider the SDE

$$dX_t = dW_t. \tag{9}$$

The solution is trivially distributed as $\mathbb{W}$. We derive Z by applying Theorem 1 on the SDEs (8) and (9) (with their corresponding probability measures);

$$Z = \exp\Big\{-\frac{1}{2}\int_0^T \alpha(X_t)^2 ds + \int_0^T \alpha(X_t) dW_t\Big\}. \tag{10}$$

Unfortunately it is difficult to work with Z in its current form. However, if we apply the Itô formula to the Itô-integral, (10) simplifies. Define $A(x)$ by

$$A(x) := \int_0^x \alpha(y)dy,\ x \in \mathbb{R}.$$

According to the Itô formula for an Itô-process:

$$A(X_T) = A(x_0) + \int_0^T A_x(X_t)dX_t + \frac{1}{2}\int_0^T A_{xx}(X_t)d[X,X]_t$$

$$\overset{dX_t = dW_t}{\Longleftrightarrow}$$

$$\int_0^T \alpha(X_t)dW_t = A(X_T) - A(x_0) - \frac{1}{2}\int_0^T \alpha'(X_t)dt,$$

thus, provided that α is differentiable, rewriting Z from the theorem above as

$$Z = \exp\{A(X_T) - A(x_0) - \frac{1}{2}\int_0^T (\alpha^2(X_t) + \alpha'(X_t))dt\}.$$

Now recall Theorem 2:

$$\frac{d\mathbb{Q}}{d\mathbb{Z}} = \frac{d\mathbb{Q}}{d\mathbb{W}}\frac{d\mathbb{W}}{d\mathbb{Z}},$$

calling for the Radon-Nikodym derivative $d\mathbb{W}/d\mathbb{Z}$.

Theorem 5 (End points, [3] p. 1079). *Let $\omega = \{\omega_t\}_{0\le t\le T}$ be an arbitrary element of $C([0,T])$, $M = \{M_t\}_{t=0}^T$ and $N = \{N_t\}_{t=0}^T$ be two stochastic processes on $C[0,T]$ with corresponding probability measures $\mathbb{M}$ and $\mathbb{N}$. Assume that f_M and f_N are the densities of the ending points M_T and N_T respectively such that*
$f_M(x), f_N(x) > 0,\ \forall x \in \mathbb{R}$. If $(M|M_T = p) \overset{\mathcal{D}}{=}$ [2] $(N|N_T = p)$ holds for all $p \in \mathbb{R}$, then

$$\frac{d\mathbb{M}}{d\mathbb{N}}(\omega) = \frac{f_M}{f_N}(\omega_T).$$

Definition 2. *Define the biased Brownian motion $\widehat{W}$ as*

$$\widehat{W} \overset{\mathcal{D}}{=} (W^{x_0}|W_T^{x_0} haspdfh)$$

[2] Equality in distribution

where $h(x) \propto$ [3] $\exp\{A(x) - (x - x_0)^2/2T\}$, $x \in \mathbb{R}$ and let $\mathbb{Z}$ be the distribution of $\widehat{W}$. Let $\mathbb{W}$ and $\mathbb{Z}$ correspond to $\mathbb{M}$ and $\mathbb{N}$ respectively in Theorem 5:

$$\frac{d\mathbb{W}}{d\mathbb{Z}} \propto \frac{\exp -(x - x_0)^2/2T}{h} = \exp\{-A(X_T)\}.$$

Finally:

$$\frac{d\mathbb{Q}}{d\mathbb{Z}} = \frac{d\mathbb{Q}}{d\mathbb{W}}\frac{d\mathbb{W}}{d\mathbb{Z}} \propto \exp\Big\{-\frac{1}{2}\int_0^T \alpha^2(X_t) + \alpha'(X_t)dt\Big\}. \tag{11}$$

For simplicity, define Θ as

$$\Theta(u) = \frac{\alpha^2(u) + \alpha'(u)}{2} - k, \tag{12}$$

where $k \in \mathbb{R}$ is chosen so that

$$k \leq \inf_{u\in\mathbb{R}}\Big\{\frac{\alpha^2(u) + \alpha'(u)}{2}\Big\}.$$

Provided that $\Theta(x)$ is bounded, we have:

$$\frac{d\mathbb{Q}}{d\mathbb{Z}} \propto \exp\Big\{-\int_0^T \Theta(X_t)dt\Big\} \leq 1,\ \mathbb{Z}\ a.s.,$$

thus fulfilling the first condition for Theorem 4. To conclude, these are the conditions needed to apply EA:

- α is differentiable (A1)
- Θ is bounded (A2)
- $h(x)$ is integrable (A3)

3.1.3 The rejection event

Left to do is to decide the rejection event I.

Theorem 6 ([3], p. 1080). *Let ω be an arbitrary element of $C([0,T])$, M an upper bound of $\Theta(\omega)$ and Φ a Poisson process with unit intensity on $[0,T] \times [0,M]$. Further, let N be the number of points of Φ found below $\Theta(\omega)$. Then, given this path ω:*

$$\mathbb{P}\Big(N = 0 \,\Big|\, \omega\Big) = \exp\Big\{-\int_0^T \Theta(\omega_t)dt\Big\}.$$

We now give an outline of the proof:
Since Φ is a Poisson process, the number of points N found below $\Theta(\omega)$ (in the same area that Φ operates) will also be Poisson distributed with parameter $\int_0^T \Theta(\omega_t)dt$. Hence, the probability of finding no points under the function $\Theta(\omega)$ is $\exp\{-\lambda_N\}0^{\lambda_N}/0! = \exp\{-\int_0^T \Theta(\omega_t)dt\}$. □

[3] $f \propto g$ if $f = C * g$ for some constant C

From Theorem 6 we now know the rejection event I; $1_{[N=0]}$. Let $((x_1, t_1), .., (x_n, t_n))$ be the Poisson process Φ mentioned above on $[0, T]$. Since we already have at which time instances the rejection event will be evaluated, it suffices to simulate the path's value ω_{t_i} only at $t_1, .., t_n$.

When simulating these ω_{t_i}s we perform a retrospective rejection sampling We simulate these ω_{t_i}s in the following manner: first we draw the endpoint ω_T from h (Definition 2). Then, since $\mathbb{Z}$ is the Wiener measure for $t \in [0, T)$, we can simulate ω_{t_i}, $i = t_1, .., t_{n-1}$ iteratively, using $\omega_{t_{i-1}}$ together with ω_T, starting with ω_{t_1}. More concrete, draw Brownian bridge proposals for ω_{t_i} from the normal distribution:

$$N\Big(\omega_{t_{i-1}} + \frac{\omega_T - \omega_{t_{i-1}}}{T - t_{i-1}}(t_i - t_{i-1}), \frac{(T - t_i)(t_i - t_{i-1})}{T - t_{i-1}}\Big). \tag{13}$$

This way we avoid drawing a whole path ω from $\mathbb{Z}$ (which is impossible!). The Idea behind this method is retrospective rejection sampling. The points ω_{t_i} are if accepted called a *skeleton* of ω, where ω is distributed according to $\mathbb{Q}$.

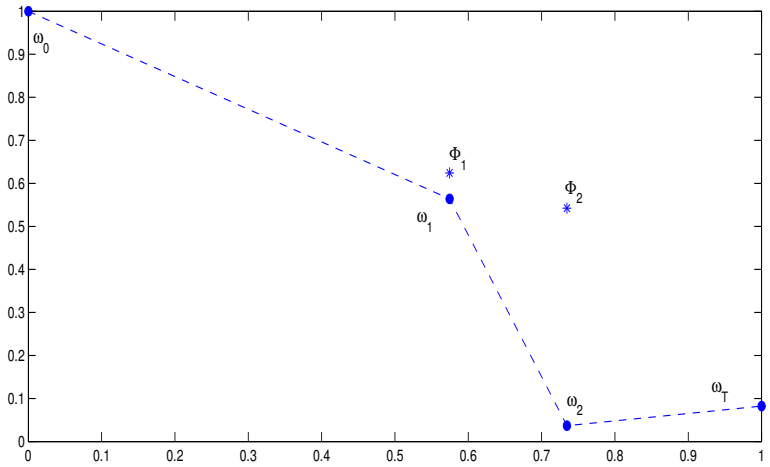

Figure 1: The skeleton points ω_0, ω_{t_1},ω_{t_2} and ω_T together with the Poisson points

Once we have the skeleton, $\omega_0, \omega_{t_1}, ..., \omega_{t_n}, \omega_T$, we can simulate ω_t at whatever time instance t desired. Just choose t_k such that $t_k \leq t \leq t_{k+1}$ and use the Brownian bridge setup above with $T = t_{k+1}$ and $t_{i-1} = t_k$. Then add ω_t into the skeleton.

Having this in mind, together with Theorem 4, we can finally define the Exact Algorithm in its complete form:

Algorithm 3 (The first Exact Algorithm).

1. *Draw the ending point* $\omega_{_T}$ *from* h, *for* $\omega \sim \mathbb{Z}$.
2. *Simulate* n, *the number of points in the Poisson process* Φ
   a: If $n = 0$, *set* $N = 0$ *and go to 5.*
   b: Else, draw n *points uniformly on* $[0, T] \times [0, M]$, $((t_1, m_1), ..., (t_n, m_n))$.
3. *for i=1 to n*
   Draw the point ω_i *at* t_i *from the normal distribution in (13).*
4. *Calculate* $N = \sum_{i=1}^{n} 1_{[m_i > \omega_{t_i}]}$.
5. *If the rejection event* $I = 1$, *output the skeleton of* ω, *otherwise go to 1.*

4 An extension of the Exact Algorithm

The constraint (A2) (Θ being bounded from above and below) limits EA to a small set of SDEs. To expand this set of SDEs (A2) will be reformulated as

$$\limsup_{x\to\infty} \Theta(x) < \infty \qquad (A2')$$

or

$$\limsup_{x\to-\infty} \Theta(x) < \infty$$

i.e. relaxing the restriction of Θ being bounded from above.

To control the proposed paths we will draw their minimum in addition to the start point and the endpoint (which is carried out as in the previous chapter). Then, by using two Bessel bridges, we connect these three points.

4.1 Drawing the minimum from a Brownian path

The distribution of the minimum m and its time of occurrence τ_m conditioned on the endpoint $z \in \mathbb{R}$ can be derived according to [7] (Chapter 2):

$$\mathbb{P}(m \in db, \tau_m \in dt | W_T = z) \propto \frac{b(b-z)}{\sqrt{t^3(T-t)}} \exp\left\{-\frac{b^2}{2t} - \frac{(b-z)^2}{2(T-t)}\right\} db\, dt \qquad (14)$$

where $b \leq min\{z, 0\}$ and $t \in [0, T]$. We use the Theorem below, which provides an algorithm that in a few steps renders points from the distribution above.

Theorem 7 ([3], p. 1083). *Let $E \sim \text{Exp}(1)$ and define $S_1 = (z - \sqrt{2ET + z^2}/2)$. If $S_1 = b$ is a realization of S_1, set $c_1 = (z-b)^2/2T$ and $c_2 = b^2/2T$.*
Let $U \sim Uni(0,1)$, $I_1 \sim InvG(\sqrt{c_1/c_2}, 2c_1)$ and $I_2 \sim 1/InvG(\sqrt{c_2/c_1}, 2c_2)$ independently, and define

$$V = \mathrm{I}_1 1_{[U<(1+1/\sqrt{c_1/c_2})]} + \mathrm{I}_2 1_{[U\geq(1+1/\sqrt{c_1/c_2})]}.$$

Then if $S_2 := T/(1+V)$, (S_1, S_2) is distributed according to (14).

Implementing this is rather straightforward in view of section 2.3 and the fact that there are numerous methods for drawing from $Exp(1)$ and $Uni(0,1)$. To simulate the remaining points of the skeleton, Bessel bridges will come in handy.

Let W^{m,τ_m} be a Brownian motion starting at 0, conditioned on ending at z and reaching its minimum m at time τ_m. Further, let $B_t^{0\to\delta}(3)$ be a three-dimensional Bessel bridge from 0 to $\delta \geq 0$ with unit length. This process will be denoted as B_t^δ, $0 \leq t \leq 1$.

Theorem 8 ([3], p. 1083). *The processes W_s^{m,τ_m}, $0 \leq s \leq t$ and W_s^{m,τ_m}, $t \leq s \leq T$ are independent and has the same distribution as*

$$\sqrt{t} B^{\delta_1}_{(t-s)/t} + m, 0 \leq s \leq t$$

and

$$\sqrt{T-t} B^{\delta_2}_{(s-t)/(T-t)} + m, t \leq s \leq T$$

respectively where $\delta_1 = -m/\sqrt{t}$ and $\delta_2 = (z-m)/\sqrt{T-t}$.

Now we are ready to present the first extension of EA, EA2.

Algorithm 4 (The second Exact Algorithm).

1. *Draw a point* $\omega_T \sim h$.
2. *Simulate the minimum* m *and its time of occurrence,* τ_m.
3. *Locate an upper bound* $M(m)$ *for* $\Theta(\omega_t)$, $0 \leq t \leq T$.
4. *Realize the unit Poison process* Φ *on* $[0, T] \times [0, M(m)]$.
5. *Construct the skeleton of* ω *at the Poisson time instances using Bessel bridges.*
6. *Evaluate* N.
7. *If* $N = 0$, *output the skeleton, otherwise go to 1.*

5 A third version of the Exact Algorithm

As a final extension of the Exact Algorithm (EA3), we aim to completely remove the constraint of upper boundedness of Θ, thus leaving Θ bounded only from below. Removal of this constraint will however create a new problem; the rectangle on which we simulate the Poisson process will be unbounded. This is solved by restricting the absolute maximum of a proposed path $\omega \sim \mathbb{Z}$.

We will now present the the most interesting version of the Exact Algorithm, EA3, but in a slightly different manner than citeBeskos1.

5.1 Boundaries

Considering the absolute maximum of a Brownian bridge, this theorem is a step towards finding a boundary for the proposed path ω.

Theorem 9 ([10] p. 154). *Denote by $\gamma(t, a, b, K)$ the probability that under $\overline{\mathbb{W}}_t^{a\to b}$ a path does not leave the interval $[-K, K]$, where $K > \max(|a|, |b|)$. Then*

$$\gamma(t, a, b, K) = 1 - \sum_{j=1}^{\infty}(v_j - w_j)$$

where

$$v_j = \exp\Big\{-\frac{2}{t}\Big(2Kj - (K+a)\Big)\Big(2Kj - (K+b)\Big)\Big\} + \exp\Big\{-\frac{2}{t}\Big(2Kj - (K-a)\Big)\Big(2Kj - (K-b)\Big)\Big\}$$

and

$$w_j = \exp\Big\{-\frac{2j}{t}\Big(4K^2 j + 2K(a-b)\Big)\Big\} + \exp\Big\{-\frac{2j}{t}\Big(4K^2 j - 2K(a-b)\Big)\Big\}.$$

Changing perspective, assume we want to find K such that for a predetermined probability p, a path does not leave $[-K, K]$. We cannot find a precise K corresponding to this p, but if we can create a positive increasing sequence $\{k_i\}_{i\geq 1}$, p will be enclosed by $\gamma(t, a, b, k_{i-1}) \leq p \leq \gamma(t, a, b, k_i)$ for some i. Set $J = k_{i-1}$ and $L = k_i$ and thus $J \leq K \leq L$.

Theorem 10 ([2], p. 8). *Let $\{S_j\}$ be constructed as follows:*

$$S_{2j-1} = \sum_{k=1}^{j-1}(v_k - w_k) + v_j,\ S_{2j} = S_{2j-1} - w_j.$$

Then S_j is a Cauchy sequence such that

$$S_{2j} < S_{2j+2} < S_{2j+1} < S_{2j-1} \quad \forall j \geq 1.$$

A sequence with these properties provides a direct way to sample $\gamma_i = \lim S_j^i,\ j \to \infty$, where γ_i is the probability that a path does not leave the interval $[-k_i, k_i]$.

Now, what will be the relation between the sequences S_j^i and K? To answer this question we first derive the distribution function for K. Denote by a the minimum of

ω_0 and ω_T, and by b the maximum. For convenience, we shift the interval $[a, b]$ to be symmetric around zero.

$$F(k_i) := p_i = \mathbb{P}(K \leq k_i) = \gamma\Big(t, \frac{a-b}{2}, \frac{b-a}{2}, \frac{|b-a|+2k_i}{2}\Big).$$

Note that

$$\mathbb{P}(K = k_i) = \mathbb{P}(F(k_{i-1}) < U \leq F(k_i)), \;\; U \sim Uni(0,1).$$

From here on we assume a and b are already symmetrized.

The relation

$$1_{[U<\gamma_i]} = 1_{[\inf j \,:\, j \text{ even},\, S^i_j > U]},$$

where $U \sim Uni(0,1)$ is the last piece we need to find K. First define $S^0_j = 0$ for all $j \geq 1$. Next, by this relation between U and S^i_j, we conclude that it is enough to find $i \geq 1$ and j odd such that $S^{i-1}_j < U < S^i_{j+1}$ in order to corner K.

Set $J = k_{i-1}$ and $L = k_i$. The paths we are interested in are all elements of

$$A_K = \Big\{\omega : \sup_{t\in[0,T]} \omega \in [b+J, b+L], \inf_{t\in[0,T]} \omega > a - L\Big\}$$
$$\bigcup \Big\{\omega : \inf_{t\in[0,T]} \omega \in [a-L, a-J], \sup_{t\in[0,T]} \omega < b + L\Big\}.$$

Although A_K seems complicated, it is nothing but the set of paths in $\mathbb{Z}$ that live on $[-L, L]$ and either have its maximum in $[J, L]$ or its minimum in $[-L, -J]$. Thus, defining $\widetilde{M}$ as

$$\widetilde{M} = \sup_{x\in[-L,L]} \Theta(x),$$

we know that the Poisson process will operate on the rectangle $[0, T] \times [0, \widetilde{M}]$ which takes us one step closer to simulate the skeleton.

5.2 Rejection sampling on a restricted probability measure

In this step of the procedure we will create the skeleton. Instead of simulating paths from $\mathbb{Z}$ until an element from A_K is obtained, we apply rejection sampling on the restricted probability measure for Brownian bridges, $\overline{\mathbb{W}}_{A_K}$. With $\mathbb{P}_E$ we denote a probability measure $\mathbb{P}$, induced by a stochastic process X_t, restricted to E. Now, define

$$M^*_{(J,L)} = \Big\{\omega : \sup_{t\in[0,T]} \omega_t \in [b+J, b+L]\Big\} \tag{15}$$

and

$$M_{(J,L)} = \Big\{\omega : \inf_{t\in[0,T]} \omega_t \in [a-J, a-L]\Big\} \tag{16}$$

and let

$$\mathbb{B} = \frac{1}{2}\overline{\mathbb{W}}_{M^*_{(J,L)}} + \frac{1}{2}\overline{\mathbb{W}}_{M_{(J,L)}}.$$

Drawing ω from $\mathbb{B}$ will ensure that either $\omega \in M^*_{(J,L)}$ or $\omega \in M_{(J,L)}$. This will increase the probability of a skeleton being drawn from $\overline{\mathbb{W}}_{A_K}$. To conclude whether a skeleton from $\mathbb{B}$ is also from $\overline{\mathbb{W}}_{A_K}$, we look at the Radon-Nikodym derivative:

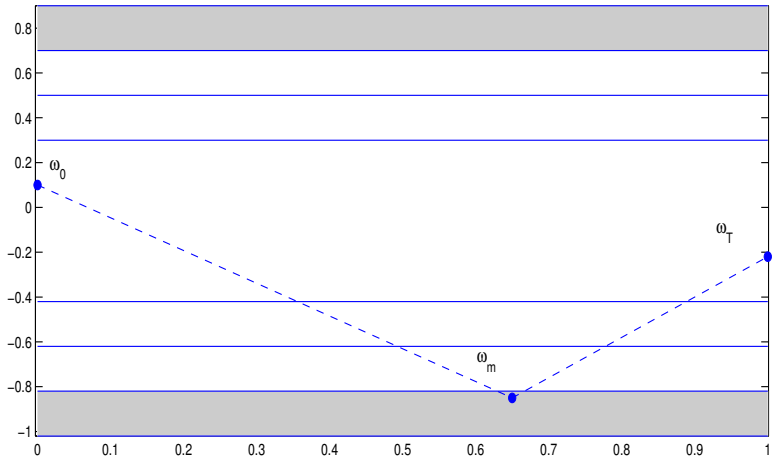

Figure 2: The starting point ω_0 and the ending point ω_T together with the minimum of ω, ω_m. The area $[0,T]\times([b+J,b+L]\cup[a-L,a-J])$ is colored gray.

Theorem 11 ([2], p. 10).

$$\frac{d\overline{\mathbb{W}}_{A_K}}{d\mathbb{B}}(\omega)=\frac{2\overline{\mathbb{W}}(\omega\in M)}{\overline{\mathbb{W}}(\omega\in A_K)}\times\frac{1_{[\omega\in A_K]}}{1+1_{[\omega\in M^*_{(J,L)}\cap M_{(J,L)}]}}.$$

5.3 Bessel proposals

From Theorem 8 we know how to create a skeleton when the minimum m (or the maximum m^*) and its time of occurrence, τ_m (or τ_{m^*}) are known. The skeleton is completed with Bessel points at the time instances simulated by the Poisson process. To accept (or reject) the skeleton as a realization of $\overline{\mathbb{W}}_{A_K}$, we turn to Theorem 11. The indicators $1_{[\omega\in A_K]}$ and $1_{[\omega\in M^*_{(J,L)}\cap M_{(J,L)}]}$ have to be simulated in order to decide on rejection/acceptance. To evaluate them for a specific skeleton, we need the probability that a Bessel bridge hits a certain level K.

(See [2] for more details) Let $a=0$ and $0<b<B<K$ for some B. Under the probability measure induced by the Bessel bridge from 0 to b on $[0,t]$ we denote by $\delta(t,b,B;K)$ the probability that a path conditioned not to leave $[0,K]$, does not leave $[0,B]$. This probability can be calculated as follows:

$$\delta(t,b,B;K)=\frac{\overline{\mathbb{W}}(\omega\text{ does not leave }[0,B])}{\overline{\mathbb{W}}(\omega\text{ does not leave }[0,K])}=\frac{b-\sum_{j=1}^{\infty}\big(u_j(t,-b,B)-u_j(t,b,B)\big)}{b-\sum_{j=1}^{\infty}\big(u_j(t,-b,K)-u_j(t,b,K)\big)},$$

where $u_j(t,b,K)=(2Kj+b)\exp\{-2Kj(Kj+b)/t\}$. We also define $\delta(t,b,B)$, the probability that a Bessel bridge does not leave the interval $[0,B]$:

$$\delta(t,b,B)=\lim_{K\to\infty}\delta(t,b,B;K)=1-\frac{1}{b}\sum_{j=1}^{\infty}\big(u_j(t,-b,B)-u_j(t,b,B)\big).$$

We can now use u_j to construct a sequence $\{S_j\}$ for finding the interval determined by δ, as we used v_j and w_j in Theorem 10.

Theorem 12 ([2], p. 9). *Let $\{S_j\}$ be constructed as follows:*

$$S_{2j-1} = \sum_{k=1}^{j-1}(u_k(t,-b,K) - u_k(t,b,K) + u_j(t,-b,K)),\ S_{2j} = S_{2j-1} - u_j(t,b,K) \quad (17)$$

with $0 < b < K$ and $3K^2 - t > 0$. Then S_j is a Cauchy sequence such that

$$S_{2j} < S_{2j+2} < S_{2j+1} < S_{2j-1}\ \forall j \geq 1.$$

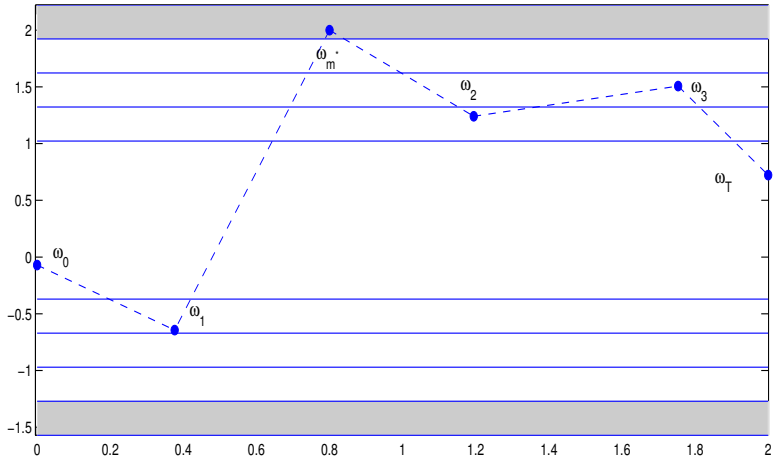

Figure 3: A skeleton of ω. The skeleton consists of six points; apart from the start and ending points and the maximum m^* also three points at the Poisson times instances.

We end this section by giving you the third Exact Algorithm, step by step.

Algorithm 5 (The third Exact Algorithm).

1. *Draw a point $\omega_T \sim h$.*
2. *Find A_K for the path, and locate an upper bound $\widetilde{M}$ for $\Theta(\omega_t)$, $0 \leq t \leq T$.*
3. *Simulate either the minimum or the maximum (each with probability 1/2), conditionally on ω being either in M or M^* and its time of occurrence, τ_m or τ_m^*. For simplicity, assume that a minimum is drawn.*
4. *Realize the unit Poison process Φ on $[0,T] \times [0,\widetilde{M}]$.*
5. *Simulate ω_{t_i} given this Poisson process.*
6. *Randomly pick $U \sim Uni(0,1)$ and evaluate $\delta(t,b,L)$. If $U < \delta$, set $1_{[\omega \in A_K]} = 1$, else set it to 0 and reject ω as coming from $\overline{\mathbb{W}}_{A_K}$.*
   - *If $1_{[\omega \in A_K]} = 1$, randomly pick $U \sim Uni(0,1)$ and evaluate $\delta(t,b,J,L)$. If $U < \delta$, set $1_{[\omega \in M_{(J,L)}]} = 1$, else set it to 0.*
     i. *If $1_{[\omega \in M_{(J,L)}]} = 0$, accept the path as a realization from $\overline{\mathbb{W}}_{A_K}$.*

ii. If $1_{[\omega \in M_{(J,L)}]} = 1$, *accept the path as a realization from* $\overline{\mathbb{W}}_{A_K}$ *with probability 1/2.*

7. If the path is accepted, evaluate N. Otherwise go to 5.

8. If $N = 0$, *output the skeleton, otherwise go to 1.*

6 Implementation of the Exact Algorithm

As a first evaluation of the Exact Algorithm, consider the SDE

$$dX_t = \sin(X_t)dt + dW_t, \quad X_0 = x_0, \;\; t \in [0, T].$$

Now follows an analysis of the problem to find out if EA is applicable in this case.

6.1 Regarding Θ(x)

Since we have drift coefficient $\alpha(x) = \sin(x)$, set

$$\Theta(x) = \frac{\sin^2(x) + \cos(x)}{2} + \frac{1}{2}, \;\; x \in \mathbb{R},$$

as $\sin^2(x) + \cos(x) \geq -1$. The maximum of $\Theta(x)$ is now 9/8, thus satisfying condition (A1) and (A2) with the demand of boundedness and differentiability. This suggests that it is the first version of the Exact Algorithm (EA1) that ought to be considered.

6.2 Regarding h(x)

According to Definition 2 this SDE's endpoint density is defined as

$$h(x) = C \exp\{1 - \cos(x) - (x - x_0)^2/2T\}, \;\; x \in \mathbb{R},$$

where C is a normalizing constant. Due to the structure of h, rejection sampling is required in order to simulate the ending points. For the simplicity of sampling from a normal density, $M\,\phi(x, \mu, \sigma)$ is chosen for enveloping (for some $M \geq 1$).

Since h in this example is symmetric around zero, μ is set to the same. In cases when it is trickier to find a suitable μ, rejection sampling once again comes in handy: simulate a large number of points uniformly in the square $[-N, N] \times [0, a]$, for some $a \geq \max h(x)$ and a large N. Investigate for which points (x_i, y_i), $y_i \leq h(x_i)$ holds and take the mean over these x_is to get an approximation of μ.

As for M and σ, two methods to find the optimal values are suggested, considering different aspects of ϕ and h. Since a desired property of the envelope function is for it to be as close to the original function as possible, it seems reasonable to consider the distance between the two functions, i.e.,

$$D(M, \sigma) = \sqrt{\int_{-\infty}^{\infty} (M * \phi(x, 0, \sigma) - h(x))^2 dx}$$

and minimize it wrt M and σ.

The other option is to optimize the quota $(\textit{accepted points})/(\textit{suggested points})$ from the rejection sampling algorithm when altering M and σ.

When using either of these methods for computing the parameters we need to make sure that

$$\min_{x \in [-N, N]} \Big(M\,\phi(x, \mu, \sigma) - h(x) \Big) \geq 0$$

for N large enough. This to ensure that the function ϕ in question is indeed an envelope function.

Now, putting these pieces together with the pseudo-code presented in the end of Chapter 4, the Exact Algorithm produces solutions to (6) in form of skeletons.

6.3 Testing and results

Two different tests were performed to check the accuracy of our precious EA1. Both originate from the null-hypothesis

H_0 : The paths produced by EM and EA1 come from the same distribution.

The first test compares the distribution of a single point at a fix time t (chosen to be t = T). The second test takes into account a path's finite dimensional density.

6.3.1 Endpoint test

This gives an indication whether the null-hypothesis stated above seems reasonable or not. We produced two equally sized samples of paths with Euler-Maruyama and EA1 respectively. Then the Kolmogorov-Smirnov test with

H_0 : The endpoint samples come from the same distribution.

was executed to get the p-values presented in the table below, i.e. the lowest levels on which we can reject H_0.

Euler-Maruyama	Elapsed time for 10^5 paths	p-value, 10^5 paths	p-value, 10^6 paths
2^{-2}	0.36 sec	0	0
2^{-3}	0.43 sec	0	0
2^{-4}	0.97 sec	0	0
2^{-5}	1.49 sec	0	0
2^{-6}	1.81 sec	0.002	0
2^{-7}	4.19 sec	0.442	0.045
2^{-8}	7.69 sec	0.624	0.324

For comparison, the elapsed time for EA1 when simulating 10^5 paths was 5.68 sec.

Observing the p-values in the table above, it might be tempting to conclude that they will continue to increase as the time increments of Euler-Maruyama are decreased. However, this tendency will decline and the p-values start to fluctuate between 0 and 1 for increments smaller than 2^{-9}.

6.3.2 Transition density test

Here we considered the distribution of a path at several time instances instead of at a single point, thus making the test stronger.

To evaluate (6) we first needed to approximate the integral

$$I = \int_0^1 G\big(r(g(y) - g(x)) + g(x) + \sqrt{t-s}\,\overline{W}_r\big),$$

which was done using Riemann sums with an underlying Brownian bridge discretized to 150 time instances on $[0, 1]$. A 100 values of I (for 100 realizations of a Brownian bridge) was then calculated to approximate the expectation

$$\mathbb{E}\Big[\exp\Big\{\frac{s-t}{2}I\Big\}\Big].$$

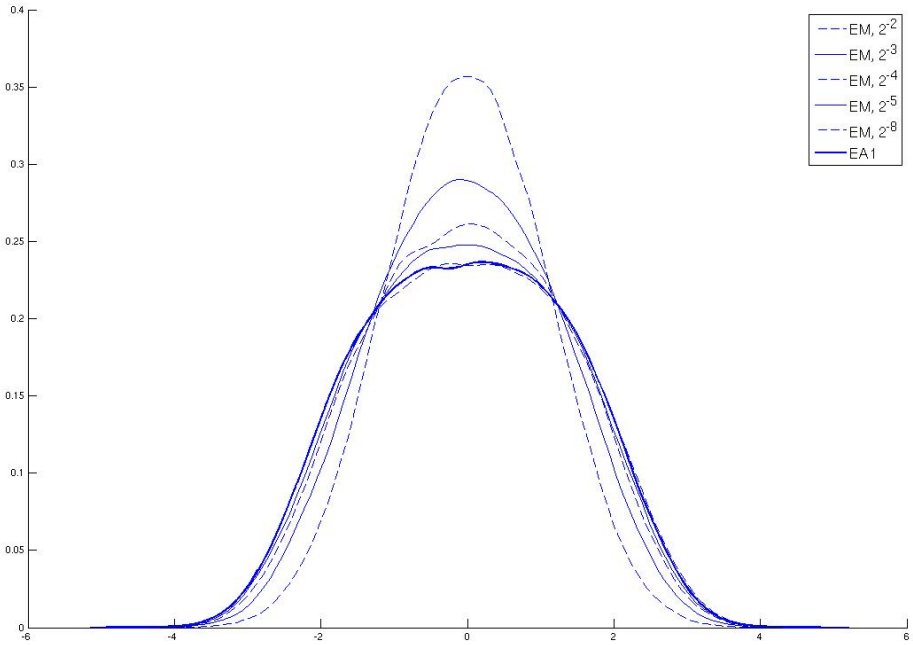

Figure 4: Comparison between the empirical endpoint densities for EA1 and Euler-Maruyama with time increments $2^{-i},\ i = 2, 3, 4, 5, 8$ with sample size $100,000$. As can be seen, there is an obvious convergence as the time increments are decreased.

The simulated Brownian bridges were kept to be used for each path in order to achieve stability of the expectation.

When it came to testing, Euler-Maruyama was assumed to give the correct solution. We calculated the transition densities for 2500 simulated paths at 100 time instances. We then evaluated one path simulated by EA1 and compared it to the empirical transition density.

As we see in Figure 5, we can not reject the null hypothesis that Euler-Maruyama and EA1 simulate paths with the same finite dimensional distribution on a significance level of 20%.

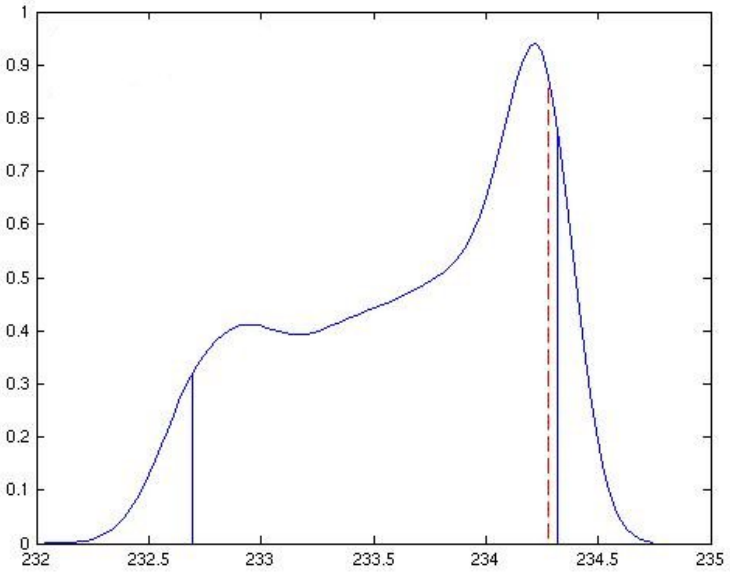

Figure 5: Log of the empirical finite dimensional density of (6) together with its 10% quantiles. The dashed line is the (logged) value of (6) for a path simulated by EA1.

7 Implementing the first extension

For the second application of EA, we consider *the logistic growth model* (LGM):

$$dY_t = rY_t(1 - \frac{Y_t}{K})dt + \beta Y_t\, dW_t,\ t \in [0,T],\ r, K, \beta > 0.$$

The process Y_t will oscillate around K for t large enough. Here K is described as the maximum population the system can hold. Altering β and r will affect the amplitude and frequency of these oscillations respectively.

To transform Y_t into a unit diffusion process we apply (5) with a slight modification. Let

$$X_t = -\int_1^{Y_t} \frac{1}{\beta u} du \ = \ -\frac{log(Y_t)}{\beta}, \tag{18}$$

so that $Y_t = \exp\{-\beta X_t\}$. It can be shown that Y_t is strictly positive with probability one for all parameter values and finite time, which makes X_t well-defined. See [3] for further references. For the rest of this chapter we will focus on

$$dX_t = \Big(\frac{re^{-\beta X_t}}{\beta K} - \frac{r}{\beta} + \frac{\beta}{2}\Big)dt - dW_t,\ t \in [0,T], \tag{19}$$

which we rewrite to the very same SDE but with positive unit diffusion coefficient.

7.1 Regarding $\Theta(x)$ and $h(x)$

From (19) we now have $\alpha(x) = \beta/2 - \beta/r + r\exp\{-\beta x\}/(\beta K)$ so

$$\Theta(x) = \Big(\frac{\beta}{2} - \frac{r}{\beta}\Big)^2 + e^{-\beta x}\frac{r(\beta^2 - 2r)}{2\beta^2 K} + e^{-2\beta x}\frac{r^2}{2\beta^2 K^2} - b,$$
$$b \le \inf_{x\in\mathbb{R}}\Big\{\frac{\alpha^2(x) + \alpha'(x)}{2}\Big\}.$$

Note that the negation in (18) is crucial in order to fulfill the condition $(A2')$ of the second Exact Algorithm (EA2), that is for $\limsup_{x\to\infty}\Theta(x)$ to be bounded.

Define h as

$$h(x) = \exp\Big\{\Big(\frac{\beta}{2} - \frac{r}{\beta}\Big)x - \frac{r}{\beta^2 K}e^{-\beta x} - \frac{(x - x_0)^2}{2T}\Big\},\ x \in \mathbb{R}.$$

Figure 6 displays the endpoint probability density function for the transformed LGM with its corresponding envelope function.

If we are interested in extending the time interval beyond [0, 1], there are strong reasons not to apply EA on all of the area [0, T] at once. For instance, it might take many proposals before accepting a skeleton. Instead we divide the interval into smaller segments, where EA is recursively applied. For this procedure to be efficient it is necessary to find an envelope function for h on closed form. If this is not the case, a re-calibration of the parameters has to be performed for each partition. This will be severely time consuming.

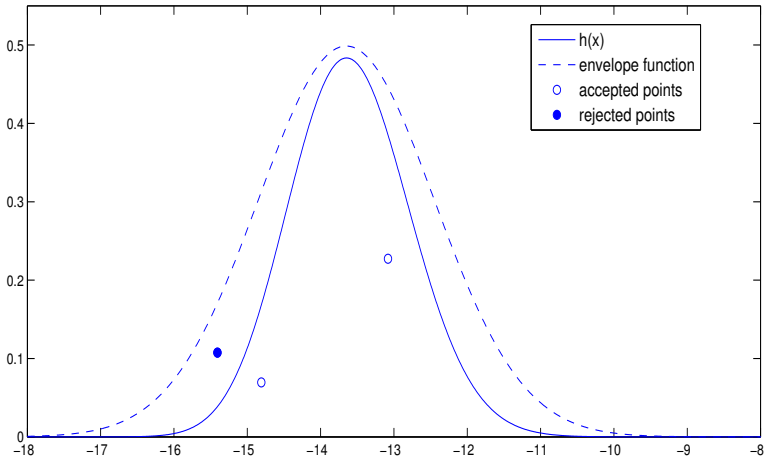

Figure 6: The endpoint density h together with an envelope function.

7.2 Results

The parameters are set to $\beta = 0.5$, $r = 0.5$, $K = 1000$, $T = 1$ and $X_0 = \log(1000)/\beta$.

The endpoint density is looked upon, as is the finite dimensional density of X_t. We notice resemblance in the endpoint pdfs of EA and EM (see Figure 7). However, performing the Kolmogorov-Smirnov test on the same level as in the previous example (2^{-8}) will yield p-values too low not to reject the null hypothesis. It required increments of the size 2^{-19} before obtaining p-values indicating non rejection.

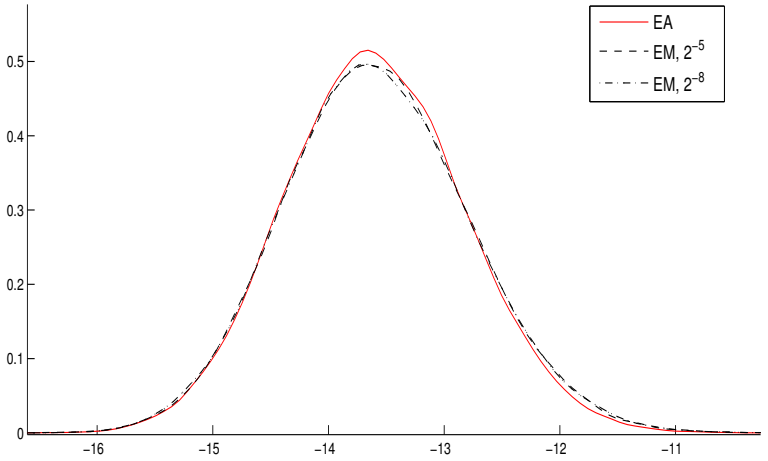

Figure 7: Displaying the empirical pdf of the endpoints for EA and EM

The elapsed time for EA2 in this problem is significantly higher than in the previous algorithm. Compared to EM with increment size 10^{-8}, they will differ with a factor four. Although, when considering the elapsed time of EM with increments of the size 2^{-19}, approximatively 3 hours, the strength of EA is once again obvious.

As for the finite dimensional distribution of X_t, the null hypothesis regarding equality can not be rejected.

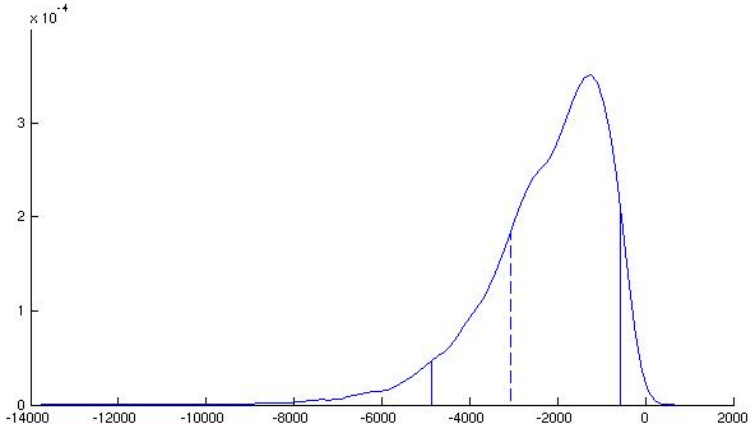

Figure 8: Log of the empirical transition density function of (19) together with its 10% quantiles. The dashed line is the (logged) value of (6) for a path simulated by EA2

8 Conclusions

An endpoint test evaluated for both a SDE with sine drift and a logistic growth model gives a first clear indication of similarity between paths simulated by the Exact Algorithm and Euler-Maruyama. In the sine example a convergence towards EA can be observed for the endpoint distribution as the time increments for EM is decreased. Furthermore, in neither the sine nor the LGM case can we reject the null hypothesis that the finite dimensional density of paths simulated by EA and EM come from the same distribution. This on a significance level of 20%.

The runtime of the two simulation algorithms alters depending on the SDE. Generally EA is faster than EM. However, for large values of the ending time T, a partitioning of $[0, T]$ is needed in order to meet satisfactory accuracy. The parameters for the endpoint function h will then have to be continuously re-calibrated, an updating sequence that will slow the simulation process down considerably.

When comparing properties with other simulation methods such as EM, one important feature is that EA is *e*xact, whereas most other algorithms are approximative. Also, EA allows us to control the range of the simulated solutions. This is particularly useful for processes restricted to the positive y-axis.

A question that naturally arises is whether it is possible to derive another rejection event that will result in a less restricted algorithm, still using the underlying idea of rejection sampling on measures.

Although the Exact Algorithm is more demanding in both a mathematical sense and at the implementational stage, its performance as a simulation algorithm can easily compete with that of Euler-Maruyama.

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