Fifteen Lectures

From LÉVY PROCESSES

To SEMIMARTINGALES

With 4 Serious Figures

Fall 02

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Contents

1 Lecture 1 6/9 - 02

The first five lectures cover classical distribution theory for Lévy processes and infinitely divisible distributions, following "*Sato: A Course on Lévy Processes*". We have added proofs when negotiable and of sufficient "general value". Most of them come from "*Sato: Lévy Processes and Infinitely Divisible Distributions*", often slightly modified. We have also added many exercises. They are all intended to be quite straightforward (and it is a mistake when they are not).

Another example of standard literature on Lévy Processes is "*Bertoin: Lévy Processes*". See also e.g., the relevant parts of "*Feller: An Introduction to Probability Theory and Its Application*", "*Fristedt & Gray: A Modern Approach to Probability Theory*" and "*Kallenberg: Foundations of Modern Probability*".

1.0 Basic Notation

In Lectures 1-7, $X=\{X(t)\}_{t\geq 0}, X_1, X_2, \ldots$ denote $\mathbb{R}^d$-valued stochastic processes, and $Y, Y_1, Y_2, \ldots$ $\mathbb{R}^d$-valued random variables (`rv's`).

Processes and `rv's` that feature are assumed defined on a common complete basic probability space $\underline{(\Omega, \mathfrak{F}, \mathbf{P})}$, when needed. A null-event is a set $N\in\mathfrak{F}$ with $\mathbf{P}\{N\}=0$.

The probability distribution (law) $\mathbf{P}\circ Y^{-1}$ of Y on $\mathbb{R}^d$ is denoted $\underline{\mathcal{L}(Y)}$.

We write $\underline{Y_1 =_d Y_2}$ when $\mathcal{L}(Y_1)=\mathcal{L}(Y_1)$, and $\underline{X_1 =_d X_2}$ when the `fidi's` (see Definition 1.9 below) of X_1 and X_2 coincide. Further, $\underline{\rightarrow_d}$ denotes weak convergence.

The Borel sets in $\mathbb{R}^d$ (the σ-algebra generated by the open sets) is denoted $\underline{\mathcal{B}(\mathbb{R}^d)}$. For $(\Sigma, \mathcal{S})$ a measurable space (see Section 6.3), $\underline{\mathbb{L}^0(\Sigma, \mathcal{S})} \equiv \{\text{measurable } f:\Sigma\rightarrow\mathbb{R}\}$.

1.1 Leŕy Processes and Additive Processes

Definition 1.1 *X has independent increments if, for $n\in\mathbb{N}$ and $0\leq t_1<\ldots<t_n$,*

$$X(t_1), X(t_2)-X(t_1), \ldots, X(t_n)-X(t_{n-1}) \qquad \textit{are independent.}$$

Definition 1.2 *X is time homogeneous if, for $t\geq 0$,*

$$\mathcal{L}(X(t+s)-X(s)) \qquad \textit{does not depend on} \quad s\geq 0.$$

Definition 1.3 *X is stochastically continuous* (also called continuous in probability or **P**-continuous)*, if, for $s\geq 0$,*

$$X(t+s)-X(s)\rightarrow_{\mathbf{P}} 0 \qquad \textit{as} \quad t\rightarrow 0.$$

Definition 1.4 *X is càdlàg* (for "continu à droit avec limites à gauche", also called rcll) *if $X(t)=X(\omega;t)$ is right continuous with left limits, except for ω in a null-event.*

Definition 1.5 *X is an additive process in law* (also called independent increment process in law) *if the following conditions hold;*

(1) *X has independent increments;*

(2) *$X(0)=0$ a.s.;*

(3) *X is stochastically continuous.*

Definition 1.6 *X is a Lévy process in law if an additive process in law such that*

(4) *X is time homogeneous.*

Definition 1.7 *X is an additive process* (also called independent increment process) *if an additive process in law such that*

(v) *X is càdlàg.*

Definition 1.8 *X is a Lévy process if a Lévy process in law such that*

(v) *X is càdlàg.*

EXAMPLE 1 Brownian motion (`Bm`) and a Poisson process (`Pp`) are Lévy processes. For independent identically distributed (`iid`) $\{Y_k\}_{k=1}^{\infty}$, $X=\sum_{k=1}^{\lfloor\cdot\rfloor} Y_k$ is additive. #

EXERCISE 1 Make computer simulations of sample paths of `Bm` and `Pp`. Discuss what sources of errors there are, if any.

Definition 1.9 *The finite dimensional distributions (`fidi's`) of a stochastic process $\{Z(t)\}_{t\in T}$ are the probability laws*

$$\{\{\mathcal{L}(Z(t_1),\ldots,Z(t_n))\}_{(t_1,\ldots,t_n)\in T^n}\}_{n\in\mathbb{N}}.$$

EXERCISE 2 Give examples of stochastic processes $\{Z_1(t)\}_{t\in T}$ and $\{Z_2(t)\}_{t\in T}$ such that $Z_1 =_d Z_2$, but $\mathbf{P}\{Z_1(t)=Z_2(t) \text{ for all } t\in T\}<1$. (In general, this "event" may not be measurable, so that the probability is not even well-defined.)

Definition 1.10 *X has stationary increments if*

$$\{X(t+h)-X(h)\}_{t\geq 0} =_d \{X(t)-X(0)\}_{t\geq 0} \qquad \text{for} \quad h\geq 0.$$

EXERCISE 3 Explain how Definition 1.10 connects to the intuitive concept of stationary increments. Show that Lévy processes in law have stationary increments.

EXERCISE 4 Show that a time homogeneous process X has mean function $\mathbf{E}\{X(t)\} = Kt$ for $t\geq 0$, for some constant $K\in\mathbb{R}^d$, when that mean is well-defined (and a measurable function of t).

EXERCISE 5 Show that a Lévy process in law X has variance function $\mathbf{Var}\{X(t)\} = Kt$ for $t\geq 0$, for some constant non-negative definite matrix $K\in\mathbb{R}_{d|d}$, when that variance is well-defined.

Definition 1.11 *Two stochastic processes $\{Z_1(t)\}_{t\in T}$ and $\{Z_2(t)\}_{t\in T}$ are modifications* (also called indistinguishable by some authors) *of each other, if*

$$\mathbf{P}\{Z_1(t)=Z_2(t)\} = 1 \qquad \text{for all} \quad t\in T.$$

REMARK 1.12 Many authors (e.g., Protter) call Z_1 and Z_2 indistinguishable if

$$\mathbf{P}\{Z_1(t)=Z_2(t) \text{ for all } t\in T\} = 1.$$

However, people in "general stochastic processes" do not use this language, since the above probability is not even well-defined without additional assumptions (e.g., cádlág, a.s. continuity, etc.). And under such assumptions, the two possible definitions of indistinguishability typically coincide (see Exercise 6 below). #

EXERCISE 6 Show that the two alternative definitions of indistinguishability indicated above coincide for cádlág processes X_1 and X_2.

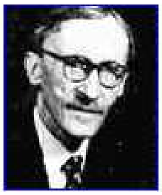
Theorem 1.13 *Each Lévy process (additive process) in law has a modification that is a Lévy process (additive process).*

Albeit quite straightforward under additional technical conditions (e.g., existence of second moments), the general proof is delicate. It would occupy us for several lectures. The usual approach is by oscillation analysis of martingales or Markov processes.

On http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Levy_Paul.html we found

Paul Pierre Lévy

Born: 15 Sept 1886 in Paris, France
Died: 15 Dec 1971 in Paris, France



Paul Lévy was born into a family containing several mathematicians. His grandfather was a professor of mathematics while Paul's father, Lucien Lévy, was an examiner with the Ecole Polytechnique and wrote papers on geometry. Paul attended the Lycée Saint Louis in Paris and he achieved outstanding success winning prizes not only in mathematics but also in Greek, chemistry and physics. He was placed first for entry to the Ecole Normale Supérieur and second for entry to the Ecole Polytechnique in the Concours d'entrée for the two institutions.

He chose to attend the Ecole Polytechnique and he while still an undergraduate there published his first paper on semiconvergent series in 1905. After graduating in first place, Lévy took a year doing military service before entering the Ecole des Mines in 1907. While he studied at the Ecole des Mines he also attended courses at the Sorbonne given by Darboux and Emile Picard. In addition he attended lectures at the Collège de France by Georges Humbert and Hadamard.

It was Hadamard who was the major influence in determining the topics on which Lévy would undertake research. Finishing his studies at the Ecole des Mines in 1910 he began research in functional analysis. His thesis on this topic was examined by Emile Picard, Poincaré and Hadamard in 1911 and he received his Docteur ès Sciences in 1912.

Lévy became professor Ecole des Mines in Paris in 1913, then professor of analysis at the Ecole Polytechnique in Paris in 1920 where he remained until he retired in 1959. During World War I Lévy served in the artillery and was involved in using his mathematical skills in solving problems concerning defence against attacks from the air. A young mathematician R Gateaux was killed near the beginning of the war and Hadamard asked Lévy to prepare Gateaux's work for publication. He did this but he did not stop at writing up Gateaux's results, rather he took Gateaux's ideas and developed them further publishing the material after the war had ended in 1919.

As we indicated above Lévy first worked on functional analysis [12]:-

> *... done in the spirit of Volterra. This involved extending the calculus of functions of a real variable to spaces where the points are curves, surfaces, sequences or functions.*

In 1919 Lévy was asked to give three lectures at the Ecole Polytechnique on (see [9]):-

> *... notions of calculus of probabilities and the role of Gaussian law in the theory of errors.*

Taylor writes in [12]:-

> *At that time there was no mathematical theory of probability - only a collection of small computational problems. Now it is a fully-fledged branch of mathematics using techniques from all branches of modern analysis and making its own contribution of ideas, problems, results and useful machinery to be applied elsewhere. If there is one person who has influenced the establishment and growth of probability theory more than any other, that person must be Paul Lévy.*

Loève, in [9], gives a very colourful description of Lévy's contributions:-

> *Paul Lévy was a painter in the probabilistic world. Like the very great painting geniuses, his palette was his own and his paintings transmuted forever our vision of reality. ... His three main, somewhat overlapping, periods were: the limit laws period, the great period of additive processes and of martingales painted in pathtime colours, and the Brownian pathfinder period.*

Not only did Lévy contribute to probability and functional analysis but he also worked on partial differential equations and series. In 1926 he extended Laplace transforms to broader function classes. He undertook a large-scale work on generalised differential equations in functional derivatives. He also studied geometry.

His main books are *Leçons d'analyse fonctionnelle* (1922), *Calcul des probabilités* (1925), *Théorie de l'addition des variables aléatoires* (1937-54), and *Processus stochastiques et mouvement brownien* (1948).

In 1963 Lévy was elected to honorary membership of the London Mathematical Society. In the following year he was elected to the Académie des Sciences.

Loève sums up his article [9] in these words:-

> *He was a very modest man while believing fully in the power of rational thought. ... whenever I pass by the Luxembourg gardens, I still see us there strolling, sitting in the sun on a bench; I still hear him speaking carefully his thoughts. I have known a great man.*

Article by: *J J O'Connor* and *E F Robertson*

1.2 Markov Property

Definition 1.14 *A family* $\mathbb{F}=\{\mathfrak{F}_t\}_{t\geq 0}$ *of* σ*-algebras* $\mathfrak{F}_t\subseteq\mathfrak{F}$ *is a* *filtration* *if it is non-decreasing, i.e.,* $\mathfrak{F}_s\subseteq\mathfrak{F}_t$ *for* $0\leq s\leq t$.

Definition 1.15 X *is* *adapted* *to a filtration* $\mathbb{F}$ *if* $X(t)$ *is* $\mathfrak{F}_t$*-measurable for* $t\geq 0$.

Definition 1.16 X *is a* *Markov process wrt. a filtration* $\mathbb{F}$, *if adapted to* $\mathbb{F}$ *with*

$$\mathbf{P}\{X(t+s)\in\cdot\,|\mathfrak{F}_s\}=\mathbf{P}\{X(t+s)\in\cdot\,|X(s)\}\qquad\textit{for}\quad t+s>s\geq 0.$$

Definition 1.17 X *is a* *Markov process wrt. itself* *if it is a Markov process wrt. the filtration* $\{\sigma(X(s):s\in[0,t])\}_{t\geq 0}$.

Exercise 7 Show that if X is a Markov process, then it is Markov wrt. itself.

Definition 1.18 $P(\cdot,t,x,s)$ *is a* *transition probability* *for a Markov process* X *if*

$$\mathbf{P}\{X(t+s)\in\cdot\,|X(s)=x\}=P(\cdot,t,x,s)\qquad a.e.\ (dF_{X(s)})\ \textit{for}\ x\in\mathbb{R}^d,\quad\textit{for}\ \ t+s>s\geq 0.$$

Definition 1.19 *A transition probability* $P(\cdot,t,x,s)$ *is* *time homogeneous* *if it does not depend on the last argument* $s\geq 0$.

Definition 1.20 *A Markov process* X *is* *time homogeneous* *if it has a time homogeneous transition probability.*

Fact 1.21 *A Lévy process in law is a time homogeneous Markov process wrt. itself with transition probability*

$$P(\cdot,t,x,s)=\mathbf{P}\{X(t)\in\cdot-x\}.$$

Proof. The Markov property follows from that, by independence of increments,

$$\mathbf{P}\{X(t{+}s)\in\cdot\,|\mathcal{F}_s\}=\mathbf{P}\{X(t{+}s)-X(s)\in\cdot-X(s)|\mathcal{F}_s\}=\mathbf{P}\{X(t{+}s)-X(s)\in\cdot-X(s)|X(s)\}.$$

We get a transition probability similarly, this time also using time homogeneity,

$$\mathbf{P}\{X(t+s)\in\cdot\,|X(s)=x\} = \mathbf{P}\{X(t+s)-X(s)\in\cdot-x\}. \quad \square$$

EXERCISE 8 Show that an additive process in law is a Markov process wrt. itself.

1.3 Infinitely Divisible Random Variables and Processes

Definition 1.22 *Y is infinitely divisible (`id`) if, for each $n\in\mathbb{N}$,*

$$Y =_d Y_1+\ldots+Y_n \qquad \textit{for some } \texttt{iid} \quad Y_1,\ldots,Y_n.$$

The `rv's` $\{Y_k\}_{k=1}^n$ in Definition 1.22 that "divides" Y must have common characteristic function (`chf`) $\varphi_{Y_k}(\theta) = \mathbf{E}\{\mathrm{e}^{i\langle\theta,Y_k\rangle}\} = \varphi_Y(\theta)^{1/n}$ for $\theta\in\mathbb{R}^d$.

Definition 1.23 *For Y `id`, $Y^{*1/n}$ denotes an `rv` with `chf` $\varphi_Y^{1/n}$ for $n\in\mathbb{N}$.*

Fact 1.24 *For a Lévy process in law X, $X(t)$ is `id` with $X(t)^{*1/n} =_d X(\frac{t}{n})$ for $t>0$ and $n\in\mathbb{N}$.*

Proof. $X(t) = \sum_{k=1}^n (X(\frac{k}{n}t)-X(\frac{k-1}{n}t))$ with $X(\frac{k}{n}t)-X(\frac{k-1}{n}t)=_d X(\frac{t}{n})$ `iid`. $\square$

Fact 1.25 *For Y `id` and $t>0$, there exists an `rv` $Y^{\star t}$ with $\varphi_{Y^{\star t}} = \varphi_Y^t$.*

Proof. By weak convergence, it is enough to check that φ_Y^t is a `chf` for $0<t=\frac{k}{\ell}\in\mathbb{Q}$. But $\varphi_Y^{k/\ell}$ is `chf` for a sum of k `iid rv's` with law $\mathcal{L}(Y^{*1/\ell})$, since $\varphi_Y^{1/\ell}=\varphi_{Y^{*1/\ell}}$. $\square$

EXERCISE 9 For Y `id`, show that $Y^{\star t}$ is `id` for $t>0$.

Fact 1.26 *The `fidi's` of a Lévy process in law X are determined by $\mathcal{L}(X(t))$ for any choice of $t>0$.*

Proof. To check that $\mathcal{L}(X(t_1),\ldots,X(t_n))$ is determined by $\mathcal{L}(X(t))$, it is enough to check that $\mathcal{L}(X(t_1),X(t_2)-X(t_1),\ldots,X(t_n)-X(t_{n-1}))$ is, for $0<t_1<\ldots<t_n$. By $\mathbf{P}$-continuity and independence and homogeneity of increments, this holds if $\mathcal{L}(X(t))$ determines $\mathcal{L}(X(t_j-t_{j-1}))$ for $t_j-t_{j-1}=\frac{k}{\ell}t\in t\mathbb{Q}$ $(t_0\equiv 0)$. This follows from

$$\varphi_{X(t_j-t_{j-1})} = \varphi_{X(\ell(t_j-t_{j-1}))}^{1/\ell} = \varphi_{X(kt)}^{1/\ell} = \varphi_{X(t)}^{k/\ell}. \quad \square$$

Corollary 1.27 *For a Lévy process in law X, we have* $\varphi_{X(s)}=\varphi_{X(t)}^{s/t}$ *for* $s,t>0$.

The next theorem is one of the most important in theory for stochastic processes.

Theorem 1.28 (Kolmogorov Consistency) *Given distribution functions* $\{\{F_t:\mathbb{R}^k\to[0,1]\}_{t\in T^k}\}_{k\in\mathbb{N}}$, *there exists a stochastic process* $\{Z(t)\}_{t\in T}$ *with these distributions as its* `fidi's`, *iff. the following two consistency conditions hold*

(1) $F_{\ldots,t_{i-1},t_j,t_{i+1},\ldots,t_{j-1},t_i,t_{j+1},\ldots}(\ldots,x_{i-1},x_j,x_{i+1},\ldots,x_{j-1},x_i,x_{j+1},\ldots)=F_t(x)$;
(2) $\lim_{x_{k+1}\to\infty}F_{t,t_{k+1}}(x,x_{k+1})=F_t(x)$.

The proof is not difficult, but requires a basic understanding of probability measures on cylinder sets, together with the general theory of weak convergence.

Fact 1.29 *For Y* `id`, *there exists a Lévy process X with* $X(1)=_d Y$.

Proof. We specify the law of $F_{t_1,\ldots,t_n}$ through that of $F_{t_1,t_2-t_1,\ldots,t_n-t_{n-1}}$, for $0<t_1<\ldots<t_n$, as that with `chf` $\varphi_Y(\theta_1)^{t_1}\varphi_Y(\theta_2)^{t_2-t_1}\ldots\varphi_Y(\theta_n)^{t_n-t_{n-1}}$. These distributions are consistent. Thus Kolmogorov's Theorem gives us a process X, with these `fidi's`, that must be a Lévy process in law with $X(1)=_d Y$. By Theorem 1.13, there exists a Lévy process with the same `fidi's`. □

Exercise 10 Explain why the 1-dimensional Kolmogorov Theorem works for d-dimensional processes.

Exercise 11 For Y `id`, why is a Lévy process in law X that satisfies $X(1)=_d Y$ unique in law? (I.e., why does any other such Lévy process have the same `fidi's`?)

Also process values of additive processes are `id`. But this is no longer an elementary observation, and the proof requires some background results and notation.

Definition 1.30 *A sequence* $\{\{Y_{n,k}\}_{k=1}^{r_n}\}_{n=1}^{\infty}$ *of* `rv's` *is a <u>null-array</u> if* $Y_{n,1},\ldots,Y_{n,r_n}$ *are independent for* $n\in\mathbb{N}$, *with*

$$\lim_{n\to\infty}\max_{1\leq k\leq r_n}\mathbf{P}\{|Y_{n,k}|>\varepsilon\}=0\qquad\text{for}\quad\varepsilon>0.$$

The three (or so) page proof of ou next theorem is not difficult. But it belongs to basic probability courses, rather than being suitable to give here.

Theorem 1.31 (KHINTCHINE) *Y is* `id` *iff., for some null-array $\{\{Y_{n,k}\}_{k=1}^{r_n}\}_{n=1}^{\infty}$, and for some constants $b_n \in \mathbb{R}^d$,*

$$\textstyle\sum_{k=1}^{r_n} Y_{n,k} - b_n \to_d Y \qquad \text{as} \quad n \to \infty.$$

EXERCISE 12 One implication in Khintchine's Theorem is trivial: Prove that part.

Lemma 1.32 *A $\mathbf{P}$-continuous X is locally uniformly $\mathbf{P}$-continuous, i.e., for $t_0 > 0$,*

$$\lim_{\delta \downarrow 0} \sup_{s,t \in [0,t_0],\, |s-t| \le \delta} \mathbf{P}\{|X(s) - X(t)| > \varepsilon\} = 0 \qquad \text{for} \quad \varepsilon > 0.$$

EXERCISE 13 Prove Lemma 1.32.

Fact 1.33 *For an additive process in law X, any process value $X(t)$ is* `id`.

Proof. With $r_n = n$ and $Y_{n,k} = X(\frac{k}{n}t) - X(\frac{k-1}{n}t)$, we trivially have $\sum_{k=1}^{r_n} Y_{n,k} = X(t) \to_d X(t)$. Thus Khintchine's Theorem shows that $X(t)$ is `id`, if $\{\{Y_{n,k}\}_{k=1}^{r_n}\}_{n=1}^{\infty}$ is a null-array. This we get from Lemma 1.32, since, as $n \to \infty$,

$$\max_{1 \le k \le r_n} \mathbf{P}\{|Y_{n,k}| > \varepsilon\} \le \sup_{r,s \in [0,t],\, |r-s| \le t/n} \mathbf{P}\{|X(r) - X(s)| > \varepsilon\} \to 0. \quad \square$$

`Id` processes are defined by an infinite dimensional version of Definition 1.22.

Definition 1.34 *A stochastic process $\{Z(t)\}_{t \in T}$ is* <u>`id`</u> *if, for each $n \in \mathbb{N}$,*

$$\{Z(t)\}_{t \in T} =_d \{Z_1(t)\}_{t \in T} + \ldots + \{Z_n(t)\}_{t \in T} \qquad \text{for some } \texttt{iid} \text{ processes} \quad Z_1, \ldots, Z_n.$$

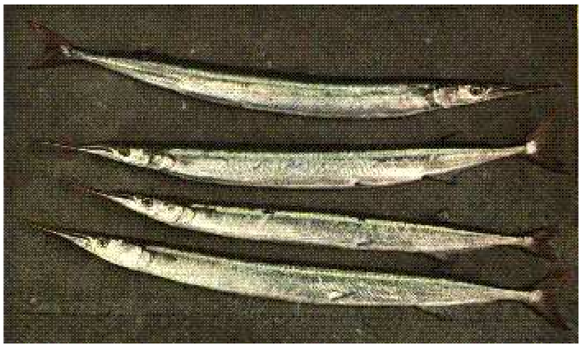
EXERCISE 14 Show that Lévy processes are `id` processes.

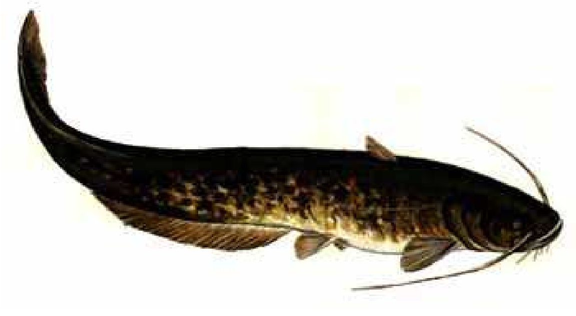
Definition 1.35 *An* `id` *Y has <u>cumulant generating function (`cgf`)</u>*

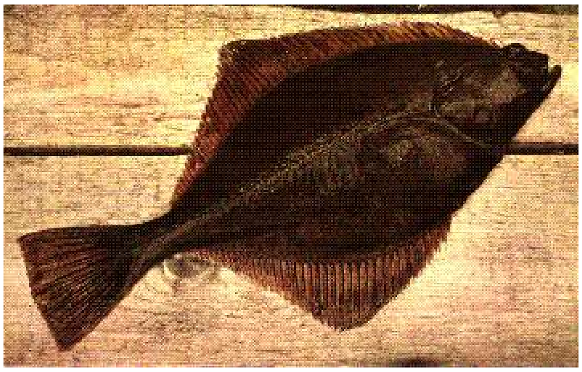
$$\underline{\Psi(-i\,\cdot)} \equiv \ln(\varphi_Y(\cdot)).$$

1.4 Something I came to think of this time ...

Vilken eller vilka av dessa fiskar är en 1) gädda, 2) hongädda, 3) horngädda, 4) näbbgädda, 5) skädda, 6) honskädda, 7) hornskädda, 8) näbbskädda?







2 Lecture 2 10/9 - 02

2.1 Lévy-Khintchine Formula

The following result is one of the most fundamentally important in probability. The proof is not really difficult, but too technical to be worthwhile doing here.

Theorem 2.1 (Lévy-Khintchine) *Y is* id *iff. there exists a triplet (A,ν,γ) of*

$$\begin{cases} A \ \text{a symmetric non-negative definite } d|d\text{-matrix (the Gaussian covariance)} \\ \nu \ \text{a measure on } \mathbb{R}^d \text{ with } \nu(\{0\})=0 \text{ and } \int_{\mathbb{R}^d}|y|^2\wedge 1\,d\nu(y)<\infty \ \text{(the Lévy measure)}, \\ \gamma\in\mathbb{R}^d \ \text{a constant} \end{cases}$$

which in that case is uniquely determined, such that, for $\theta\in\mathbb{R}^d$,

$$\mathbf{E}\{e^{i\langle\theta,Y\rangle}\} = \exp\{-\tfrac{1}{2}\langle\theta,A\theta\rangle + i\langle\theta,\gamma\rangle + \int_{\mathbb{R}^d}(e^{i\langle\theta,y\rangle}-1-\mathbf{1}_{\{|y|\le 1\}}i\langle\theta,y\rangle)\,d\nu(y)\}.$$

That the integral in the exponent is well-defined follows from the requirements on ν and the fact that $|e^{i\langle\theta,y\rangle}-1-\mathbf{1}_{\{|y|\le1\}}i\langle\theta,y\rangle| = O(|y|^2)$ $[O(1)]$ as $|y|\to 0$ $[|y|\to\infty]$.

The function $\mathbf{1}_{\{|y|\le1\}}$ in Lévy-Khintchine Formula may be (often is) replaced with, e.g., $y/(1\wedge|y|)$, or any other measurable function that is $y+O(|y|^2)$ $[O(1)]$ as $|y|\to 0$ $[|y|\to\infty]$. The only (other) effect of this replacement is that the value of γ changes.

Exercise 15 Explain the claim about replacement of $\mathbf{1}_{\{|y|\le1\}}$ above.

Corollary 2.2 *If Y is* id *with triplet (A,ν,γ) such that*

$$\text{(the drift)} \qquad \gamma_0 \equiv \gamma - \int_{|y|\le1} y\,d\nu(y) \qquad \text{is well-defined and finite,}$$

then we may rewrite the Lévy-Khintchine Formula with a new triplet $(A,\nu,\gamma_0)_0$, as

$$\mathbf{E}\{e^{i\langle\theta,Y\rangle}\} = \exp\{-\tfrac{1}{2}\langle\theta,A\theta\rangle + i\langle\theta,\gamma_0\rangle + \int_{\mathbb{R}^d}(e^{i\langle\theta,y\rangle}-1)\,d\nu(y)\} \qquad \text{for} \quad \theta\in\mathbb{R}^d.$$

Corollary 2.3 *If Y is* id *with triplet (A,ν,γ) such that*

$$\text{(the center)} \qquad \gamma_1 \equiv \gamma + \int_{|y|>1} y\,d\nu(y) \qquad \text{is well-defined and finite,}$$

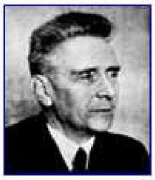
then we may rewrite the Lévy-Khintchine Formula with a new triplet $(A,\nu,\gamma_1)_1$, as

$$\mathbf{E}\{e^{i\langle\theta,Y\rangle}\} = \exp\{-\tfrac{1}{2}\langle\theta,A\theta\rangle + i\langle\theta,\gamma_1\rangle + \int_{\mathbb{R}^d}(e^{i\langle\theta,y\rangle}-1-i\langle\theta,y\rangle)\,d\nu(y)\} \qquad \text{for} \quad \theta\in\mathbb{R}^d.$$

On http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Khinchin.html we found

Aleksandr Yakovlevich Khinchin

Born: 19 July 1894 in Kondrovo, Kaluzhskaya guberniya, Russia
Died: 18 Nov 1959 in Moscow, USSR



Aleksandr Yakovlevich Khinchin's father was an engineer. Khinchin attended the technical high school in Moscow where he became fascinated by mathematics. However mathematics was certainly not his only interest when he was at secondary school for he also had a passionate love of poetry and of the theatre. He completed his secondary education in 1911 and entered the Faculty of Physics and Mathematics of Moscow University in that year.

At university in Moscow Khinchin worked with Luzin and others. He was an outstanding student being particularly interested in the metric theory of functions and before he graduated in 1916 he had already written his first paper on a generalisation of the Denjoy integral. This first paper began a series of publications by Khinchin on properties of functions which are retained after deleting a set of density zero at a given point. He summarised his contributions to this area with the paper *Recherches sur la structure des fonctions measurables* in *Fundamanta mathematica* in 1927.

After graduating in 1916, Khinchin remained at Moscow University undertaking research for his dissertation which would allow him to become a university teacher. After a couple of years he began teaching in a number of different colleges both in Moscow and Ivanovo. The town of Ivanovo, east of Moscow, was a centre for the textile industry and it plays a surprisingly important part in the development of Russian mathematics with several of the major figures teaching in the town.

Around 1922 Khinchin took up new mathematical interests when he began to study the theory of numbers and probability theory. In the following year he strengthened results of Hardy and Littlewood with his introduction of the iterated logarithm published in *Mathematische Zeitschrift*. With these ideas he also strengthened the law of large numbers due to Borel.

In 1927 Khinchin was appointed as a professor at Moscow University and, in the same year, he published *Basic laws of probability theory*. Between 1932 and 1934 he laid the foundations for the theory of stationary random processes culminating in a major paper in *Mathematische Annalen* in 1934. Khinchin left Moscow in 1935 to spend two years at Saratov University but returned to Moscow University in 1937 to continue his role of building the school of probability theory there in partnership with Kolmogorov and others, including in particular their student Gnedenko. From the 1940s his work changed direction again and this time he became interested in the theory of statistical mechanics. In the last few years of his life his interests turned to developing Shannon's ideas on information theory.

We shall look at some of Khinchin's major publications and in this way get a feel for the large number of important contributions he made in a remarkably large range of topics. Some of these publications we have already mentioned in the brief description of his career which we gave above.

Khinchin first published the book *Continued Fractions* in 1936 with a second edition being published in 1949. The book consists of three chapters, the first two of which present the classical theory of continued fractions. The third chapter, the longest and most important, contains an account of Khinchin's own contributions to the topic of the metrical theory of Diophantine approximations. Another contribution by Khinchin to number theory is the short book *Three pearls of number theory* which appeared in an English translation in 1952.

The book *Eight lectures on mathematical analysis* by Khinchin ran to several editions. It was first published in 1943 and the eight lectures it contains are: Continuum; Limits; Functions; Series; Derivative; Integral; Series expansions of functions; and Differential equations. The book was designed to be used to supplement a standard course on the calculus and gives a careful treatment of some of the basic notions of mathematical analysis. Ivanov, reviewing the fourth edition, wrote:-

> *The presentation is smooth, elegant and interesting and makes very enjoyable reading ...*

Khinchin published *Mathematical Principles of Statistical Mechanics* in 1943. It showed how to make classical statistical mechanics into a mathematically rigorous subject, developing a consistent presentation of the topic. In 1951 he extended the work of this 1943 book when he published *Mathematical foundations of quantum statistics*. This new publication on the topic appeared in a German translation in 1956 and then in an English translation in 1960. The book was written in such a way as to be useful both to mathematicians who wanted to become better acquainted with some applications of analysis to physics, and also to physicists who wanted to understand more about the mathematical foundations for their subject. Topics covered included: local limit theorems for sums of identically distributed random variables; the foundations of quantum mechanics; general principles of quantum statistics; the foundations of the statistics of photons; entropy; and the second law of thermodynamics. The book has been rated as being equal in quality to von Neumann's masterpiece *Mathematical foundations of quantum mechanics*.

Khinchin's book *Mathematical Foundations of Information Theory*, translated into English from the original Russian in 1957, is important. It consists of English translations of two articles: *The entropy concept in probability theory* and *On the basic theorems of information theory* which were both published earlier in Russian. The second of these articles provides a refinement of Shannon's concepts of the capacity of a noisy channel and the entropy of a source. Khinchin generalised some of Shannon's results in this book which was written in an elementary style yet gave a comprehensive account with full details of all the results.

In [6] Gnedenko, who was a student of Khinchin, lists 151 publications by Khinchin on the mathematical theory of probability (the list is given again in [4]).

Among the many honours which Khinchin received for his work was election to the Soviet Academy of Sciences in 1939 and the award of a State Prize for scientific achievements in the following year.

Vere-Jones writes [9]:-

> *Khinchin was a fascinating figure ..., not least because of his early enthusiasms for poetry and acting, and his links with such figures of the revolution as the poet Mayakovsky and members of the Moscow Arts Theatre.*

Article by: *J J O'Connor* and *E F Robertson*

Definition 2.4 *The generating triplet of a Lévy process is the triplet of* $X(1)$.

Exercise 16 Compute all well-defined generating triplets for `Bm` and for `Pp`.

Definition 2.5 *A Lévy process with generating triplet* (A,ν,γ) *is of*

$$\begin{cases} \underline{\textit{type A}} & \textit{if } A=0 \textit{ and } \nu(\mathbb{R}^d)<\infty, \\ \underline{\textit{type B}} & \textit{if } A=0 \textit{ and } \nu(\mathbb{R}^d)=\infty \textit{ but } \int_{|x|\le 1}|x|\,d\nu(x)<\infty, \\ \underline{\textit{type C}} & \textit{if } A\neq 0 \textit{ or } \int_{|x|\le 1}|x|\,d\nu(x)=\infty. \end{cases}$$

Many important properties of Lévy processes and `id rv's` vary with the type.

Exercise 17 Determine the types of `Bm` and `Pp`.

2.2 Compound Poisson Processes

Definition 2.6 *A compound* `Pp` *is a Lévy process with generating triplet* $(0,c\sigma,0)_0$, *where* $c>0$ *is a constant and* σ *a probability measure on* $\mathbb{R}^d$ *with* $\sigma(\{0\})=0$.

Compound `Pp` are crucail in the proof of many theorems on `id` phenomena.

Exercise 18 Determine the generating triplets $(\cdot,\cdot,\cdot)$ and $(\cdot,\cdot,\cdot)_1$ for a compound `Pp`. Show that a Lévy process is a compound `Pp` iff. it has generating triplet $(0,\nu,0)_0$ with $\nu(\mathbb{R}^d)\in(0,\infty)$.

Theorem 2.7 *Let* $\{N(t)\}_{t\ge 0}$ *be a* `Pp` *with intensity* c, *and* $\{Y_k\}_{k=1}^{\infty}$ `iid rv's`, *independent of* N, *with* $\mathcal{L}(Y_k)=\sigma$ *where* $\sigma(\{0\})=0$. *Denoting* $S_n=\sum_{k=1}^n Y_k$ *for* $n\in\mathbb{N}$, $X(t)\equiv S_{N(t)}$ *is a compound* `Pp` *with generating triplet* $(0,c\sigma,0)_0$.

Proof. Càdlàg sample paths and $X(0)=_d 0$ are immediate. $\mathbf{P}$-continuity follows from

$$\mathbf{P}\{|X(t+s)-X(s)|>\varepsilon\}\le\mathbf{P}\{|N(t+s)-N(s)|>0\}=1-\mathrm{e}^{-c|t|}\to 0 \qquad \text{as} \quad t\to 0.$$

Independence and homogeneity of increments come by conditioning on the values of N involved, see Exercise 19 below. The generating triplet is the claimed one, since

$$\mathbf{E}\{\mathrm{e}^{i\langle\theta,X(1)\rangle}\}=\sum_{n=0}^{\infty}\mathbf{E}\{\mathrm{e}^{i\langle\theta,S_1\rangle}\}^n\frac{c^n}{n!}\mathrm{e}^{-c}=\exp\{c\int_{\mathbb{R}^d}(\mathrm{e}^{i\langle\theta,y\rangle}-1)\,d\sigma(y)\}. \quad \square$$

EXERCISE 19 Show that the process $S_{N(t)}$ in Theorem 2.7 has independent and time homogeneous increments.

EXERCISE 20 The $\boxed{\Leftarrow}$ part of Lévy-Khintchine Formula is easy: Show that it is enough to prove that $\varphi(\theta) = \exp\{\int_{\mathbb{R}^d}(e^{i\langle\theta,y\rangle}-1-\mathbf{1}_{\{|y|\leq 1\}}i\langle\theta,y\rangle)\,d\nu(y)\}$ is a `chf` for any Lévy measure ν [i.e., measure with $\int_{\mathbb{R}^d}|y|^2\wedge 1\,d\nu(y)<\infty$]. Prove that φ is a `chf` by considering $\exp\{\int_{|y|>\varepsilon}(e^{i\langle\theta,y\rangle}-1-\mathbf{1}_{\{|y|\leq 1\}}i\langle\theta,y\rangle)\,d\nu(y)\}$ and sending $\varepsilon\downarrow 0$.

2.3 Moments

Definition 2.8 *A function $g:\mathbb{R}^d\to[0,\infty)$ is <u>submultiplicative</u> if*

$$g(x+y)\leq a\,g(x)g(y) \qquad \text{for} \quad x,y\in\mathbb{R}^d, \quad \text{for some constant} \quad a>0.$$

EXERCISE 21 Show that products of submultiplicative functions are submultiplicative, as are $g(x)=|x|\vee 1$ and $g(x)=e^{|x|^b}$ for $b\in(0,1]$.

Theorem 2.9 (KRUGLOV) *For g locally bounded submultiplicative measurable, and X a Lévy process with Lévy measure ν, the following conditions are equivalent:*

(1) $\mathbf{E}\{g(X(t))\}<\infty$ *for some $t>0$;*

(2) $\mathbf{E}\{g(X(t))\}<\infty$ *for each $t>0$;*

(3) $\int_{|x|>1}g(x)\,d\nu(x)<\infty$.

EXERCISE 22 Show that, for a Lévy process X with Lévy measure ν, $\mathbf{E}\{|X(t)|\}<\infty$ for some (each) $t>0$ iff. $\int_{|x|>1}|x|\,d\nu(x)<\infty$.

Corollary 2.10 *An* `id` *Y with triplet (A,ν,γ) has well-defined expected value, which in that case coincides with the center γ_1, iff.*

$$\int_{|x|>1}|x|\,d\nu(x)<\infty.$$

Proof. The existence issue is contained in Exercise 22. Assuming existence, expressing $\mathbf{E}\{e^{i\langle\theta,Y\rangle}\}$ with the triplet $(A,\nu,\gamma_1)_1$, we readily get $\frac{\partial}{\partial\theta_k}\mathbf{E}\{e^{i\theta_k Y_k}\}\,|_{\theta_k=0}=i(\gamma_1)_k$. □

Existence of an odd derivative at zero for a `chf` is necessary but not sufficient for existence of the corresponding odd moment. However, existence of an even derivative at zero for a `chf` is equivalent with existence of the corresponding even moment.

Lemma 2.11 *$g(x) \le b\mathrm{e}^{C|x|}$ for some constants $b, C > 0$.*

Proof. With $b \equiv \max\{1/a, \sup_{x\in[0,1]} g(x)\}$, we get, for $|x| \in [n-1, n]$,

$$g(x) = g(\textstyle\sum_{k=1}^n \frac{x}{n}) \le a^{n-1} g(\frac{x}{n})^n \le a^{n-1} b^n \le b(ab)^{|x|} = b\mathrm{e}^{\ln(ab)|x|}. \quad \square$$

Lemma 2.12 *An $\mathbb{R}$-valued `id` `rv` with triplet $(\hat{A}, \hat{\nu}, \hat{\gamma}_1)_1$ such that $\hat{\nu}$ has bounded support, has an entire `chf`.*

Proof. Since $|(\mathrm{e}^{iy(\theta+h)} - 1 - iy(\theta+h)) - (\mathrm{e}^{iy\theta} - 1 - iy\theta)|/|h| = O(|\theta|^2)$ as $|\theta| \to 0$, uniformly for $|h|$ small enough, the following `chf` is differentiable, by elementary arguments,

$$\varphi(\theta) = \exp\{-\tfrac{1}{2}\hat{A}\theta^2 + i\hat{\gamma}_1\theta + \textstyle\int_{\mathbb{R}} (\mathrm{e}^{iy\theta} - 1 - iy\theta)\, d\hat{\nu}(y)\}, \quad \theta \in \mathbb{C}. \quad \square$$

Lemma 2.13 *For an $\mathbb{R}$-valued `rv` Y with entire `chf`, $\mathbf{E}\{\mathrm{e}^{C|Y|}\} < \infty$ for $C > 0$.*

Proof. Existence of $\varphi^{(2n)}(0)$ gives $\mathbf{E}\{Y^{2n}\} < \infty$ for $n \in \mathbb{N}$. Specifically,

$$\varphi(\theta) = \textstyle\sum_{n=0}^\infty \varphi^{(n)}(0)\, \theta^n/(n!) = \sum_{n=0}^\infty i^n\, \mathbf{E}\{Y^n\}\, \theta^n/(n!) \qquad \text{for} \quad \theta \in \mathbb{C}.$$

This power series is absolutely convergent on $\mathbb{C}$, so that

$$\textstyle\sum_{n=0}^\infty |\mathbf{E}\{Y^n\}|\, |\theta|^n/(n!) < \infty \;\Rightarrow\; \sum_{n=0}^\infty \mathbf{E}\{Y^{2n}\}\, |\theta|^{2n}/((2n)!) < \infty \qquad \text{for} \quad \theta \in \mathbb{C}.$$

Clearly, it is enough to show that $\mathbf{E}\{\cosh(C|Y|)\} < \infty$. However,

$$\mathbf{E}\{\cosh(C|Y|)\} = \mathbf{E}\{\textstyle\sum_{n=0}^\infty (CY)^{2n}/((2n)!)\} = \sum_{n=0}^\infty \mathbf{E}\{Y^{2n}\}\, C^{2n}/((2n)!) < \infty. \quad \square$$

Proof of Kruglov's Theorem. Let X have generating triplet (A, ν, γ). Denote $\nu_0 \equiv \nu \mathbf{1}_{\{|x|\le 1\}}$ and $\nu_1 \equiv \nu \mathbf{1}_{\{|x|>1\}}$. Let X^0 and X^1 be independent Lévy processes with generating triplets (A, ν_0, γ) and $(0, \nu_1, 0)_0$, respectively, so that $X =_d X^0 + X^1$.

$(1) \Rightarrow (3)$ If $\mathbf{E}\{g(X(t))\} < \infty$, since $X(t) =_d X^0(t) + X^1(t)$, we conclude that

$$\textstyle\int_{\mathbb{R}^d \times \mathbb{R}^d} g(x+y)\, dF_{X^1(t)}(y)\, dF_{X^0(t)}(x) < \infty \;\Rightarrow\; \int_{\mathbb{R}^d} g(x+y)\, dF_{X^1(t)}(y) < \infty$$

for some $x \in \mathbb{R}^d$. By Lemma 2.11, $g(y) \le a g(-x) g(x+y) \le ab\, \mathrm{e}^{C|x|} g(x+y)$, so that

$$\textstyle\int_{\mathbb{R}^d} g(y)\, dF_{X^1(t)}(y) = \mathbf{E}\{g(X^1(t))\} < \infty.$$

For $\nu_1(\mathbb{R}^d) > 0$, since $\nu_1(\mathbb{R}^d) < \infty$, by Exercise 18, X^1 is a compound Pp. Thus

$$\infty > \mathbf{E}\{g(X^1(t))\} = \mathbf{E}\{g(S_{N(t)})\} = \textstyle\sum_{n=0}^\infty \mathbf{E}\{g(S_n)\} \frac{(ct)^n}{n!} \mathrm{e}^{-ct}.$$

In particular, it follows that

$$\infty > \mathbf{E}\{g(S_1)\} = \textstyle\int_{\mathbb{R}^d} g(x)\, d\sigma(x) = c^{-1} \int_{\mathbb{R}^d} g(x)\, d\nu_1(x) = c^{-1} \int_{|x|>1} g(x)\, d\nu(x).$$

Of course, for $\nu_1(\mathbb{R}^d) = 0$, the right-hand side is trivially finite.

(3) $\Rightarrow$ (2) When $\int_{\mathbb{R}^d} g(x)\, d\nu_1(x) < \infty$, submultiplicativity gives

$$\textstyle\int_{(\mathbb{R}^d)^n} g(x_1 + \ldots + x_n)\, d\nu_1(x_1) \ldots d\nu_1(x_n) \le a^{n-1} [\int_{\mathbb{R}^d} g(x)\, d\nu_1(x)]^n.$$

From this we immediately get (cf. above)

$$\mathbf{E}\{g(X^1(t))\} = \textstyle\sum_{n=0}^\infty \mathbf{E}\{g(S_n)\} \frac{(ct)^n}{n!} \mathrm{e}^{-ct} \le \sum_{n=0}^\infty a^{n-1} [\int_{\mathbb{R}^d} g(x)\, d\nu_1(x)]^n \frac{t^n}{n!} \mathrm{e}^{-ct} < \infty.$$

Thus we are done if $\mathbf{E}\{\mathrm{e}^{C|X^0(t)|}\} < \infty$ since, by submultiplicativity and Lemma 2.11,

$$\mathbf{E}\{g(X(t))\} \le a \mathbf{E}\{g(X^0(t)) g(X^1(t))\} \le ab \mathbf{E}\{\mathrm{e}^{C|X^0(t)|}\} \mathbf{E}\{g(X^1(t))\}.$$

However, by Hölder's inequality together with Lemmas 2.12 and 2.13, we have

$$\mathbf{E}\{\mathrm{e}^{C|X^0(t)|}\} \le \mathbf{E}\{\mathrm{e}^{C \sum_{k=1}^d |X_k^0(t)|}\} \le \textstyle\prod_{k=1}^d \mathbf{E}\{\mathrm{e}^{C|X_k^0(t)|}\} < \infty. \quad \square$$

Exercise 23 Explain the representation $X =_d X^0 + X^1$ in the above proof.

Exercise 24 Explain why Lemmas 2.12 and 2.13 really apply to $X_k^0(t)$.

2.4 Something I came to think of this time ...

...JUST WHAT IS INDIAN SUMMER AND DID INDIANS REALLY HAVE ANYTHING TO DO WITH IT?...
By: Bill Deedler, Weather Historian, WFO Detroit/Pontiac Mi

An early American writer described Indian Summer well when he wrote, "The air is perfectly quiescent and all is stillness, as if Nature, after her exertions during the Summer, were now at rest." This passage belongs to the writer John Bradbury and was written nearly an "eternity" ago, back in 1817. But this passage is as relevant today as it was way back then. The term "Indian Summer" dates back to the 18th century in the United States. It can be defined as "any spell of warm, quiet, hazy weather that may occur in October or even early November." Basically, autumn is a transition season as the thunderstorms and severe weather of the summer give way to a tamer, calmer weather period before the turbulence of the winter commences.

The term "Indian Summer" is generally associated with a period of considerably above normal temperatures, accompanied by dry and hazy conditions ushered in on a south or southwesterly breeze. Several references make note of the fact that a true Indian Summer can not occur until there has been a killing frost/freeze. Since frost and freezing temperatures generally work their way south through the fall, this would give credence to the possibility of several Indian Summers occurring in a fall, especially across the northern areas where frost/freezes usually come early.

While almost exclusively thought of as an autumnal event, I was surprised to read that Indian Summers have been given credit for warm spells as late as December and January (but then, just where does that leave the "January Thaw" phenomenon?). Another topic of debate about Indian Summer has been "location, location". Evidently, some writers have made reference to it as native only to New England, while others have stated it happens over most of the United States, even along the Pacific coast. Probably the most common or accepted view on location for an Indian Summer would be from the Mid-Atlantic states north into New England, and than west across the Ohio Valley, Great Lakes, Midwest and Great Plains States. In other words, locations that generally have a winter on the horizon! But then, what about the king of winter weather in the United States, Alaska? Do they have an "Indian Summer", or something similar? Some places in Alaska are lucky to have a "summer", let alone an Indian Summer! One would certainly have to throw out the notion of it usually happening in October or November, when, winter generally has already taken an aggressive foothold on much of the state. What about other locations that come to mind, The Rocky Mountain States and parts of Canada, particularity in the east and south? Note: If anyone reading this has any information on Indian Summers in those areas questioned, or just thoughts on Indian Summers in general, leave us a note in our "guestbook" section.

A typical weather map that reflects Indian Summer weather involves a large area of high pressure along or just off the East Coast. Occasionally, it will be this same high pressure that produced the frost/freeze conditions only a few nights before, as it moved out of Canada across the Plains, Midwest and Great Lakes and then finally, to the East Coast. Much warmer temperatures, from the deep South and Southwest, are then pulled north on southerly breezes resulting from the clockwise rotation of wind around the high pressure. It is characteristic for these conditions to last for at least a few days to well over a week and there may be several cases before winter sets in. Such a mild spell is usually broken when a strong low pressure system and attending cold front pushes across the region. This dramatic change results from a sharp shift in the upper winds or "jet stream" from the south or southwest to northwest or north. Of course, there can be some modifications to the above weather map scenario, but for simplicity and common occurrence sake, this is the general weather map.

Now we come to the origin of the term itself, "Indian Summer". Over the years, there has been a considerable amount of interest given to this topic in literature. Probably one of the most intensive studies occurred way back around the turn of the century. A paper by Albert Matthews, written in 1902, made an exhaustive study of the historical usage of the term. Evidently, the credit for the first usage of the term was mistakenly given to a man by the name of Major Ebenezer Denny, who used it in his "Journal", dated October 13th, 1794. The journal was kept at a town called Le Boeuf, which was near the present day city of Erie, Pennsylvania. Matthews however, uncovered an earlier usage of the term in 1778 by a frenchman called St. John de Crevecoeur. It appeared in a letter Crevecoeur wrote dated "German-flats, 17 Janvier, 1778." The following is a translation of a portion of the letter:

"Sometimes the rain is followed by an interval of calm and warmth which is called the Indian Summer; its characteristics are a tranquil atmosphere and general smokiness. Up to this epoch the approaches of winter are doubtful; it arrives about the middle of November, although snows and brief freezes often occur long before that date."

3 Lecture 3 13/9 - 02

3.1 Stable Random Variables and Processes

Definition 3.1 *Y is stable if, for each $n\in\mathbb{N}$, with $Y_1,\ldots,Y_n$* `iid` *copies of Y,*

$$Y_1+\ldots+Y_n =_d bY+c \qquad \text{for some constants} \quad b=b(n)>0 \quad \text{and} \quad c=c(n)\in\mathbb{R}^d.$$

Y is strictly stable if it is possible to take $c(n)=0$ for $n\in\mathbb{N}$.

Definition 3.1 may be rewritten $Y^{\star 1/n} =_d \frac{1}{b}Y - \frac{c}{n}$. And so stable `rv's` are `id`.

Definition 3.2 *A Lévy process X with $X(1)$ (strictly) stable is called a (strictly) stable Lévy motion.*

Definition 3.3 *An $\mathbb{R}$-valued stochastic process $\{Z(t)\}_{t\in T}$ is stable if, for each $n\in\mathbb{N}$, with $Z_1,\ldots,Z_n$* `iid` *copies of Z,*

$$Z_1+\ldots+Z_n =_d b_n Z + c_n \qquad \text{for some constant} \quad b_n>0 \quad \text{and function} \quad c_n : T\to\mathbb{R}.$$

Z is strictly stable if it is possible to take $c_n\equiv 0$ for $n\in\mathbb{N}$.

Sato uses the old-fashioned language to call stable Lévy motions stable processes. This was standard some decades ago, when stable processes in the sense of Definition 3.3 had not been studied. Now most authors have switched to the language we use.

Stable distributions are among the few most important `id` distributions. Two reasons are their stability under additon (Definition 3.1), and the explicitness of their `chf` (see below). Stable processes occupy a similar position among `id` processes. They have become fashionable since naturally having heavy tails and long range dependence.

Exercise 25 Show that stable Lévy motions are stable processes.

Definition 3.4 *Y is trivial* (or degenerate), *if $Y =_d c$ for some constant $c\in\mathbb{R}^d$.*

Definition 3.5 *A Lévy process X is trivial if $X(1)$ is trivial.*

Exercise 26 Show that a Lévy process X is trivial iff. $X =_d \{ct\}_{t\geq 0}$ for some constant $c\in\mathbb{R}^d$.

Theorem 3.6 *For Y non-trivial stable, there exists a unique constant $\alpha \in (0,2]$ such that*

$$Y^{\star t} =_d t^{1/\alpha} Y + c \qquad \text{for} \quad t>0, \quad \text{for some constant} \quad c = c(t) \in \mathbb{R}^d.$$

For Y non-trivial strictly stable, $c(t)=0$ for $t>0$.

The proof is not difficult, but a bit technical. It belongs to basic probability courses.

Definition 3.7 *A stable Y is called α-stable, $\alpha \in (0,2]$, whenever*

$$Y^{\star t} =_d t^{1/\alpha} Y + c \qquad \text{for} \quad t>0, \quad \text{for some constant} \quad c = c(t) \in \mathbb{R}^d.$$

Y is called strictly α-stable if $c(t)=0$ for $t>0$.

EXERCISE 27 For Y (strictly) α-stable, show that $Y^{\star t}$ is (strictly) α-stable for $t>0$. Show that values of (strictly) α-stable Lévy motions are (strictly) α-stable.

Definition 3.8 *X is self-similar with index $\kappa > 0$ if*

$$\{X(\lambda t)\}_{t\geq 0} =_d \{\lambda^{\kappa} X(t)\}_{t\geq 0} \qquad \text{for} \quad \lambda > 0.$$

Fact 3.9 *A Lévy process is self-similar iff. it is a strictly stable Lévy motion.*

Proof. $\boxed{\Rightarrow}$ Self-similarity and Corollary 1.27 give $X(1)^{\star 1/n} =_d X(\frac{1}{n}) =_d n^{-\kappa} X(1)$.

$\boxed{\Leftarrow}$ Take Y strictly stable and X a Lévy process with $X(1) =_d Y$. For Y trivial, self-similarity with $\kappa = 1$ follows from Exercise 26. For Y non-trivial, so that Y is strictly α-stable (Theorem 3.6), we get $X(\lambda t) - X(\lambda s) =_d \lambda^{1/\alpha}(X(t) - X(s))$, since

$$X(\lambda(t-s)) =_d X(1)^{\star\lambda(t-s)} =_d (\lambda(t-s))^{1/\alpha} X(1) =_d \lambda^{1/\alpha} X(1)^{\star(t-s)} =_d \lambda^{1/\alpha} X(t-s).$$

From this we conclude that, for $0 < t_1 < \ldots < t_n$,

$$(X(\lambda t_1), X(\lambda(t_2 - t_1)), \ldots, X(\lambda(t_n - t_{n-1}))) =_d \lambda^{1/\alpha}(X(t_1), X(t_2 - t_1), \ldots, X(t_n - t_{n-1})),$$

which by a familiar argument is equivalent with sought after

$$(X(\lambda t_1), \ldots, X(\lambda t_n)) =_d (\lambda^{1/\alpha} X(t_1), \ldots, \lambda^{1/\alpha} X(t_n)). \quad \square$$

We now give the two most important representations of stable `chf`. Historically, the literature in this area, articles as well as books, is heavily polluted by technical errors, see "*Hall: A comedy* *of errors: the cannonical form for a stable characteristic function*". The book "*Samorodnitsky & Taqqu: Stable Non-Gaussian Processes*" from 1994 has put things in order. (Be careful with earlier literature!)

Theorem 3.10 *Y is 2-stable iff. it is Gaussian. Y is α-stable, $\alpha\in(0,2)$, iff.*

$$\mathbf{E}\{e^{i\langle\theta,Y\rangle}\} = \begin{cases} \exp\{-\int_{S^d} |\langle\theta,y\rangle|^\alpha(1-i\tan(\frac{\pi\alpha}{2})\,\mathrm{sign}(\langle\theta,y\rangle))\,d\mu(x)+i\langle\theta,\tau\rangle\}, & \alpha\neq 1 \\ \exp\{-\int_{S^d} |\langle\theta,y\rangle|\,(1+i\frac{2}{\pi}\ln(|\langle\theta,y\rangle|)\,\mathrm{sign}(\langle\theta,y\rangle))\,d\mu(x)+i\langle\theta,\tau\rangle\}, & \alpha=1 \end{cases}$$

for $\theta\in\mathbb{R}^d$, where μ is a unique finite measure on $\underline{S^d}\equiv\{x\in\mathbb{R}^d:|x|=1\}$, and $\tau\in\mathbb{R}^d$ a unique constant that is the drift γ_0 for $\alpha\in(0,1)$ and the center γ_1 for $\alpha\in(1,2)$.

Y is α-stable, $\alpha\in(0,2)$, iff. it is `id` *with triplet $(0,\nu,\gamma)$ such that*

$$\nu(B) = \int_{x\in S^d}\int_{x=0}^{x=\infty} \mathbf{1}_B(rx)\,r^{-(\alpha+1)}\,dr\,d\lambda(x) \qquad \text{for} \quad B\in\mathcal{B}(\mathbb{R}^d),$$

where λ is a unique finite measure on S^d.

The proof is somewhat difficult, and will not be attempted here. The result is only marginally important for us, so there is really no reason to do it.

Exercise 28 In one direction Theorem 3.10 is trivial: Prove that part.

Corollary 3.11 *An $\mathbb{R}$-valued* `id rv` *Y is α-stable, $\alpha\in(0,2)$, iff. it has triplet $(0,\nu,\gamma)$ such that, for some unique constants $c_1,c_2\geq 0$,*

$$d\nu(x) = (c_1\mathbf{1}_{(0,\infty)}(x)+c_2\mathbf{1}_{(-\infty,0)}(x))\,|x|^{-(\alpha+1)}\,dx \qquad \text{for} \quad x\in\mathbb{R}.$$

An $\mathbb{R}$-valued `rv` *Y is α-stable, $\alpha\in(0,2)$, iff.*

$$\mathbf{E}\{e^{i\theta Y}\} = \begin{cases} \exp\{-c|\theta|^\alpha(1-i\beta\tan(\frac{\pi\alpha}{2})\,\mathrm{sign}(\theta))+i\theta\tau\}, & \alpha\neq 1 \\ \exp\{-c|\theta|(1+i\beta\frac{2}{\pi}\ln(|\theta|)\,\mathrm{sign}(\theta))+i\theta\tau\}, & \alpha=1 \end{cases} \qquad \text{for} \quad \theta\in\mathbb{R},$$

with $c\geq 0$, $\beta=\frac{c_1-c_2}{c_1+c_2}\in[-1,1]$ and $\tau\in\mathbb{R}$ constants that are unique for Y non-trivial. Here τ is the drift γ_0 for $\alpha\in(0,1)$ and the center (mean) γ_1 for $\alpha\in(1,2)$.

For Y with the `chf` in the second part of Corollary 3.11 we write $\underline{Y\sim S_\alpha(c,\beta,\tau)}$.

Exercise 29 Derive Corollary 3.11 from Theorem 3.10.

Corollary 3.12 *For an $\mathbb{R}$-valued α-stable* `rv`, *$\alpha\in(0,2)$, $\mathbf{E}\{|Y|^p\}<\infty$ iff. $p<\alpha$.*

Exercise 30 Prove Corollary 3.12.

There are four univariate α-stable distributions that are known explicitely in closed form, one of which is the trivial distribution. The other three are given in Exercises 31-33 below. Further, the distribution with $\alpha=\frac{3}{2}$ and $\beta=0$ is called Holtsmark distribution.

EXERCISE 31 Show that an rv with the density below is $S_1(c,0,\tau)$-distributed

$$f(y) = (c/\pi)\,((y-\tau)^2+c^2)^{-1} \qquad \text{for} \quad y\in\mathbb{R} \qquad (\text{CAUCHY DISTRIBUTION}).$$

EXERCISE 32 Show that an rv with the density below is $S_{1/2}(c,1,\tau)$-distributed

$$f(y) = \sqrt{c/(2\pi)}\,(y-\tau)^{-3/2}\,\mathrm{e}^{-c/(2(y-\tau))} \qquad \text{for} \quad y>\tau \qquad (\text{LÉVY DISTRIBUTION}).$$

EXERCISE 33 Show that Gaussian `rv's` are 2-stable. Are they strictly stable?

There exists a simple explicit formula to simulate univariate α-stable `rv's` in a computer. Multivariate α-stable `rv's` are considerably more difficicult to simulate.

Theorem 3.13 *For V and W independent* `rv's` *with uniform distribution over $(-\frac{\pi}{2},\frac{\pi}{2})$ and standard exponential distribution, respectively, and for constants $\alpha\in(0,2)$, $c\geq 0$, $\beta\in[-1,1]$ and $\tau\in\mathbb{R}$, we have*

$$c\,\frac{\sin[\alpha V+\tan^{-1}(\beta\tan(\frac{\pi\alpha}{2}))]\,(\cos[(1-\alpha)V-\tan^{-1}(\beta\tan(\frac{\pi\alpha}{2}))])^{1/\alpha-1}}{(\cos[\tan^{-1}(\beta\tan(\frac{\pi\alpha}{2}))])^{1/\alpha}\,(\cos(V))^{1/\alpha}\,W^{1/\alpha-1}}+\tau\sim S_\alpha(c,\beta,\tau).$$

The proof is not difficult at all, but is of little interest from our point of view.

EXERCISE 34 Derive a formula for simulating Gaussian `rv's` by sending $\alpha\uparrow 2$.

3.2 Something I came to think of this time ...

Vi väcker er inte då ni skall gå hem!

4 **Lecture 4 17/9 - 02**

4.1 Continuity Properties of `id` Distributions

Definition 4.1 *A measure ρ on $\mathbb{R}^d$ is*

- *discrete if $\rho(\mathbb{R}^d \setminus C) = 0$ for some countable $C \subseteq \mathbb{R}^d$;*
- *continuous if $\rho(\{x\}) = 0$ for $x \in \mathbb{R}^d$;*
- *absolutely continuous if $\rho(B) = 0$ for null-events $B \in \mathcal{B}(\mathbb{R}^d)$.*

Lemma 4.2 *For $\rho = \rho_1 \star \rho_2$ with ρ_1 and ρ_2 non-zero finite measures on $\mathbb{R}^d$,*

(1) *ρ is discrete iff. ρ_1 and ρ_2 are discrete;*

(2) *ρ is continuous iff. ρ_1 or ρ_2 is continuous;*

(3) *ρ is absolutely continuous iff. ρ_1 or ρ_2 is absolutely continuous.*

Exercise 35 Prove Lemma 4.2.

Theorem 4.3 (Döblin) *For a Lévy process X the following three conditions are equivalent*

(1) *$\mathcal{L}(X(t))$ is continuous for each $t > 0$;*

(2) *$\mathcal{L}(X(t))$ is continuous for some $t > 0$;*

(3) *X is of type B or type C.*

Proof. $\neg(3) \Rightarrow \neg(2)$ Let X be type A, i.e., $A = 0$ and $\nu(\mathbb{R}^d) < \infty$. Then either $\nu(\mathbb{R}^d) = 0$, so that X is trivial and $\mathcal{L}(X(t))$ not continuous, or $\nu(\mathbb{R}^d) > 0$, so that $X(t) - \gamma_0 t$ is compound `Pp` and has an atom at 0 with mass $\mathbf{P}\{N(t) = 0\} = \mathrm{e}^{-\nu(\mathbb{R}^d)t}$.

$(3) \Rightarrow (1)$ If $A \neq 0$, then X has a non-trivial Gaussian component, which is continuously distributed (albeit not necessarily absolutely continuous). Hence (1) follows from Lemma 4.2.2. In the rest of the proof we may thus assume that $\nu(\mathbb{R}^d) = \infty$.

For ν discrete, let $\{x_j\}_{j=1}^{\infty}$ be the atoms with $\nu(\{x_j\}) \equiv m_j > 0$. Put $m_j' = m_j \wedge 1$, so that $\sum_{j=1}^{\infty} m_j' = \infty$ (since $\sum_{j=1}^{\infty} m_j = \infty$). Let $X =_d X_n^0 + X_n^1$, with X_n^0 and X_n^1 independent, and X_n^0 compound `Pp` with Lévy measure $\nu_n \equiv \sum_{j=1}^{n} m_j' \delta_{x_j}$. Put $D(Y) \equiv \sup\{\mathbf{P}\{Y = x\} : x \in \mathbb{R}^d\}$. Here $D(Y_1 + Y_2) \leq D(Y_1)$ for Y_1 and Y_2 independent, since

$$\mathbf{P}\{Y_1 + Y_2 = x\} = \int_{\mathbb{R}^d} \mathbf{P}\{Y_1 = x - y_2\}\, dF_{Y_2}(y_2) \leq D(Y_1) \qquad \text{for} \quad x \in \mathbb{R}^d.$$

Hence $D(X(t)) \leq D(X_n^0(t))$. Let $c_n = \nu_n(\mathbb{R}^d)$ and $\sigma_n = c_n^{-1}\nu_n$. Since $D(\sigma_n\star\ldots\star\sigma_n) \leq D(\sigma_n) \leq c_n^{-1}$, by the construction of ν_n, we have, for any $x\in\mathbb{R}^d$, as $n\to\infty$,

$$\mathbf{P}\{X_n^0(t)=x\} = \mathrm{e}^{-c_n t}\sum_{k=0}^{\infty}\frac{(c_n t)^k}{k!}(\sigma_n\star\ldots\star\sigma_n)(\{x\}) \leq \mathrm{e}^{-c_n t}+c_n^{-1} \to 0.$$

For $\underline{\nu \text{ continuous}}$, let X_n^0 be a compound Pp with Lévy measure $\nu_n = \nu\mathbf{1}_{\{|x|>1/n\}}$. Since $\nu_n\star\ldots\star\nu_n$ is continuous (by Lemma 4.2.2), $X_n^0(t)$ only has an atom at zero with mass $\mathrm{e}^{-c_n t}\to 0$ as $n\to\infty$, so that again $D(X(t))=0$ [since $\leq D(X_n^0(t))$].

General case. Since $\nu\mathbf{1}_{\{|x|>1/n\}}$ is finite, $\nu=\nu_c+\nu_d$ with ν_c continuous and ν_d discrete, and one of them infinite. To that guy we apply one of the above arguments, giving continuity for that component, and thus for $X(t)$ by Lemma 4.2.2. □

EXERCISE 36 Explain in detail what is going on in the last part of the proof.

Corollary 4.4 *For a Lévy process X the following three conditions are equivalent*

(1) $\mathcal{L}(X(t))$ *is discrete for each* $t>0$;

(2) $\mathcal{L}(X(t))$ *is discrete for some* $t>0$;

(3) X *is of type A with discrete Lévy measure* ν.

EXERCISE 37 Derive Corollary 4.4 from Theorem 4.3 (and Lemma 4.2).

Lemma 4.5 *Given $n\in\mathbb{N}$, Y has a density $f_Y\in\mathbb{C}^n(\mathbb{R}^d)$, with corresponding partial derivatives that all tend to zero at infinity, provided that*

$$\int_{\mathbb{R}^d}|\theta|^n\,|\varphi_Y(\theta)|\,d\theta < \infty.$$

Proof. By elementary theory of chf, Y has a continuous density

$$f_Y(y) = (2\pi)^{-d}\int_{\mathbb{R}^d}\mathrm{e}^{-i\langle\theta,y\rangle}\,\varphi_Y(\theta)\,d\theta \qquad \text{for} \quad y\in\mathbb{R}^d.$$

Using the assumption of the lemma together with dominated convergence, we readily get $f_Y\in\mathbb{C}^n(\mathbb{R}^d)$. The claim at infinity is the Riemann-Lebesgue Lemma. □

Theorem 4.6 (OREY) *An $\mathbb{R}$-valued* id *Y with Lévy measure ν such that*

$$\liminf_{r\downarrow 0} r^{\alpha-2}\int_{[-r,r]}x^2\,d\nu(x) > 0 \qquad \textit{for some} \quad \alpha\in(0,2),$$

has a density $f_Y\in\mathbb{C}^\infty(\mathbb{R})$, all derivatives of which tends to zero at infinity.

Proof. We check that the hypothesis of Lemma 4.5 holds for all n: Pick $c_1 > 0$ such that $\int_{[-r,r]} x^2\,d\nu(x) \geq c_1 r^{2-\alpha}$ for $r>0$ small enough. Pick $c_2>0$ such that $1-\cos(u) \geq c_2 u^2$ for $|u|\leq 1$. By inspection of Lévy-Khintchin Formula, we get

$$|\varphi_Y(\theta)| \leq \exp\{\int_{\mathbb{R}}(\cos(\theta x)-1)\,d\nu(x)\} \leq \exp\{-c_2\int_{|x|\leq 1/|\theta|}\theta^2 x^2\,d\nu(x)\} \leq \mathrm{e}^{-c_1c_2|\theta|^\alpha}. \quad \square$$

Exercise 38 Show that non-trivial univariate α-stable `rv's` have $\mathbb{C}^\infty$-densities.

4.2 Selfdecomposable Distributions

Definition 4.7 *An* `id` Y *is selfdecomposable* (or of class L), *if there exists a function* $k: S^d\times(0,\infty)\to[0,\infty)$, *that is measurable in its first argument and non-increasing in the second, such that, for some finite measure* λ *on* S^d, Y *has Lévy measure*

$$\nu(B) = \int_{x\in S^d}\int_{x=0}^{x=\infty} \mathbf{1}_B(rx)\,k(x,r)\,dr/r\,d\lambda(x) \qquad \textit{for} \quad B\in\mathcal{B}(\mathbb{R}^d).$$

By Therem 3.10, univariate α-stable `rv's` are selfdecomposable with $k(x,r)=r^{-\alpha}$.

Exercise 39 Show that exponential `rv's` are selfdecomposable but not stable.

Definition 4.8 *A Lévy process* X *with* $X(1)$ *selfdecomposable is called a selfdecomposable process.*

Definition 4.9 *An* $\mathbb{R}$*-valued* Y *is unimodal with mode* $a\in\mathbb{R}$, *if it has probability distribution* $dF_Y(y) = c\delta_a(y)+f(y)\,dy$, *for some constant* $c\in[0,1]$ *and subprobability density* f *that is non-decreasing on* $(-\infty,a)$ *and non-increasing on* (a,∞).

Unimodality is an important property that simplifies or even "saves" many proofs.

Theorem 4.10 (Yamazato) *Selfdecomposable distributions on* $\mathbb{R}$ *are unimodal.*

The proof is technical, and not really of interest to us. It is famous for having been preceeded by several faulty proofs, as well as disproofs (i.e., counterexamples).

Theorem 4.11 (Sato) *Non-trivial selfdecomposable distributions on* $\mathbb{R}$ *are absolutely continuous.*

Againg the proof is technical, and not of interest to us.

Theorem 4.12 *Y is selfdecomposable iff., for each $b>1$,*

$$\varphi_Y(\theta)=\varphi_Y(\theta/b)\varphi_{\rho_b}(\theta) \quad \text{for} \quad \theta\in\mathbb{R}^d, \quad \text{for some probability measure } \rho_b \text{ on } \mathbb{R}^d.$$

Proof. $\boxed{\Leftarrow}$ This is quite long and technical and cannot be done here.

$\boxed{\Rightarrow}$ For the Lévy measure ν of Y, we have, for $b>1$,

$$\nu(bB)=\int_{S^d}\int_0^\infty \mathbf{1}_{bB}(rx)k(x,r)\,dr/r\,d\lambda(x)=\int_{S^d}\int_0^\infty \mathbf{1}_B(\hat{r}x)k(x,\hat{r}b)\,d\hat{r}/\hat{r}\,d\lambda(x)\leq\nu(B).$$

Write $\nu=\nu_0+\nu_1$, where $\nu_0(B)=\nu(bB)$ and $\nu_1(B)=\nu(B)-\nu_0(B)$ are both Lévy measures. Then Lévy-Khintchine Formula together with routine calculations give

$$\varphi_Y(\theta)=\varphi_Y(\theta/b)\,\varphi_{((1-b^{-2})A\,,\,\nu_1\,,\,\ldots)}(\theta). \quad \square$$

Exercise 40 Check the last formula in the proof of Theorem 4.12.

4.3 Subordination

Lemma 4.13 *The* `chf` *of an $\mathbb{R}$-valued* `id` *Y cannot be zero.*

Proof. Let $\{\varphi_n(\theta)\}_{n=1}^\infty$ be univariate `chf` such that $\varphi_n(\theta)\to 1$ as $n\to\infty$ for θ in an open interval around zero. For $a_1=a_3=1$, $a_2=-2$, $t_1=0$, $t_2=t$ and $t_3=2t$, we get

$$0\leq\sum_{i,j=1}^3 a_i a_j\varphi_n(t_j-t_i)=6\varphi_n(0)-4(\varphi_n(t)+\varphi_n(-t))+(\varphi_n(2t)+\varphi_n(-2t)),$$

by non-negative definiteness. Taking real parts and rearranging, this gives

$$\Re\varphi_n(\pm 2t)=\tfrac{1}{2}\Re(\varphi_n(2t)+\varphi_n(-2t))\geq 2\Re(\varphi_n(t)+\varphi_n(-t))-3\varphi_n(0)=4\Re\varphi_n(\pm t)-3\varphi_n(0).$$

By iteration we therefore readily conclude that $\varphi_n(\theta)\to 1$ for $\theta\in\mathbb{R}$.

We have $\varphi_n(\theta)\equiv\varphi_Y(\theta)^{1/n}\to 1$ as $n\to\infty$, for θ in an open interval around zero, by continuity of φ_Y. Thus $\varphi_Y^{1/n}\to 1$ on $\mathbb{R}$. This means that $\varphi_Y(\theta)\neq 0$ for $\theta\in\mathbb{R}$. $\square$

Fact 4.14 *An $\mathbb{R}$-valued* `id` *Y is the weak limit of a sequence of compound Poison distributions.*

Proof. By Lemma 4.13, $\varphi_Y(\theta)^{1/n}=1+n^{-1}\ln\varphi_Y(\theta)+O(n^{-2})\to 1$, so that

$$\varphi_Y(\theta)\sim\exp\{n(\varphi_Y(\theta)^{1/n}-1)\}\sim\exp\{n\textstyle\int_{\mathbb{R}}(\mathrm{e}^{i\theta x}-1)\,dF_{Y^{\star 1/n}}(x)\} \qquad \text{as} \quad n\to\infty. \quad \square$$

EXERCISE 41 Can Fact 4.14 be useful to simulate Y in a computer?

Corollary 4.15 *A non-negative $\mathbb{R}$-valued* `id` *Y is the weak limit of a sequence of non-negative compound Poison distributions.*

EXERCISE 42 Derive Corollary 4.15 from Theorem 4.14.

Definition 4.16 *An $\mathbb{R}$-valued Lévy process X is a subordinator if it is non-decreasing a.s.*

EXERCISE 43 Show that an $\mathbb{R}$-valued Lévy process X is a subordinator iff. its generating `id` distribution $X(1)$ is non-negative a.s.

The following theorem is of fundamental importance.

Theorem 4.17 *An $\mathbb{R}$-valued Lévy process X is a subordinator iff. it is of type A or B, with Lévy measure supported on $(0,\infty)$ and non-negative drift γ_0, i.e., iff.*

$$\mathbf{E}\{e^{i\theta X(1)}\} = \exp\{i\theta\gamma_0 + \textstyle\int_{(0,\infty)}(e^{i\theta x}-1)\,d\nu(x)\} \qquad \textit{with} \qquad \gamma_0 \geq 0.$$

Proof. $\boxed{\Leftarrow}$ Let the right statement in the theorem hold. Clearly, it is enough to show that the triplet $(0,\nu,0)_0$ (with γ_0 removed) corresponds to a non-negative `rv`. For $\nu(\mathbb{R})=0$ this is trivial. For $\nu(\mathbb{R})\neq 0$ we put $\nu_n \equiv \nu\mathbf{1}_{\{|x|>1/n\}}$. Then, as $n\to\infty$,

$$\int_{(0,\infty)}(e^{i\theta x}-1)\,d\nu_n(x) = \int_{(0,\infty)}(e^{i\theta x}-1)\mathbf{1}_{\{|x|>1/n\}}\,d\nu(x) \to \int_{(0,\infty)}(e^{i\theta x}-1)\,d\nu(x), \qquad (\star)$$

i.e., $(0,\nu_n,0)_0 \to_d (0,\nu,0)_0$: Recall that X is type A or B when $A=0$ and $\int_{|x|\leq 1}|x|\,d\nu(x)<\infty$. Thus $\int_{\mathbb{R}}|x|\wedge 1\,d\nu(x)<\infty$, which gives dominated convergence in $(\star)$.

Now $(0,\nu,0)_0$ is a weak limit of non-negative compound Poisson `rv's`, since ν_n is supported on $(0,\infty)$. Therefore $(0,\nu,0)_0$ is non-negative.

$\boxed{\Rightarrow}$ Clearly, for the generating triplet (A,ν,γ) of X, we cannot have $A\neq 0$, see Exercise 44 below. Hence, with $\gamma_\eta \equiv \gamma - \int_{\eta<|x|\leq 1} x\,d\nu(x)$, $\Psi(-i\theta)$ is given by

$$\int_{|x|\leq\eta}(e^{i\theta x}-1-i\theta x)\,d\nu(x)+\int_{|x|>\eta}(e^{i\theta x}-1)\,d\nu(x)+i\theta\gamma_\eta \equiv \Psi_1(-i\theta)+\Psi_2(-i\theta)+i\theta\gamma_\eta \qquad (\star\star)$$

for $\eta\in(0,1]$. Let Y_1 and Y_2 be `id` with `cgf's` Ψ_1 and Ψ_2, respectively. Since Y_2 is compound Poisson, $\nu\mathbf{1}_{\{|x|>\eta\}}$ must be supported on $(0,\infty)$ for $\eta>0$ (Exercise 45 below), i.e., ν is supported on $(0,\infty)$. Since $\mathbf{E}\{Y_1\}=0$ (Exercise 46), similar arguing give that $\gamma_\eta \geq 0$ for $\eta > 0$ (Exercise 47). Hence $\int_{|x|\leq 1} |x|\, d\nu(x) < \infty$, since otherwise $\gamma_\eta \to -\infty$ as $\eta \downarrow 0$. Therefore $Y_1 \to_d 0$ as $\eta \downarrow 0$, by dominated convergence. And so $X(1)$ is of type A or B, with triplet $(0, \nu, \gamma_0)_0$, since we must have $\gamma_\eta \to \gamma_0$ (by uniqueness of triplets). Consequently, $\gamma_0 \geq 0$. $\square$

EXERCISE 44 Why is the Gaussian component zero for a non-negative $\mathbb{R}$-valued `id rv`?

EXERCISE 45 In $(\star\star)$, why has $\nu \mathbf{1}_{\{|x|>\eta\}}$ support $(0, \infty)$ for $X(1)$ non-negative?

EXERCISE 46 In $(\star\star)$, why is $\mathbf{E}\{Y_1\} = 0$?

EXERCISE 47 In $(\star\star)$, for $X(1)$ non-negative, why is $\gamma_\eta \geq 0$ for $\eta > 0$?

EXERCISE 48 Show that an α-stable Lévy motion is a subordinator iff. $\alpha < 1$, $\beta = 1$ and $\tau = \gamma_0 \geq 0$.

EXERCISE 49 In Exercise 48, why is $\tau = \gamma_0$?

Theorem 4.18 *Let X_1 be a Lévy process with generating triplet (A, ν, γ), and X_2 an independent subordinator with generating triplet $(0, \rho, \beta)_0$. Then $X \equiv X_1 \circ X_2$ is a Lévy process with generating triplet $(\hat{A}, \hat{\nu}, \hat{\gamma})$ given by*

$$\begin{cases} \hat{A} = \beta A \\ \hat{\nu}(B) = \beta \nu(B) + \int_{(0,\infty)} \mathbf{P}\{X_1(1)^{\star s} \in B\}\, d\rho(s) \\ \hat{\gamma} = \beta\gamma + \int_{(0,\infty)} \int_{|x|\leq 1} x\, dF_{X_1(1)^{\star s}}(x)\, d\rho(s) \end{cases}.$$

EXERCISE 50 Explain why X is a Lévy process in Theorem 4.18.

EXERCISE 51 Try to derive the triplet $(\hat{A}, \hat{\nu}, \hat{\gamma})$ in Theorem 4.18.

EXERCISE 52 Show that $X(t)$ really is a `rv` for $t \geq 0$ in Theorem 4.18.

EXERCISE 53 In Theorem 4.18, show that X is a subordinator if X_1 is.

Definition 4.19 *In Theorem 4.18, the transformation of X_1 to X is called <u>subordination by X_2</u>. Further, any Lévy process with the law of X, for a suitable X_2, is called <u>subordinate to X_1</u>.*

4.4 Something I came to think of this time ...

SAMHALL
MÖJLIGHETERNAS FÖRETAG

Vi väcker er då ni skall gå hem! OBS: Limmet är hälsovådligt att indandas.

5 Lecture 5 20/9 - 02

5.1 Ornstein-Uhlenbeck Processes

Here we give an introductory coverage of `id` Ornstein-Uhlenbeck processes.

Let $\{L(t)\}_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued Lévy process. This means that L satisfies an obvious version of Definition 1.8 [including $L(0)=_d 0$], extended to the whole real line.

For a constant $c>0$, the $\mathbb{R}^d$-valued Ornstein-Uhlenbeck (`OU`) process is given by

$$Z(t) \equiv \int_{-\infty}^{t} \mathrm{e}^{-c(t-s)}\, dL(s) \qquad \text{for} \quad t\in\mathbb{R}.$$

The stochastic integral can be defined as a limit of Riemann sums (see Lecture 6).

If Z is well-defined, it will be a stationary process, since by homogeneity of L,

$$\{Z(t+h)\}_{t\in\mathbb{R}} = \{\int_{-\infty}^{t+h} \mathrm{e}^{-c(t-(s-h))}\, dL(s)\}_{t\in\mathbb{R}} = \{\int_{-\infty}^{t} \mathrm{e}^{-c(t-s)}\, dL(s+h)\}_{t\in\mathbb{R}} =_d \{Z(t)\}_{t\in\mathbb{R}}.$$

By Riemman sums and `id` of L (Definition 1.34, Exercise 14), we see that Z is `id`.

Further, Z is a (continuous time) Markov [AR(1)-] process, because (trivially)

$$Z(t) = \mathrm{e}^{-c(t-t_0)} Z(t_0) + \int_{t_0}^{t} \mathrm{e}^{-c(t-s)}\, dL(s) \qquad \text{for} \quad t\geq t_0,$$

and is a strong weak solution to the Langevin stochastic differential equation (`sde`)

$$dZ(t) = -c\,Z(t)\,dt + dL(t) \qquad \text{for} \quad t>0, \quad Z(0) = \int_{-\infty}^{0} \mathrm{e}^{cs}\, dL(s),$$

because the corresponding integrated equation holds, simply by insertion of Z,

$$\begin{aligned}
&Z(0) - c\int_0^t Z(s)\,ds + \int_0^t dL(s)\\
&= \int_{-\infty}^{0} \mathrm{e}^{cs}\, dL(s) - c\int_{s=0}^{s=t}\int_{r=-\infty}^{r=s} \mathrm{e}^{c(r-s)}\, dL(r) ds + \int_0^t dL(s)\\
&= \int_{-\infty}^{0} \mathrm{e}^{cs}\, dL(s) - c\int_{r=-\infty}^{r=0}\int_{s=0}^{s=t} \mathrm{e}^{c(r-s)}\, ds dL(r) - c\int_{r=0}^{r=t}\int_{s=r}^{s=t} \mathrm{e}^{c(r-s)}\, ds dL(r) + \int_0^t dL(s)\\
&= \int_{-\infty}^{0} \mathrm{e}^{cs}\, dL(s) - \int_{-\infty}^{0} (\mathrm{e}^{cr} - \mathrm{e}^{c(r-t)})\, dL(r) - \int_0^t (1-\mathrm{e}^{c(r-t)})\, dL(r) + \int_0^t dL(s)\\
&= \int_{-\infty}^{t} \mathrm{e}^{c(r-t)}\, dL(r) = Z(t).
\end{aligned}$$

Since $\Psi_{L(s)} = s\Psi_{L(1)}$, the `cgf` of $Z(t)$ (independent of t by stationarity) is

$$\Psi_{Z(t)}(-i\theta) = \ln \mathbf{E}\{\mathrm{e}^{i\langle\theta, \int_{-\infty}^{0} \mathrm{e}^{cs} dL(s)\rangle}\} = \lim \sum \Psi_{L(s_{n+1}-s_n)}(i\,\mathrm{e}^{cs_n}\theta) = \int_{-\infty}^{0} \Psi_{L(1)}(-i\,\mathrm{e}^{cs}\theta)\, ds.$$

Using Theorem 4.12, it follows that $Z(t)$ is selfdecomposable, since, for $b>1$,

$$\varphi_{Z(t)}(\theta)/\varphi_{Z(t)}(\theta/b) = \mathrm{e}^{\int_{-\infty}^{0} \Psi_{L(1)}(-i\mathrm{e}^{cs}\theta)\, ds - \int_{-\infty}^{0} \Psi_{L(1)}(-i\mathrm{e}^{cs}\theta/b)\, ds} = \mathrm{e}^{\int_{-\ln(b)/c}^{0} \Psi_{L(1)}(-i\mathrm{e}^{cs}\theta)\, ds}.$$

Here the right-hand side is a `chf` (that defines ρ_b in Theorem 4.12), because (continuous at zero and) the limit of a product of `chf's` by Riemann sum approximation.

To check when Z is well-defined, let L have generating triplet (A,ν,γ), so that

$$\begin{aligned}\int_{-\infty}^0 \Psi_{L(1)}(-i\,\mathrm{e}^{cs}\theta)\,ds = -\tfrac{1}{2}\int_{-\infty}^0 \langle \mathrm{e}^{cs}\theta, A\mathrm{e}^{cs}\theta\rangle\,ds + i\int_{-\infty}^0\langle \mathrm{e}^{cs}\theta,\gamma\rangle\,ds \\ + \int_{-\infty}^0\big[\int_{\mathbb{R}^d} i\langle \mathrm{e}^{cs}\theta, y\rangle(\mathbf{1}_{\{|\mathrm{e}^{cs}y|\le 1\}}-\mathbf{1}_{\{|y|\le 1\}})\,d\nu(y)\big]\,ds \\ + \int_{-\infty}^0\big[\int_{\mathbb{R}^d}(\mathrm{e}^{i\langle \mathrm{e}^{cs}\theta,y\rangle}-1-i\langle \mathrm{e}^{cs}\theta,y\rangle)\,\mathbf{1}_{\{|\mathrm{e}^{cs}y|\le 1\}}\,d\nu(y)\big]\,ds \\ + \int_{-\infty}^0\big[\int_{\mathbb{R}^d}(\mathrm{e}^{i\langle \mathrm{e}^{cs}\theta,y\rangle}-1)\,\mathbf{1}_{\{|\mathrm{e}^{cs}y|>1\}}\,d\nu(y)\big]\,ds.\end{aligned}$$

Here the two first integrals on the right-hand side are trivially well-defined. The third integral is well-defined, since it is "dominated" by the integral

$$|\theta|\int_{\mathbb{R}^d}\big[\int_{-\infty}^{-(\ln(|y|)/c)^+}\mathrm{e}^{cs}|y|\,ds\big]\,\mathbf{1}_{\{|y|>1\}}\,d\nu(y) = |\theta|\,\nu(\{y\in\mathbb{R}^d : |y|>1\})<\infty,$$

while, for some constant $C>0$, the fourth is dominated by

$$C|\theta|^2\int_{\mathbb{R}^d}\big[\int_{-\infty}^{-(\ln(|y|)/c)^+}\mathrm{e}^{2cs}|y|^2\,ds\big]\,d\nu(y) = C|\theta|^2\int_{\mathbb{R}^d}1\wedge|y|^2\,d\nu(y)<\infty.$$

As for the fifth integral on the right, it is dominated by

$$2\int_{\mathbb{R}^d}\big[\int_{-(\ln(|y|)/c)^+}^0 ds\big]\,d\nu(y) = (2/c)\int_{|y|>1}\ln(|y|)\,d\nu(y).$$

Thus the requirement ensuring that the OU process is well-defined becomes

$$\boxed{\int_{\{y\in\mathbb{R}^d : |y|>1\}}\ln(|y|)\,d\nu(y)<\infty.}$$

5.2 Weak Convergence of ID Distributions

Theorem 5.1 *For* id $\{Y_n\}_{n=1}^\infty$ *with triplets* (A_n,ν_n,γ_n) *we have* $Y_n\to_d Y$ *for some* Y, *iff.* Y *is* id *with triplet* (A,ν,γ) *such that the following conditions hold;*

(1) $\lim_{n\to\infty}\int_{|y|>\varepsilon}f(y)\,d\nu_n(y)=\int_{|y|>\varepsilon}f(y)\,d\nu(y)$ *for bounded* $f\in\mathbb{C}(\mathbb{R}^d)$ *and* $\varepsilon>0$;

(2) $\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}|\langle\theta,(A_n-A)\theta\rangle+\int_{|y|\le\varepsilon}\langle\theta,y\rangle^2\,d\nu_n(y)|=0$ *for* $\theta\in\mathbb{R}^d$;

(3) $\lim_{n\to\infty}\gamma_n=\gamma$.

Exercise 54 In one direction Theorem 5.1 is immediate: Prove that part.

5.3 Lévy-Itô Decomposition

The Lévy-Itô decomposition expresses the sample function of an additive process as a sum of a continuous part and an independent compensated sum of independent jumps. It is very important! The proof is long and difficult and cannot be given here. We state the result for Lévy processes only. The general formulation with additive processes is only notationally more complicated.

On http://www-history.mcs.st-andrews.ac.uk/history/Mathematicians/Ito.html we found

Kiyosi Ito

Born: 7 Sept 1915 in Hokusei-cho, Mie Prefecture, Japan

Kiyosi Ito studied mathematics in the Faculty of Science of the Imperial University of Tokyo. It was during his student years that he became attracted to probability theory. In [3] he explains how this came about:-

> *Ever since I was a student, I have been attracted to the fact that statistical laws reside in seemingly random phenomena. Although I knew that probability theory was a means of describing such phenomena, I was not satisfied with contemporary papers or works on probability theory, since they did not clearly define the random variable, the basic element of probability theory. At that time, few mathematicians regarded probability theory as an authentic mathematical field, in the same strict sense that they regarded differential and integral calculus. With clear definition of real numbers formulated at the end of the19th century, differential and integral calculus had developed into an authentic mathematical system. When I was a student, there were few researchers in probability; among the few were Kolmogorov of Russia, and Paul Levy of France.*

In 1938 Ito graduated from the University of Tokyo and in the following year he was appointed to the Cabinet Statistics Bureau. He worked there until 1943 and it was during this period that he made his most outstanding contributions:-

> *During those five years I had much free time, thanks to the special consideration given me by the then Director Kawashima ... Accordingly, I was able to continue studying probability theory, by reading Kolmogorov's Basic Concept of Probability Theory and Levy's Theory of Sum of Independent Random Variables. At that time, it was commonly believed that Levy's works were extremely difficult, since Levy, a pioneer in the new mathematical field, explained probability theory based on his intuition. I attempted to describe Levy's ideas, using precise logic that Kolmogorov might use. Introducing the concept of regularisation, developed by Doob of the United States, I finally devised stochastic differential equations, after painstaking solitary endeavours. My first paper was thus developed; today, it is common practice for mathematicians to use my method to describe Levy's theory.*

In 1940 he published *On the probability distribution on a compact group* on which he collaborated with Yukiyosi Kawada. The background to Ito's famous 1942 paper *On stochastic processes (Infinitely divisible laws of probability)* which he published in the *Japanese Journal of Mathematics* is given in [2]:-

> *Brown, a botanist, discovered the motion of pollen particles in water. At the beginning of the twentieth century, Brownian motion was studied by Einstein, Perrin and other physicists. In 1923, against this scientific background, Wiener defined probability measures in path spaces, and used the concept of Lebesgue integrals to lay the mathematical foundations of stochastic analysis. In 1942, Dr. Ito began to reconstruct from scratch the concept of stochastic integrals, and its associated theory of analysis. He created the theory of stochastic differential equations, which describe motion due to random events.*

Although today we see this paper as a fundamental one, it was not seen as such by mathematicians at the time it was published. Ito, who still did not have a doctorate at this time, would have to wait several years before the importance of his ideas would be fully appreciated and mathematicians would begin to contribute to developing the theory. In 1943 Ito was appointed as Assistant Professor in the Faculty of Science of Nagoya Imperial University. This was a period of high activity for Ito, and when one considers that this occurred during the years of extreme difficulty in Japan caused by World War II, one has to find this all the more remarkable. Volume 20 of the *Proceedings of the Imperial Academy of Tokyo* contains six papers by Ito: (1) *On the ergodicity of a certain stationary process;* (2) *A kinematic theory of turbulence;* (3) *On the normal stationary process with no hysteresis;* (4) *A screw line in Hilbert space and its application to the probability theory;* (5) *Stochastic integral;* and (6) *On Student's test.*

In 1945 Ito was awarded his doctorate. He continued to develop his ideas on stochastic analysis with many important papers on the topic. Among them were *On a stochastic integral equation* (1946), *On the stochastic integral* (1948), *Stochastic differential equations in a differentiable manifold* (1950), *Brownian motions in a Lie group* (1950), and *On stochastic differential equations* (1951).

In 1952 Ito was appointed to a Professorship at Kyoto University. In the following year he published his famous text *Probability theory.* In this book, Ito develops the theory on a probability space using terms and tools from measure theory. The years 1954-56 Ito spent at the Institute for Advanced Study at Princeton University. An important publication by Ito in 1957 was *Stochastic processes.* This book contained five chapters, the first providing an introduction, then the remaining ones studying processes with independent increments, stationary processes, Markov processes, and the theory of diffusion processes. In 1960 Ito visited the Tata Institute in Bombay, India, where he gave a series of lectures surveying his own work and that of other on Markov processes, Levy processes, Brownian motion and linear diffusion.

Although Ito remained as a professor at Kyoto University until he retired in 1979, he also held positions as professor at Aarhus University from 1966 to 1969 and professor at Cornell University from 1969 to 1975. During his last three years at Kyoto before he retired, Ito was Director of the Research Institute for Mathematical Sciences there. After retiring from Kyoto University in 1979 he did not retire from mathematics but continued to write research papers. He was also appointed at Professor at Gakushuin University.

Ito gives a wonderful description mathematical beauty in [3] which he then relates to the way in which he and other mathematicians have developed his fundamental ideas:-

> *In precisely built mathematical structures, mathematicians find the same sort of beauty others find in enchanting pieces of music, or in magnificent architecture. There is, however, one great difference between the beauty of mathematical structures and that of great art. Music by Mozart, for instance, impresses greatly even those who do not know musical theory; the cathedral in Cologne overwhelms spectators even if they know nothing about Christianity. The beauty in mathematical structures, however, cannot be appreciated without understanding of a group of numerical formulae that express laws of logic. Only mathematicians can read "musical scores" containing many numerical formulae, and play that "music" in their hearts. Accordingly, I once believed that without numerical formulae, I could never communicate the sweet melody played in my heart. Stochastic differential equations, called "Ito Formula," are currently in wide use for describing phenomena of random fluctuations over time. When I first set forth stochastic differential equations, however, my paper did not attract attention. It was over ten years after my paper that other mathematicians began reading my "musical scores" and playing my "music" with their "instruments." By developing my "original musical scores" into more elaborate "music," these researchers have contributed greatly to developing "Ito Formula."*

Ito received many honours for his outstanding mathematical contributions. He was awarded the Asahi Prize in 1978, and in the same year he received the Imperial Prize and also the Japan Academy Prize. In 1985 he received the Fujiwara Prize and in 1998 the Kyoto Prize in Basic Sciences from the Inamori Foundation. These prizes were all from Japan, and a further Japanese honour was his election to the Japan Academy. However, he also received many honours from other countries. He was elected to the National Academy of Science of the United States and to the Académie des Sciences of France. He received the Wolf Prize from Israel and honorary doctorates from the universities of Warwick, England and ETH, Zurich, Switzerland.

In [2] this tribute is paid to Ito:-

> *Nowadays, Dr. Ito's theory is used in various fields, in addition to mathematics, for analysing phenomena due to random events. Calculation using the "Ito calculus" is common not only to scientists in physics, population genetics, stochastic control theory, and other natural sciences, but also to mathematical finance in economics. In fact, experts in financial affairs refer to Ito calculus as "Ito's formula." Dr. Ito is the father of the modern stochastic analysis that has been systematically developing during the twentieth century. This ceaseless development has been led by many, including Dr. Ito, whose work in this regard is remarkable for its mathematical depth and strong interaction with a wide range of areas. His work deserves special mention as involving one of the basic theories prominent in mathematical sciences during this century.*

A recent monograph entitled *Ito's Stochastic Calculus and Probability Theory* (1996), dedicated to Ito on the occasion of his eightieth birthday, contains papers which deal with recent developments of Ito's ideas:-

> *Professor Kiyosi Ito is well known as the creator of the modern theory of stochastic analysis. Although Ito first proposed his theory, now known as Ito's stochastic analysis or Ito's stochastic calculus, about fifty years ago, its value in both pure and applied mathematics is becoming greater and greater. For almost all modern theories at the forefront of probability and related fields, Ito's analysis is indispensable as an essential instrument, and it will remain so in the future. For example, a basic formula, called the Ito formula, is well known and widely used in fields as diverse as physics and economics.*

Article by: *J J O'Connor* and *E F Robertson*

Definition 5.2 *Let $(S, \mathfrak{S}, \sigma)$ be a σ-finite measure space. An $\overline{\mathbb{N}}$-valued stochastic process $\{N(B)\}_{B\in\mathfrak{S}}$ is a* <u>*Poisson random measure on S with intensity measure σ*</u>*, if the following conditions hold;*

(1) *$N(B)$ is $\mathrm{Po}(\sigma(B))$-distributed for $B\in\mathfrak{S}$;*

(2) *$\{N(B_i)\}_{i=1}^n$ are independent for disjoint $\{B_i\}_{i=1}^n \subseteq \mathfrak{S}$;*

(3) *$N(\cdot,\omega)$ is a measure on S for $\omega\in\Omega$.*

Theorem 5.3 (Lévy-Itô Decomposition) *Let X be a Lévy process with generating triplet (A,ν,γ). Define*

$$N(B) \equiv \#\{s : (s, X(s)-X(s^-)) \in B\} \qquad \text{for Borel} \quad B\subseteq (0,\infty)\times\mathbb{R}^d.$$

The following assertions hold:

(1) *N is a Poisson random measure with intensity measure σ given by*

$$\sigma((0,t]\times B) \equiv t\nu(B) \qquad \text{for} \quad t>0 \quad \text{and} \quad B\in\mathcal{B}(\mathbb{R}^d).$$

(2) *With a.s. locally uniform convergence in t, the following limit X_1 is a Lévy process with generating triplet $(0,\nu,0)$,*

$$X_1(t) = \lim_{\varepsilon\downarrow 0} \int_{(0,t]\times\{\varepsilon<|x|\le 1\}} [x\, dN(s,x) - x\, d\sigma(s,x)] + \int_{(0,t]\times\{|x|>1\}} x\, dN(s,x).$$

(3) *$X_2 \equiv X - X_1$ is a Lévy process with generating triplet $(A,0,\gamma)$. It is a.s. continuous and independent of X_1.*

Theorem 5.4 (Lévy-Itô Decomposition) *Let X be a Lévy process with generating triplet $(A,\nu,\gamma_0)_0$. With the notation of Theorem 5.3, the following assertions hold:*

(1) *The following process X_3 is a Lévy process with generating triplet $(0,\nu,0)_0$,*

$$X_3(t) = \int_{(0,t]\times\mathbb{R}^d} x\, dN(s,x).$$

(2) *$X_4 \equiv X - X_3$ is a Lévy process with generating triplet $(A,0,\gamma_0)$. It is a.s. continuous and independent of X_3.*

5.4 Sample Function Behaviour

Theorem 5.5 *A Lévy process is a.s. continuous iff. its Lévy measure is zero.*

Proof. By Theorem 5.3, $J\equiv\#\{s\in(0,t]:|X(s)-X(s^-)|>\varepsilon\}$ satisfies

$$\mathbf{E}\{J\}=\mathbf{E}\{N((0,t]\times\{x\in\mathbb{R}^d:|x|>\varepsilon\})\}=t\int_{|x|>\varepsilon}d\nu(x).\quad\square$$

Definition 5.6 *A function $f:[0,\infty)\to\mathbb{R}^d$ is piecewise constant if there exist $0=t_0<t_1<\ldots$ with $\lim_{n\to\infty}t_n=\infty$ and f constant on the intervals $\{[t_{n-1},t_n)\}_{n=1}^{\infty}$.*

Theorem 5.7 *A Lévy process is a.s. piecewise constant iff. it is type A with $\gamma_0=0$.*

Proof. $\boxed{\Leftarrow}$ For $A=0$, $\nu(\mathbb{R}^d)<\infty$ and $\gamma_0=0$, by Theorem 5.4, the number of jumps of $X=_d X_3$ in $(0,t]$ has a $\mathrm{Po}(t\nu(\mathbb{R}^d))$-law, and thus is finite a.s.

$\boxed{\Rightarrow}$ Since $\#\{s\in(0,t]:|X(s)-X(s^-)|>\varepsilon\}$ has a $\mathrm{Po}(t\nu(\{x\in\mathbb{R}^d:|x|>\varepsilon\}))$-law, by Theorem 5.3, $\nu(\mathbb{R}^d)<\infty$, since otherwise the number of jumps in $(0,t]$ is infinite a.s., by sending $\varepsilon\downarrow 0$. For $\nu(\mathbb{R}^d)<\infty$, X_3 is piecewise constant by $\boxed{\Leftarrow}$. Now Theorem 5.4 gives $A=0$ and $\gamma_0=0$, since X_4 is not piecewise constant, if non-zero. $\square$

Theorem 5.8 (1) *A Lévy process with infinite Lévy measure has jumping times that a.s. are countable and dense in $[0,\infty)$.*

(2) *A Lévy process with finite Lévy measure ν has jumping times that a.s. are countable in increasing order, with the time to the first jump $\exp(\nu(\mathbb{R}^d))$-distributed.*

Proof. Jumps are countable for càdlàg sample paths. Let $T_\varepsilon\equiv\inf\{s>0:|X(s)-X(s^-)|>\varepsilon\}$. By Theorem 5.3, unless zero or infinite, T_ε is exp-distributed with

$$\mathbf{P}\{T_\varepsilon\le t\}=\mathbf{P}\{N((0,t]\times\{x\in\mathbb{R}^d:|x|>\varepsilon\})\ge 1\}=1-\exp\{-t\int_{|x|>\varepsilon}d\nu(x)\}.$$

For $\nu(\mathbb{R}^d)=\infty$, $\lim_{\varepsilon\downarrow 0}\mathbf{P}\{T_\varepsilon\le t\}=1$ for $t>0$, giving assertion (1) (see Exercise 55 below). For $\nu(\mathbb{R}^d)<\infty$, we get an $\exp(\nu(\mathbb{R}^d))$-law as $\varepsilon\downarrow 0$, and Theorem 5.3 gives the claimed countability, since the number of jumps in $(0,t]$ has a $\mathrm{Po}(t\nu(\mathbb{R}^d))$-law. $\square$

Exercise 55 Explain the argument in the proof of Theorem 5.8 for $\nu(\mathbb{R}^d)=\infty$.

Definition 5.9 *The variation $v(t,f)$ over $(0,t]$ of $f:[0,\infty)\to\mathbb{R}^d$ is defined*

$$v(t,f)\equiv\sup\{\textstyle\sum_{k=1}^n|f(t_k)-f(t_{k-1})|:0=t_0<t_1<\ldots<t_n=t,\ n\in\mathbb{N}\}.$$

The following important result is again proved making crucial use of the Lévy-Itô Decomposition. But now arguments are quite long and difficult.

Theorem 5.10 (1) *A Lévy process X of type A or B with generating triplet $(0,\nu,\gamma_0)_0$ has a.s. locally finite variation $V(t)\equiv v(t,X)$, that is a subordinator with generating triplet $(0,\rho,|\gamma_0|)_0$, where*

$$\rho(B)=\nu(\{x\in\mathbb{R}^d : |x|\in B\}) \qquad \textit{for} \quad B\in\mathcal{B}(\mathbb{R}).$$

(2) *A Lévy process of type C has a.s. locally infinite variation.*

Exercise 56 Exercise E 22.1 in Sato's book.

Exercise 57 Exercise E 22.5 in Sato's book.

5.5 Recurrence and Transience

There are many results on recurrence and transience for Lévy processes. We only display a single result, to give some flavour of the topic (see Sato's book and tutorials on more information). An application is given to stable Lévy motions.

Definition 5.11 *A Lévy process X is reccurent if*

$$\liminf_{t\to\infty}|X_t|=0 \qquad a.s.,$$

and transient if

$$\lim_{t\to\infty}|X_t|=\infty \qquad a.s.$$

Theorem 5.12 *Given an $a>0$, a Lévy process X is reccurent iff.*

$$\int_0^\infty \mathbf{P}\{|X(t)|<a\}\,dt=\infty \qquad a.s.,$$

and transient iff.

$$\int_0^\infty \mathbf{P}\{|X(t)|<a\}\,dt<\infty \qquad a.s.$$

Exercise 58 Show that an $\mathbb{R}$-valued non-trivial α-stable Lévy motion X, $\alpha\neq 1$, may be written $X(t)=L(t)+\tau t$, where L is strictly stable and $\tau\in\mathbb{R}$ a constant. Conclude that X is strictly α-stable iff. $\tau=0$.

Example 2 Let $X(t)=L(t)+\tau t$ be an $\mathbb{R}$-valued non-trivial α-stable, $\alpha\in(1,2]$, Lévy motion, with L strictly stable and $\tau\in\mathbb{R}$ a constant (cf. Exercise 58). Recall that $L(1)$ has a density function $f_{L(1)}$, by Exercise 38. Assume that $f_{L(1)}(0)>0$.

By self-similarity of L (this is Fact 3.9), we have

$$\int_0^\infty \mathbf{P}\{|L(t)|<a\}\,dt = \int_0^\infty \mathbf{P}\{|L(1)|<t^{-1/\alpha}a\}\,dt \geq \int_{t_0}^\infty f_{L(1)}(0)\,t^{-1/\alpha}a\,dt = \infty$$

(for some constant $t_0>0$). Hence L is reccurent, by Theorem 5.12. I.e., strictly α-stable Lévy motions with $\alpha\in(1,2]$ are reccurent.

If X is not strictly α-stable, so that $\tau\neq 0$, then we get

$$\mathbf{P}\{|X(t)|<a\} = \mathbf{P}\{|t^{1/\alpha}L(1)+\tau t|<a\} = \mathbf{P}\{-t^{-1/\alpha}a-\tau t^{1-1/\alpha}<L(1)<t^{-1/\alpha}a-\tau t^{1-1/\alpha}\},$$

i.e., $L(1)$ belongs to an interval of width $2t^{-1/\alpha}a$ located "around" $-\tau t^{1-1/\alpha}$. By unimodality of α-stable distributions (recall Section 4.2), this is at most

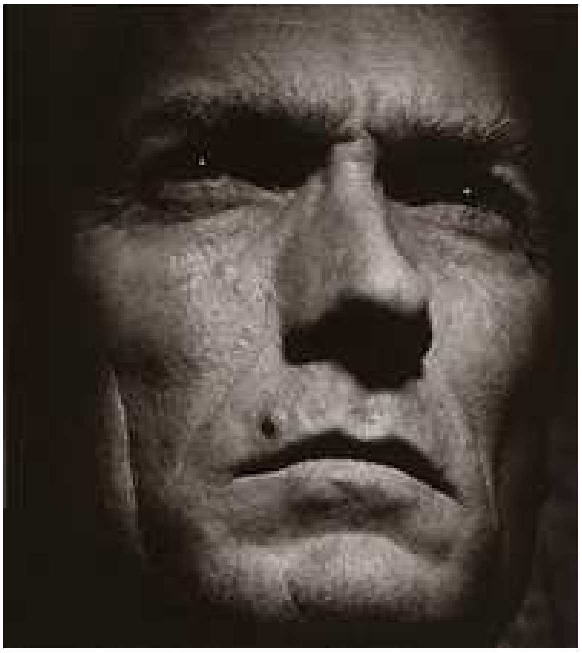
$$2t^{-1/\alpha}a\,f_{L(1)}(\pm t^{-1/\alpha}a-\tau t^{1-1/\alpha}) \leq 2t^{-1/\alpha}a\,f_{L(1)}(-\tfrac{1}{2}\tau t^{1-1/\alpha})$$

for t large enough. Hence X is transient, by Theorem 5.12, since

$$\int_0^\infty t^{-1/\alpha} f_{L(1)}(-\tfrac{1}{2}\tau t^{1-1/\alpha})\,dt = \int_0^\infty 2/(\tau(1-1/\alpha))\,f_{L(1)}(-\hat{t})\,d\hat{t} < \infty. \quad \#$$

EXERCISE 59 Show that a non-zero α-stable Lévy motion, $\alpha\in(0,1)$, is transient.

5.6 Something I came to think of this time ...



Go ahead, make my day punk!

6 Lecture 6 27/9 - 02

In Lectures 6 and 7, as preparation for `sde`, we study stochastic integrals of non-random functions wrt. additive processes, following the standard text "*Rajput & Rosiński: Spectral Representation of Infinitely Divisible Processes*". The treatment has been slightly "watered down", since it is estimated that most participants do not benifit from the full truth.

6.1 Outline of Stochastic Integration wrt. Semi-martingales

Let G be an $\mathbb{R}$-valued Gaussian additive process, where $G(t)$ has triplet $(A(t),0,\gamma(t))$, with γ of finite variation. For measurable $x:[0,\infty)\to\mathbb{R}$ the stochastic integral

$$\int_0^t x\,dG \quad\text{exists}\qquad\Leftrightarrow\qquad \|\mathbf{1}_{[0,t]}x\|_{(A,0,\gamma)}\equiv\Big[\int_0^t A(s)x(s)^2\,ds\Big]^{1/2}+\int_0^t|x(s)|\,d|\gamma|(s)<\infty.$$

We may integrate a predictable (essentially meaning left-continuous and adapted to the filtration associated with G) $\mathbb{R}$-valued process X wrt. G,

$$\int_0^t X\,dG \quad\text{exists}\qquad\Leftrightarrow\qquad \mathbf{P}\{\|\mathbf{1}_{[0,t]}X\|_{(A,0,\gamma)}<\infty\}=1.$$

An continuous semi-martingale S is a time-changed version $S=G\circ T$ of G, where $\{T(t)\}_{t\geq 0}$ is an non-decreasing, predictable and continuous stochastic process. Formally, it is an additive process with time dependent stochastic triplet $(A\circ T,0,\gamma\circ T)$. We may integrate a predictable process X wrt. S,

$$\int_0^t X\,dS \quad\text{exists}\quad(\text{for}\ \ t\geq 0)\qquad\Leftrightarrow\qquad \mathbf{P}\{\|\mathbf{1}_{[0,t]}X\|_{(A\circ T,0,\gamma\circ T)}<\infty\}=1\quad(\text{for}\ \ t\geq 0).$$

The stochastic integral process $\{\int_0^t X\,dS\}_{t\geq 0}$ is again a continuous semi-martingale.

For an $\mathbb{R}$-valued additive process L such that $L(t)$ has triplet $(A(t),\nu(t),\gamma(t))$ [recall that $L(t)$ is `id` by Fact 1.33], and measurable $x:[0,\infty)\to\mathbb{R}$, the integral

$$\int_0^t x\,dL \quad\text{exists}\qquad\Leftrightarrow\qquad \|\mathbf{1}_{[0,t]}x\|_{(A,\nu,\gamma)}<\infty,$$

where $\|\cdot\|_{(A,\nu,\gamma)}$ is a F-pseudo-norm on a Musielak-Orlicz space (see Section 6.3 below), that reduces to $\|\cdot\|_{(A,0,\gamma)}$ for L Gaussian $\nu(t)=0$. We may integrate a predictable (wrt. the filtration associated with L) process X wrt. L,

$$\int_0^t X\,dL \quad\text{exists}\qquad\Leftrightarrow\qquad \mathbf{P}\{\|\mathbf{1}_{[0,t]}X\|_{(A,\nu,\gamma)}<\infty\}=1.$$

A not necessarily continuous semi-martingale S formally is an additive process with (time dependent) stochastic triplet $(\hat{A}(t),\hat{\nu}(t),\hat{\gamma}(t))$. For a predictable X,

$$\int_0^t X\,dS \quad\text{exists}\quad(\text{for}\ \ t\geq 0)\qquad\Leftrightarrow\qquad \mathbf{P}\{\|\mathbf{1}_{[0,t]}X\|_{(\hat{A},\hat{\nu},\hat{\gamma}),t}<\infty\}=1\quad(\text{for}\ \ t\geq 0).$$

The process $\{\int_0^t X\,dS\}_{t\geq 0}$ is again a semi-martingale. (This is the general theory!)

6.2 Representation of Additive Processes

Fact 6.1 *For X additive in law, $X(t)-X(s)$ is* `id` *for $0\leq s\leq t$.*

Proof. For $s\geq 0$, $X'\equiv X(\cdot+s)-X(s)$ is an additive process in law, because

$$\begin{aligned}&(X(s),X'(t_1),X'(t_2)-X'(t_1),\ldots,X'(t_n)-X'(t_{n-1}))\\ =_d\, &(X(s),X(t_1+s)-X(s),X(t_2+s)-X(t_1+s),\ldots,X(t_n+s)-X(t_{n-1}+s))\end{aligned}$$

are independent for $0\leq t_1\leq\ldots\leq t_n$. The assertion thus follows from Fact 1.33. □

Fact 6.2 *Given a family of probability densities $\{\mu_{s,t}\}_{0\leq s\leq t<\infty}$ on $\mathbb{R}^d$, there exists X that is additive in law with $\mathcal{L}(X(t)-X(s))=\mu_{s,t}$ for $0\leq s\leq t$, iff. the following conditions hold;*

(1) $\mu_{r,s}\star\mu_{s,t}=\mu_{r,t}$ *for* $0\leq r\leq s\leq t$;

(2) $\mu_{s,t}\rightarrow_d\delta_0$ *as* $s\uparrow t>0$ *and as* $t\downarrow s\geq 0$ *[in particular,* $\mu_{t,t}=\delta_0$ *for* $t\geq 0$*].*

Exercise 60 Prove $\boxed{\Rightarrow}$ in Fact 6.2.

Exercise 61 Prove $\boxed{\Leftarrow}$ in Fact 6.2.

Definition 6.3 *An X that is additive in law such that $X(t)$ has triplet (A_t,ν_t,γ_t) for $t\geq 0$, has* <u>*system of triplets* $\{(A_t,\nu_t,\gamma_t)\}_{t\geq 0}$</u>.

Corollary 6.4 *An additive process in law is uniqely determined in law by its system of triplets.*

Proof. Let X and X' be additive with common systems of triplets. With $\mu_{s,t}\equiv\mathcal{L}(X(t)-X(s))$ and $\mu'_{s,t}\equiv\mathcal{L}(X'(t)-X'(s))$ for $0\leq s\leq t$, we have $\mu_{0,t}=\mu'_{0,t}$ for $t\geq 0$, by assumption. This gives

$$\mu_{0,s}\star\mu'_{s,t}=\mu'_{0,s}\star\mu'_{s,t}=\mu'_{0,t}=\mathcal{L}(X'(t))=\mathcal{L}(X(t))=\mu_{0,t}=\mu_{0,s}\star\mu_{s,t}\qquad\text{for}\quad 0\leq s\leq t.$$

Hence $\mu'_{s,t}=\mu_{s,t}$, by using `chf` [since the `chf` of $\mu_{0,s}$ is non-zero, by Lemma 4.13, since $X(s)$ is `id`, by Fact 6.1]. It follows that $X=_d X'$ (see Exercise 62 below). □

Exercise 62 Explain why $\mu_{s,t}=\mu'_{s,t}$ for $0\leq s\leq t$, gives the $X=_d X'$ requested in the proof of Corollary 6.4.

Theorem 6.5 *The triplets* $\{(A_t,\nu_t,\gamma_t)\}_{t\geq 0}$ *is system of triplets for an additive process in law, iff. the following conditions hold;*

(1) $A_0=0$, $\nu_0=0$ *and* $\gamma_0=0$;

(2) *for* $0\leq s\leq t$, $\langle\theta,A_s\theta\rangle\leq\langle\theta,A_t\theta\rangle$ *for* $\theta\in\mathbb{R}^d$, *and* $\nu_s(B)\leq\nu_t(B)$ *for* $B\in\mathcal{B}(\mathbb{R}^d)$;

(3) *as* $s\to t\geq 0$, $A_s\to A_t$, $\nu_s(B)\to\nu_t(B)$ *for* $B\in\mathcal{B}(\mathbb{R}^d)$ *with* $0\notin\mathrm{clos}(B)$, *and* $\gamma_s\to\gamma_t$.

Proof. $\boxed{\Leftarrow}$ For $\mu_{s,t}$ the `id` law with triplet $(A_t-A_s,\nu_t-\nu_s,\gamma_t-\gamma_s)$, for $0\leq s\leq t$, conditions (1) and (2) of Fact 6.2 hold.

$\boxed{\Rightarrow}$ Property (1) expresses that $X(0)=_d 0$. Further, $\mathcal{L}(X(t)-X(s))$ is `id`, by Fact 6.1, with triplet $(A_t-A_s,\nu_t-\nu_s,\gamma_t-\gamma_s)$, since $\varphi_{X(t)}=\varphi_{X(s)}\varphi_{X(t)-X(s)}$, so that $\varphi_{X(t)-X(s)}=\varphi_{X(t)}/\varphi_{X(s)}$. This gives claim (2). As $s\to t$, $X(s)\to_d X(t)$, by stochastic continuity. Hence, by Theorem 5.1.3, $\gamma_s\to\gamma_t$, and by Theorem 5.1.2,

$$\lim_{\varepsilon\downarrow 0}\limsup_{s\to t}|\langle\theta,(A_s-A_t)\theta\rangle+\textstyle\int_{|y|\leq\varepsilon}\langle\theta,y\rangle^2\,d\nu_s(y)|=0\qquad\text{for}\quad\theta\in\mathbb{R}^d.$$

Thus $A_s\to A_t$, since taking $\hat{t}>t$, the property (2) and Dominated Convergence give

$$\lim_{\varepsilon\downarrow 0}\limsup_{s\to t}\int_{|y|\leq\varepsilon}\langle\theta,y\rangle^2\,d\nu_s(y)\leq\lim_{\varepsilon\downarrow 0}\int_{|y|\leq\varepsilon}\langle\theta,y\rangle^2\,d\nu_{\hat{t}}(y)\leq\lim_{\varepsilon\downarrow 0}\int_{|y|\leq\varepsilon}|\theta|^2|y|^2\,d\nu_{\hat{t}}(y)=0.$$

Moreover, $\nu_s\leq\nu_{\hat{t}}$ gives $d\nu_s(x)=g_s(x)d\nu_{\hat{t}}(x)$ for some measurable $g_s:\mathbb{R}^d\to\mathbb{R}$, for $s\leq\hat{t}$, by Radon-Nikodym, with $g_s\uparrow g_{s'}$ and $g_s\downarrow g_{s'}$ a.e. $(\nu_{\hat{t}})$, as $s\uparrow s'\leq\hat{t}$ and $s\downarrow s'<\hat{t}$, by Exercise 64 below. Hence Monotone Convergence gives, as $s\uparrow t$, for $\varepsilon>0$,

$$\nu_s(\mathbf{1}_{\{|x|\geq\varepsilon\}}B)=\int_{|y|>\varepsilon}\mathbf{1}_B(x)g_s(x)\,\nu_{\hat{t}}(x)\to\int_{|y|>\varepsilon}\mathbf{1}_B(x)g_t(x)\,\nu_{\hat{t}}(x)=\nu_t(\mathbf{1}_{\{|x|\geq\varepsilon\}}B).$$

Similarly, $\nu_s(\mathbf{1}_{\{|x|\geq\varepsilon\}}B)\downarrow\nu_t(\mathbf{1}_{\{|x|\geq\varepsilon\}}B)$ as $s\downarrow t$. This finishes the proof of (3). □

Exercise 63 Explain why (3) [together with a bit of (2)] gives stochastic continuity of X in the proof of $\boxed{\Leftarrow}$ in Theorem 6.5.

Exercise 64 Explain why $g_s\uparrow g_{s'}$ and $g_s\downarrow g_{s'}$ a.e. $(\nu_{\hat{t}})$ as $s\uparrow s'\leq\hat{t}$ and $s\downarrow s'<\hat{t}$, respectively, in the proof of $\boxed{\Rightarrow}$ in Theorem 6.5.

Corollary 6.6 *A family of triplets* $\{(0,\nu_t,0)\}_{t\geq 0}$ *is system of triplets for an additive process in law, iff.*

$$\nu([0,t]\times B)\equiv\nu_t(B)\qquad\textit{for}\quad t\geq 0\quad\textit{and}\quad B\in\mathcal{B}(\mathbb{R}^d)$$

defines a measure on $[0,\infty)\times\mathbb{R}^d$, *such that*

$$\nu(\{t\}\times\mathbb{R}^d)=0\qquad\textit{and}\qquad\int_{[0,t]\times\mathbb{R}^d}1\wedge|x|^2\,d\nu(s,x)<\infty\qquad\textit{for}\quad t\geq 0.$$

EXERCISE 65 Prove $\boxed{\Rightarrow}$ in Corollary 6.6.

EXERCISE 66 Prove $\boxed{\Leftarrow}$ in Corollary 6.6.

6.3 Some Facts from Measure Theory and Functional Analysis

Definition 6.7 *A family $\mathcal{S}$ of subsets of a set S is a <u>semi-ring</u> if the following conditions hold;*

(1) $\emptyset \in \mathcal{S}$;

(2) $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$;

(3) $A, B \in \mathcal{S} \Rightarrow B \setminus A = \bigcup_{k=1}^n E_k$ *for some disjoint* $\{E_k\}_{k=1}^n \subseteq \mathcal{S}$.

Definition 6.8 *A family $\mathcal{R}$ of subsets of a set S is a <u>ring</u> if the following conditions hold;*

(1) $A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}$;

(2) $A, B \in \mathcal{R} \Rightarrow A \Delta B \equiv (A \setminus B) \cup (B \setminus A) \in \mathcal{R}$.

EXERCISE 67 Show that rings are semi-rings. Show that there exists a smallest ring, the <u>generated ring</u>, that contains any given semi-ring.

Definition 6.9 *A family $\mathfrak{R}$ of subsets of a set S is a <u>δ-ring</u> if it is a ring such that*

$$\{A_k\}_{k=1}^\infty \subseteq \mathfrak{R} \Rightarrow \bigcap_{k=1}^\infty A_k \in \mathfrak{R}.$$

Definition 6.10 *A family $\mathfrak{R}$ of subsets of a set S is a <u>σ-ring</u> if it is a ring such that*

$$\{A_k\}_{k=1}^\infty \subseteq \mathfrak{R} \Rightarrow \bigcup_{k=1}^\infty A_k \in \mathfrak{R}.$$

Definition 6.11 *A famliy $\mathcal{S}$ of subsets of a set S is <u>σ-$\mathcal{R}$</u> if*

$$S = \bigcup_{k=1}^\infty A_k \qquad \textit{for some} \quad \{A_k\}_{k=1}^\infty \subseteq \mathcal{S}.$$

EXAMPLE 3 A σ-ring $\mathfrak{R}$ is a δ-ring, since for $\{A_k\}_{k=1}^\infty \subseteq \mathfrak{R}$, with $A = \bigcup_{k=1}^\infty A_k$,

$$\mathfrak{R} \ni A \Delta \bigcup_{k=1}^\infty (A \Delta A_k) = A \cap [\bigcup_{k=1}^\infty (A \cap A_k^c)]^c = A \cap \bigcap_{k=1}^\infty (A^c \cup A_k) = \bigcap_{k=1}^\infty A_k.$$

A "typical" δ-ring (that is not a σ-ring) is $\{B \in \mathcal{B}(\mathbb{R}^d) : \int_B dx < \infty\}$.

σ-finite integration theory can be developed by considering measures (σ-additive set functions) over the σ-$\mathcal{R}$ δ-ring of sets with well-defined and finite measure (set function values). (See below for notation!) In advanced integration theory and in stochastic integration, this is the standard. #

Definition 6.12 *A family $\mathcal{A}$ of subsets of a set S is an <u>algebra</u> if it is a ring such that $S \in \mathcal{A}$.*

EXERCISE 68 Show that a ring $\mathcal{R}$ is an algebra iff. it is closed under taking complements. Show that there exists a smallest algebra, the generated algebra, that contains $\mathcal{R}$.

Definition 6.13 *A family $\mathfrak{A}$ of subsets of a set S is a σ-algebra if it is a σ-ring such that $S\in\mathfrak{A}$. A measurable space $(S,\mathfrak{A})$ is a set S together with a σ-algebra $\mathfrak{A}$ of subsets S.*

EXERCISE 69 For a ring $\mathcal{R}$, show that there exists a smallest δ-ring, the generated δ-ring, a smallest σ-ring, the generated σ-ring, that is a σ-algebra if $\mathcal{R}$ is σ-$\mathcal{R}$, and a smallest σ-algebra, the generated σ-algebra $\sigma(\mathcal{R})$, that contains $\mathcal{R}$. What happens if $\mathcal{R}$ is a semi-ring instead?

Definition 6.14 *A function $\mu:\mathcal{S}\to\overline{\mathbb{R}}\equiv[-\infty,\infty]$ on a family $\mathcal{S}$ of subsets of a set S is an additive set function if the following conditions hold;*

(1) $\mu(\emptyset)=0$;

(2) $\mu(\bigcup_{k=1}^n A_k)=\sum_{k=1}^n \mu(A_k)$ *for disjoint* $\{A_k\}_{k=1}^n\subseteq\mathcal{S}$ *with* $\bigcup_{k=1}^n A_k\in\mathcal{S}$.

EXERCISE 70 Show that $(2)\Rightarrow(1)$, in Definition 6.14, if $\mu(A)\in\mathbb{R}$ for some $A\in\mathcal{S}$. Why cannot an additive set function on a ring take both the values $-\infty$ and ∞?

Definition 6.15 *An additive set function μ on a family $\mathcal{S}$ of subsets of a set S is σ-additive if*

$$\mu(\textstyle\bigcup_{k=1}^\infty A_k)=\sum_{k=1}^\infty \mu(A_k) \quad \text{for disjoint } \{A_k\}_{k=1}^\infty\subseteq\mathcal{S} \quad \text{with } \bigcup_{k=1}^\infty A_k\in\mathcal{S}.$$

Definition 6.16 *A measure is a non-negative σ-additive set function. A measure space $(S,\mathfrak{A},\mu)$ is a set S together with a measure μ on a σ-algebra $\mathfrak{A}$ of subsets of S.*

Definition 6.17 *A function $\mu:\mathcal{S}\to\overline{\mathbb{R}}$ on a family $\mathcal{S}$ of subsets of a set S is σ-finite if*

$$B\subseteq\textstyle\bigcup_{k=1}^\infty A_k \quad \text{for some } \{A_k\}_{k=1}^\infty\subseteq\mathcal{S} \quad \text{with } \mu(A_k)\in\mathbb{R}, \quad \text{for each } B\in\mathcal{S}.$$

EXERCISE 71 Which of Definitions 6.14-6.17 apply to $\mu(A)\equiv\#\{t:t\in A\}$ for $A\subseteq S$?

Theorem 6.18 *An additive set function μ on a semi-ring $\mathcal{S}$ of subsets of a set S that is finite, or non-negative (non-positive), has a unique extension to the generated ring* (see Exercise 67). *If μ is σ-additive, then so is the extension.*

Theorem 6.19 *A σ-additive set function on a ring $\mathcal{R}$ of subsets of a set S that is finite, or non-negative (non-positive) and σ-finite, has a unique extension to the generated δ-ring* (see Exercise 69) *that is finite, or non-negative (non-positive) and σ-finite, respectively.*

Theorem 6.20 *A measure on a ring $\mathcal{R}$ of subsets of a set S has an extension to the generated σ-ring* (see Exercise 69)*. The extension is unique and σ-finite if the mesure is σ-finite.*

Definition 6.21 *A function $\rho : T\times T \to [0,\infty)$ is a* <u>*pseudo-metric*</u> *on the set T if the following conditions hold;*

(0) $\rho(t,t)=0$ *for* $t\in T$;

(1) $\rho(r,t) \leq \rho(r,s)+\rho(s,t)$ *for* $r,s,t\in T$ (Δ-inequality);

(2) $\rho(s,t) = \rho(t,s)$ *for* $s,t\in T$ (symmetry).

EXERCISE 72 Explain why, on the space of `rv's` with well-defined mean, $\rho(Y_1,Y_2) \equiv \mathbf{E}\{|Y_1 - Y_2|\}$ is a pseudo-metric, but not a metric.

Definition 6.22 *A pseudo-metric ρ on a set T is a* <u>*metric*</u> *if*

$$\rho(s,t)=0 \;\Rightarrow\; s=t \qquad \text{for} \quad s,t\in T.$$

Virtually all results for metric spaces have versions for pseudo-metric spaces. In fact, often results extends immediately, without any changes at all.

Definition 6.23 *A function $\|\cdot\| : T \to [0,\infty)$ is a* <u>*pseudo-norm*</u> (also called <u>semi-norm</u>) *on a linear space T* (over $\mathbb{R}$ say)*, if the following conditions hold;*

(0) $\|0\|=0$;

(1) $\|s+t\| \leq \|s\|+\|t\|$ *for* $s,t\in T$ (Δ-inequality);

(2) $\|\lambda t\| = |\lambda|\,\|t\|$ *for* $\lambda\in\mathbb{R}$ *and* $t\in T$ (homogeneity).

Definition 6.24 *A pseudo-norm $\|\cdot\|$ on T is a* <u>*norm*</u> *if*

$$\|t\|=0 \;\Rightarrow\; t=0 \qquad \text{for} \quad t\in T.$$

EXERCISE 73 Show that for $\|\cdot\|$ a pseudo-norm (norm), $\|\cdot-\cdot\|$ is a pseudo-metric (metric).

Definition 6.25 *A function $\|\cdot\| : T\to[0,\infty)$ is a* <u>*F-pseudo-norm*</u> (also called <u>F-semi-norm</u>) *on a linear space T* (over $\mathbb{R}$ say)*, if the following conditions hold;*

(0) $\|0\|=0$;

(1) $\|s+t\| \leq \|s\|+\|t\|$ *for* $s,t\in T$;

(2) $\|\lambda_k t_k - \lambda t\| \to 0$ *whenever* $\mathbb{R}\ni\lambda_k\to\lambda\in\mathbb{R}$ *and* $t_1,t_2,\ldots,t\in T$ *with* $\|t_k-t\|\to 0$.

Definition 6.26 *An F-pseudo-norm $\|\cdot\|$ is an* <u>*F-norm*</u> *if*

$$\|t\|=0 \;\Rightarrow\; t=0 \qquad \text{for} \quad t\in T.$$

EXERCISE 74 Show that a pseudo-norm (norm) is an F-pseudo-norm (F-norm).

Definition 6.27 *A pseudo-modular of moderate growth* (pmmg) *on a linear space T* (over $\mathbb{R}$ say), *is a function $\Phi : T \to [0,\infty)$ such that the following conditions hold;*

(0) $\Phi(0)=0$;

(1) $\mathbb{R} \ni \lambda \to \Phi(\lambda t)$ *is continuous, even, non-incresing on $(-\infty,0]$, and non-decreasing on $[0,\infty)$, for $t \in T$;*

(2) $\Phi(s+t) \le C(\Phi(s)+\Phi(t))$ *for $s,t \in T$, for some constant $C>0$.*

EXERCISE 75 Define "modular of moderate growth". Verify that $\Phi(Y) \equiv \mathbf{E}\{|Y|^2\}$ is a pmmg, but not a pseudo-norm, on the space of Y that satisfy $\mathbf{E}\{|Y|^2\} < \infty$.

Theorem 6.28 *A pmmg Φ on T induces a pseudo-metrizable topology on T with open subbasis*

$$\{\{t \in T : \Phi(t-t_1) < r_1, \ldots, \Phi(t-t_n) < r_n\}\}_{t_1,\ldots,t_n \in T,\, r_1,\ldots,r_n > 0,\, n \in \mathbb{N}}.$$

Further, addition and scalar multiplication are continuous (i.e., T is a topological vector space).

Theorem 6.29 (MUSIELAK-ORLICZ) *Let $(S,\mathfrak{A},\mu)$ be a complete σ-finite measure space, with $\mathfrak{A}$ the Borel σ-algebra of a separable topological space. Let $\psi : S \times \mathbb{R} \to [0,\infty)$ be a function such that the following conditions hold;*

(1) $\psi(s,0)=0$ *for $s \in S$;*

(2) $\psi(s,\cdot)$ *is continuous, even, non-incresing on $(-\infty,0]$, and non-decreasing on $[0,\infty)$, for $s \in S$;*

(3) $\psi(\cdot,t) \in \mathbb{L}^0(S,\mathfrak{A})$ *for $t \in \mathbb{R}$;*

(4) $\psi(s,C_1 t) \le C_2 \psi(s,t)$ *for $s \in S$ and $t \in \mathbb{R}$, for some constants $C_1, C_2 > 1$.*

Then $\psi(\cdot, f(\cdot)) \in \mathbb{L}^0(S,\mathfrak{A})$ for $f \in \mathbb{L}^0(S,\mathfrak{A})$, and

$$\Phi(f) \equiv \int_S \psi(s,f(s))\,d\mu(s) \qquad \textit{is a pmmg on} \qquad \mathbb{L}^{\psi}(S,\mathfrak{A},\mu) \equiv \{f \in \mathbb{L}^0(S,\mathfrak{A}) : \Phi(f) < \infty\}.$$

The induced topology is complete pseudo-metric, with addition and scalar multiplication continuous. If in addition $\Phi(1)<\infty$, then $\mathbb{L}^{\psi}(S,\mathfrak{A},\mu)$ is separable, with simple functions as a dense subset.

Definition 6.30 *The space $\mathbb{L}^{\psi}(S,\mathfrak{A},\mu)$ in Theorem 6.29 is a Musielak-Orlicz space. If $\psi(s,\cdot)=\psi(\cdot)$ does not depend on $s \in S$, then it is an Orlicz space.*

EXERCISE 76 Show that the space of `rv's` with the modular $\Phi(Y) \equiv \mathbf{E}\{|Y| \wedge 1\}$ is an Orlicz space, and that the convergence $Y_n \to Y \Leftrightarrow_{\mathrm{def}} \Phi(Y_n - Y) \to 0$ is convergence in probability.

EXERCISE 77 Many important Musielak-Orlicz spaces do not satsify $\Phi(1)<\infty$. Give an example of this. However, the criterion $\Phi(1)<\infty$ for denseness of simple functions can often be used anyway, by a σ-finiteness type of argument. Explain how!

6.5 Presentation av Examensarbete, Fredag 27 September kl. 11.15, S1

7 Lecture 7 1/10 - 02

7.1 Stochastic Integrals wrt. ID Random Measures

In Lecture 7, $\underline{\mathfrak{S}}$ is a σ-$\mathcal{R}$ δ-ring of subsets of a set $\underline{S}$. (Simply view $\mathfrak{S}$ as a σ-algebra!)

Definition 7.1 *An independently scattered random measure (`isrm`) on $(S,\mathfrak{S})$ is an $\mathbb{R}$-valued stochastic process $\{\Lambda(A)\}_{A\in\mathfrak{S}}$, such that the following conditions hold;*

(1) $\{\Lambda(A_k)\}_{k=1}^n$ are independent for disjoint $\{A_k\}_{k=1}^n\subseteq\mathfrak{S}$;

(2) $\Lambda(\bigcup_{k=1}^\infty A_k) =_{\text{a.s.}} \sum_{k=1}^\infty \Lambda(A_k)$ for disjoint $\{A_k\}_{k=1}^\infty\subseteq\mathfrak{S}$ with $\bigcup_{k=1}^\infty A_k\in\mathfrak{S}$.

Definition 7.2 *An `id isrm` (`idisrm`) is an `isrm` with `id` process values.*

Theorem 7.3 *A function $\Lambda:\mathfrak{S}\to\mathbb{L}^0(\Omega,\mathfrak{F})$ is an `idisrm` on $(S,\mathfrak{S})$ iff.*

$$\mathbf{E}\{e^{i\theta\Lambda(A)}\} = \exp\{i\theta\nu_0(A) - \tfrac{1}{2}\theta^2\nu_1(A) + \textstyle\int_{\mathbb{R}}(e^{i\theta x}-1-i\theta x\mathbf{1}_{\{|x|\leq 1\}})\,dF_A(x)\}$$

for $\theta\in\mathbb{R}$ and $A\in\mathfrak{S}$, for some (necessarily unique)

$$\begin{cases}\text{signed finite measure } \nu_0 \text{ on } \mathfrak{S},\\ \text{finite measure } \nu_1 \text{ on } \mathfrak{S},\\ \text{Lévy measures } \{F_A\}_{A\in\mathfrak{S}} \text{ on } \mathbb{R} \text{ with } \{F_{(\cdot)}(B)\}_{B\in\mathcal{B}(\mathbb{R}),0\notin\text{clos}(B)} \text{ finite measures on } \mathfrak{S}.\end{cases}$$

Conversely, given such ν_0, ν_1 and F_A, there exists an `idisrm` Λ with the `chf` above.

Exercise 78 Show that an `idisrm` on $(S,\mathfrak{S})$ is an `id` stochastic process. Explain why an `isrm` on $\mathbb{R}$ or $\mathbb{R}^+$ must be an `idisrm`.

Exercise 79 Motivate (or prove) Theorem 7.3.

Definition 7.4 *For an `idisrm` on $(S,\mathfrak{S})$ with `chf` as in Theorem 7.3, the control measure λ is the unique extension to $\sigma(\mathfrak{S})$ of the measure on $\mathfrak{S}$*

$$\lambda = |\nu_0| + \nu_1 + \textstyle\int_{\mathbb{R}} 1\wedge x^2\,dF_{(\cdot)}(x).$$

Corollary 7.5 *For an `idisrm` Λ on $(S,\mathfrak{S})$ with control measure λ, we have*

$$\lambda(A_n)\to 0 \quad \text{for} \quad \{A_n\}_{n=1}^\infty\subseteq\mathfrak{S} \quad\Leftrightarrow\quad \Lambda(A_n')\to 0 \quad \text{whenever} \quad \mathfrak{S}\ni A_n'\subseteq A_n \quad \text{for} \quad n\in\mathbb{N}.$$

EXERCISE 80 Prove Corollary 7.5.

Definition 7.6 *For an* `idisrm` *on* $(S,\mathfrak{S})$ *with* `chf` *as in Theorem 7.3 and control measure* λ, *we define the* characteristic (σ^2,ρ,b) *by*

$$\sigma^2 = d\nu_1/d\lambda, \quad \rho(\cdot\,,B) = dF_{(\cdot)}(B)/d\lambda \quad \text{and} \quad b = d\nu_0/d\lambda.$$

Fact 7.7 *For the characteristic* (σ^2,ρ,b) *of an* `idisrm` *with control measure* λ,

$$|b(s)| + \sigma^2(s) + \int_{\mathbb{R}} 1\wedge x^2\,\rho(s,dx) = 1 \quad a.e. \ (\lambda).$$

Proof. For $A\in\mathfrak{S}$, we have

$$\int_A [|b|+\sigma^2+\int_{\mathbb{R}}1\wedge x^2\,\rho(\cdot\,,dx)]\,d\lambda = |\nu_0(A)|+\nu_1(A)+\int_{\mathbb{R}}1\wedge x^2\,dF_A(x) = \lambda(A) = \int_A d\lambda. \ \square$$

Fact 7.8 *For an* `idisrm` *on* $(S,\mathfrak{S})$ *with control measure* λ *and characteristic* (σ^2,ρ,b), *we have, for* $\theta\in\mathbb{R}$ *and* $A\in\mathfrak{S}$,

$$\mathbf{E}\{e^{i\theta\Lambda(A)}\} = \exp\{\int_{s\in A}[i\theta b(s) - \tfrac{1}{2}\theta^2\sigma^2(s) + \int_{\mathbb{R}}(e^{i\theta x}-1-i\theta x\mathbf{1}_{\{|x|\leq 1\}})\,\rho(s,dx)]\,d\lambda(s)\}.$$

EXERCISE 81 Prove Fact 7.8.

Definition 7.9 *For an* `idisrm` Λ *on* $(S,\mathfrak{S})$, *and a* $\mathfrak{S}$*-simple function* $f = \sum_{k=1}^n a_k\mathbf{1}_{A_k}$, *with* $\{A_k\}_{k=1}^n \subseteq \mathfrak{S}$ *disjoint, the* stochastic integral $\int f\,d\Lambda$ *is defined*

$$\int f\,d\Lambda = \sum_{k=1}^n a_k\,\Lambda(A_k).$$

Fact 7.10 *For an* `idisrm` Λ *on* $(S,\mathfrak{S})$ *with characteristic* (σ^2,ρ,b) *and control measure* λ, *we have for* $\mathfrak{S}$*-simple* $f:S\to\mathbb{R}$,

$$\mathbf{E}\{e^{i\theta\int f\,d\Lambda}\} = \exp\{\int_S[i\theta f b - \tfrac{1}{2}\theta^2 f^2\sigma^2 + \int_{\mathbb{R}}(e^{i\theta f x}-1-i\theta f x\mathbf{1}_{\{|x|\leq 1\}})\,\rho(\cdot\,,dx)]\,d\lambda\}.$$

Hence $\int f\,d\Lambda$ *is* `id` *with triplet* (A_f,ν_f,γ_f) *given by*

$$\begin{cases} A_f = \int_S f^2\sigma^2\,d\lambda \\ \nu_f = \int_S\int_{\mathbb{R}}\mathbf{1}_{(\cdot)}(f(s)x)\,\rho(s,dx)d\lambda(s) \\ \gamma_f = \int_S[fb+\int_{\mathbb{R}}fx(\mathbf{1}_{\{|fx|\leq 1\}}-\mathbf{1}_{\{|x|\leq 1\}})\,\rho(\cdot\,,dx)]\,d\lambda \end{cases}.$$

Proof. For $f=\sum_{k=1}^n a_k \mathbf{1}_{A_k}$, Fact 7.8 shows that $\mathbf{E}\{e^{i\theta\int f\,d\Lambda}\}$ is

$$\prod_{k=1}^n \exp\{\int_{A_k}[i\theta a_k b-\tfrac{1}{2}\theta^2 a_k^2\sigma^2+\int_{\mathbb{R}}(e^{i\theta a_k x}-1-i\theta a_k x\mathbf{1}_{\{|x|\leq 1\}})\,\rho(s,dx)]\,d\lambda(s)\}. \quad \Box$$

EXERCISE 82 Verify the claim regarding the triplet (A_f,ν_f,γ_f) in Fact 7.10.

Theorem 7.11 *For an* `idisrm` Λ *on* $(S,\mathfrak{S})$ *with characteristic* (σ^2,ρ,b) *and control measure* λ, *the following function* ψ *satisfies the hypothesis of Theorem 6.29,*

$$\psi(s,t)\equiv[\sup_{|\tau|\leq|t|}|\tau|\,|b(s)+\int_{\mathbb{R}}x(\mathbf{1}_{\{|\tau x|\leq 1\}}-\mathbf{1}_{\{|x|\leq 1\}})\,\rho(s,dx)|]+t^2\sigma^2(s)+\int_{\mathbb{R}}1\wedge|tx|^2\,\rho(s,dx).$$

The Musielak-Orlicz space $\mathbb{L}^\psi(S,\mathfrak{A},\lambda)$ *is metrized by the complete* F*-pseudo-norm*

$$\|f\|_\psi\equiv\inf\{c>0:\Phi(f/c)\leq c\}\qquad \textit{where}\qquad \Phi(f)=\int_S\psi(s,f(s))\,d\lambda(s).$$

We have $f\in\mathbb{L}^\psi(S,\mathfrak{A},\lambda)$ *iff.* $\|f\|_\psi<\infty$ *iff.* $\Phi(f)<\infty$ *iff.*

$$\int_S[|f|\,|b+\int_{\mathbb{R}}x(\mathbf{1}_{\{|fx|\leq 1\}}-\mathbf{1}_{\{|x|\leq 1\}})\,\rho(\cdot,dx)|+f^2\sigma^2+\int_{\mathbb{R}}1\wedge|fx|^2\,\rho(\cdot,dx)]\,d\lambda<\infty.$$

Definition 7.12 *With the notation of Theorem 7.11, we write* $\underline{\mathbb{L}^\Lambda}$ *for* $\mathbb{L}^\psi(S,\mathfrak{A},\lambda)$.

Fact 7.13 *For an* `idisrm` Λ *on* $(S,\mathfrak{S})$, *with the notation of Theorem 7.11,* $\|\int f\,d\Lambda-\int g\,d\Lambda\|_\psi\equiv\|f-g\|_\psi$ *is an* F*-pseudo-norm on* $I^\Lambda\equiv\{\int f\,d\Lambda:\text{simple } f:S\to\mathbb{R}\}$. *The completion of* I^Λ *can be topologically identified with* $\mathbb{L}^\Lambda$.

EXERCISE 83 Provide arguments for Fact 7.13.

Lemma 7.14 *For an* `idisrm` Λ *on* $(S,\mathfrak{S})$, *if* $\{f_n\}_{n=1}^\infty$ *and* $\{f_n'\}_{n=1}^\infty$ *are* $\mathfrak{S}$*-simple sequences with* $\|f_n-f\|_\psi,\|f_n'-f\|_\psi\to 0$ *for* $f\in\mathbb{L}^\Lambda$, *then* $\int f_n\,d\Lambda-\int f_n'\,d\Lambda\to_{\mathbf{P}}0$.

Proof. By Fact 7.10, $\int f_n\,d\Lambda-\int f_n'\,d\Lambda$ is `id` with triplet (A_n,ν_n,γ_n) given by

$$\begin{cases} A_n=\int_S(f_n-f_n')^2\sigma^2\,d\lambda\\ \nu_n=\int_S\int_{\mathbb{R}}\mathbf{1}_{(\cdot)}(f_n(s)x-f_n'(s)x)\,\rho(s,dx)d\lambda(s)\\ \gamma_n=\int_S[(f_n-f_n')b+\int_{\mathbb{R}}(f_n-f_n')x(\mathbf{1}_{\{|(f_n-f_n')x|\leq 1\}}-\mathbf{1}_{\{|x|\leq 1\}})\,\rho(\cdot,dx)]\,d\lambda\end{cases}.$$

It follows from Theorem 5.1, and the definition of Φ, that $\int f_n\,d\Lambda-\int f_n'\,d\Lambda\to_d 0$ when $\Phi(f_n-f_n')\to 0$. This in turn holds by Theorem 7.11, since $\|f_n-f_n'\|_\psi\to 0$. $\Box$

EXERCISE 84 Elaborate on the details in the proof of Lemma 7.14.

Definition 7.15 *For an* `idisrm` Λ *on* $(S,\mathfrak{S})$*, with the notation of Theorem 7.11, the* <u>*stochastic integral* $\int f\,d\Lambda$</u> *of* $f\in\mathbb{L}^{\Lambda}$ *is defined as* $\lim_{n\to\infty}\int f_n\,d\Lambda$*, for any* $\mathfrak{S}$*-simple sequence* $\{f_n\}_{n=1}^{\infty}$ *such that* $\|f_n-f\|_{\psi}\to 0$.

Corollary 7.16 *For an* `idisrm` Λ *on* $(S,\mathfrak{S})$*,* $\int f\,d\Lambda$ *is* `id` *with triplet* (A_f,ν_f,γ_f) *as in Fact 7.10, and* $\int f_n\,d\Lambda\to_{\mathbf{P}}\int f\,d\Lambda$ *for some* $\mathfrak{S}$*-simple* $\{f_n\}_{n=1}^{\infty}$*, for* $f\in\mathbb{L}^{\Lambda}$.

For an `idisrm` Λ on $\mathbb{R}$, the following process L is an additive process on $\mathbb{R}$,

$$L(t)=\Lambda((0,t])\quad\text{for }\ t\geq 0\qquad\text{and}\qquad L(t)=\Lambda((t,0])\quad\text{for }\ t<0.$$

Conversely, for $\mathbb{R}$-valued additive process $\{L(t)\}_{t\in\mathbb{R}}$, there exists an `idisrm` Λ on $\mathbb{R}$ with $\{\Lambda((s,t])\}_{\mathbb{R}\ni s<t\in\mathbb{R}}=_d\{L(t)-L(s)\}_{\mathbb{R}\ni s<t\in\mathbb{R}}$. The integral is denoted $\int f\,dL$.

For an $\mathbb{R}$-valued Lévy process $\{L(t)\}_{t\in\mathbb{R}}$, the characteristic of dL coincide with the generating triplet, and the control measure a multiple of Lebesgue measure.

Exercise 85 Verify the claims above for additive processes and Lévy processes.

Remark 7.17 In the sequel, we will neither emphasize nor make crucial use of the mathematical aspects of stochastic integration brought up in Lectures 6 and 7. Rather, they are there to give a complementary view on things. #

7.2 Stochastic Integrals wrt. α-stable Lévy Motion

Let $\{L(t)\}_{t\in\mathbb{R}}$ be α-stable Lévy motion, $\alpha\in(0,1)\cup(1,2)$, with $L(1)\sim S_\alpha(c,\beta,0)$. For $\alpha<1$, since $\gamma_0=0$ (see Corollary 3.11), the last integral in Theorm 7.11 is

$$\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}f(s)x\mathbf{1}_{\{|f(s)x|\leq 1\}}\,d\nu(x)\Big|\,ds+\int_{\mathbb{R}}\int_{\mathbb{R}}1\wedge|f(s)x|^2\,d\nu(x)ds.$$

Exercise 86 For $\alpha<1$ and $c>0$, show that $\mathbb{L}^{dL}=\mathbb{L}^{\alpha}(\mathbb{R})$.

For $\alpha>1$, since $\gamma_1=0$ (see Corollary 3.11), the last integral in Theorm 7.11 is

$$\int_{\mathbb{R}}\Big|\int_{\mathbb{R}}f(s)x\mathbf{1}_{\{|f(s)x|>1\}}\,d\nu(x)\Big|\,ds+\int_{\mathbb{R}}\int_{\mathbb{R}}1\wedge|f(s)x|^2\,d\nu(x)ds.$$

Exercise 87 For $\alpha>1$ and $c>0$, show that $\mathbb{L}^{dL}=\mathbb{L}^{\alpha}(\mathbb{R})$.

Exercise 88 Determine $\mathbb{L}^{dB}$ for $\{B(t)\}_{t\in\mathbb{R}}$ $\mathbb{R}$-valued `Bm`.

7.3 Några tankar om forskningens psykopatografi ...

ur "Sigmund Freud (1910): Leonardo da Vinci - ett barndomsminne":

"Betänker vi Leonardos kombination av övermäktig forskardrift och förtvinat sexualliv, inskränkt till så kallad ideell homosexualitet, är vi benägna att använda honom som mönsterfall för vår tredje typ. Det skulle vara kärnan och hemligheten i hans natur, att han - efter att infantilt ha använt vetgirigheten för sexuella intressen - lyckades sublimera största delen av sin libido till forskarsträvan. Men att presentera bevis för denna uppfattning är förvisso inte lätt."

"Låt oss uttryckligen betona att vi aldrig räknat Leonardo till neurotikerna, eller de "nervsjuka" som den klumpiga termen lyder. Den som beklagar att vi över huvud taget vågar ha patografiska aspekter på honom, den klamrar vid fördomar som vi i dag med rätta har övergett. Vi tror inte längre att hälsa och sjukdom, normal och nervöst, klart kan skiljas åt, och att neurotiska drag måste uppfattas som bevis för allmänn mindervärdighet. Vi vet i dag att neurotiska symptom är ett substitut för vissa bortträngningar som vi måste göra under vår utveckling från barn till kulturmänniska. Att vi alla utför sådana substitututbildningar och att det bara är dessa substituts antal, intensitet och fördelning som motiverar det praktiska begreppet sjukdom och slutsatsen om konstitutionell mindervärdighet. Utifrån små symptom i Leonardos personlighet kan vi nu föra honom till den neurotikertyp som vi kallar "tvångstypen". Och kan jämföra hans forskande med neurotikerns "grubbeltvång" och hans hämningar med dessas så kallade abuli (viljesvaghet, ö.a.)."

"I blomman av sin ungdom verkar Leonardo till en början arbeta utan hämningar. När han i sin livsföring utåt gör fadern till förebild, så upplever han i Milano en tid av manlig skaparkraft och konstnärlig produktivitet. Där erbjuds han också av en slump ett fadersubstitut i form av hertig Lodovico Moro. Men snart gör hans erfarenhet sig påmind: hans nästan totala undertryckande av egentligt sexualliv är inte bästa förutsättningen för att hans sexualitet ska kunna sublimeras i annan aktivitet. Sexuallivet gör sig gällande som något positivt, driftigheten och förmågan till snabba beslut börjar förlamas, tendensen att fundera och tveka uppträder redan vid Nattvarden som störande och avgör tillsammans med tekniken detta storslagna verks öde. Inom honom pågår nu långsamt en process som man bara kan jämföra med regressionerna hos neurotiker. Hans pubertetsutveckling till konstnär överflyglas av hans tidiga infantilt betingade utveckling till forskare. Och den andra sublimeringen av hans erotiska drifter ger nu vika för den ursprungliga sublimeringen, förberedd för den första bortträngningen. Han blir forskare, i början fortfarande i sin konsts tjänst, senare oberoende av den och på väg bort från den. Sedan han förlorat den faderssubstituerande skyddspatronen och livet blivit alltmer dystert så utbreder sig denna regressiva substituering mer och mer. Han blir "mycket irriterad av att måla", som en brevskrivare till markgrevinnan Isabella d'Este berättar, som ovillkorligen ville ha en målning av honom. Hans förflutna som barn har tagit makten över honom. Men forskandet, som nu ersätter hans konstnärliga skapande, tycks ha några drag som är typiska för hur omedvetna drifter aktiveras: omättligheten, den otyglade halsstarrigheten, oförmågan att anpassa sig till realiteter."

8 **Lectures 8-9 4-8/10 - 02:** Sections I.1-I.5 in Protter

The remaining lectures cover Chapters I-III of "*Protter: Stochastic Integration and Differential Equations*". Since Protter's book is more readable than Sato's, we mainly indicate selection of material. We have added explanations when we found that we were in need of such ourselves, or thought we had better arguments.

We do list many exercises. There are no exercises in Protter's book, labeled as such. But Protter leaves many verifications of minor details to the reader, and to check these do in general make very useful exercises. Many of our exercises have this origin.

Our initial ambition is to cover Protter's whole book, with few omissions.

See e.g., Chapter 1 in "*Chung: Lectures from Markov processes to Brownian Motion*" for more on stopping times, and Chapter 6 in "*Kallenberg: Foundations of Modern Probability*" on martingales.

8.1 Basic Definitions and Notation: Section I.1 in Protter

Definition of filtration. (See also Definition 1.14.) Notice that Protter includes $\mathfrak{F}_\infty$ in the filtration. Filtrations with other parametersets than $[0,\infty]$ are defined similarly.

Definition of usual conditions.

Convention: We always assume the usual conditions! But notice that the usual conditions are not assumed for the natural filtrations introduced below.

Definition of stopping time. In the literature, stopping times are called Markov times, optional times, predictable times, and strong stopping times, strong ... . (Without the usual conditions, $\{T<t\}\in\mathfrak{F}_t$ is not the same as $\{T\leq t\}\in\mathfrak{F}_t$, and names are needed for for both these properties.)

Theorem 1, with proof. *Proof:* $\{T<t\}=\bigcup_{u<t}\{T\leq u\}$ and $\{T\leq t\}=\bigcap_{u>t}\{T<u\}$. $\square$

Definition of modification. (See also Definition 1.11.)

Definition of indistinguishable. Recall our Remark 1.12 on this language.

Definition of hitting time.

Theorems 3-4, with proof of Theorem 3. The analogue result for any Borel set is mentioned.

Theorem 5.

Exercise 89 Prove Theorem 5.

Definition of the stopping time σ-algebra $\mathfrak{F}_T$. $\mathfrak{F}_T$ has many properties one intuitively expects (see e.g., Exercise 90 below), but they do in general require a non-trivial proof.

Exercise 90 Show that $\mathfrak{F}_T$ is a σ-algebra for T a stopping time.

Theorem 6, with proof of $\boxed{\subseteq}$. (An "adapted" is missing at one place in the hypothesis.)

EXERCISE 91 Elaborate on the details in the proof of $\boxed{\supseteq}$ in Theorem 6.

EXERCISE 92 For S and T stopping times, show that $\mathfrak{F}_S \subseteq \mathfrak{F}_T$ if $S \leq T$ a.s., and that $\mathfrak{F}_S \cap \mathfrak{F}_T = \mathfrak{F}_{S \wedge T}$.

Because of the usual conditions, $S \leq T$ a.s. is as good as $S \leq T$ for all $\omega \in \Omega$, e.g., when checking memberships in filtrations. (Readers should make sure to take this in!)

Notation $\underline{\Delta X}$. Notice that $\underline{\Delta X}$ is not cádlág, except if it is the zero process.

EXERCISE 93 Show that ΔX is adapted when X is.

Theorem 7 (possibly with proof).

Corollary to Theorem 7. (That this really is a corollary is trivial.)

8.2 Martingales: Section I.2 in Protter

Definition of martingale, supermartingale and submartingale. Many results are proved e.g., for supermartingales, and thus hold automatically for martingales. They often follow also for submartingales, by Exercise 94 below. Martingales with other parametersets are defined similarly.

EXERCISE 94 Show that $-X$ is a supermartingale when X is a submartingale.

Definition of closed martingale. This is much more important than one may initially belive.

Theorem 9. Notice that the final statement *"Such a modification is unique."* in the theorem means that such a modification is indistinguishable from any other such modification. (This is a direct consequence of the fact that modifications of cádlág processes are indistinguishable.)

EXERCISE 95 Show that modifications of martingales, supermartingales and submartingales are still martingales, supermartingales and submartingales, respectively.

Corollary 1 to Theorem 9. (That this is a corollary is of course trivial.)

Convention: We always assume that martingales are right-continuous!

Theorem 10. (The *"Moreover ......"*-part of the result may be skipped, at least for now.)

EXERCISE 96 Why are (right-continuous) martingales cádlág?

Notation $\underline{\bigvee}$ for σ-algebras (filtrations).

Definition of uniform integrability (ui).

EXERCISE 97 Show that $\{Y_\alpha\}_{\alpha\in\mathfrak{A}}$ is ui if $\sup_{\alpha\in\mathfrak{A}}\mathbf{E}\{|Y_\alpha|^p\}<\infty$ for some $p>1$.

Theorem 13. (Theorem 12 is included in Theorem 13.)

Theorem 16. This is really very important!

Theorem 17. For martingales, this is a version of Theorem 16 with other technical conditions.

EXERCISE 98 Why is $T\wedge t$ a stopping time for T a stopping time and $t\in[0,\infty]$?

Definition of stopped process.

> Lemma 8.1 For S and T stopping times, $\{S\le T\}$, $\{S<T\}$, $\{S=T\}\in\mathfrak{F}_{S\wedge T}$.

Proof: By Exercise 99 below, it suffices to check that $\{S<T\}\in\mathfrak{F}_S,\mathfrak{F}_T$. Here $\{S<T\}\in\mathfrak{F}_T$, since

$$\{S<T\}\cap\{T\le t\}=\bigcap_{\mathbb{Q}\ni s\in(t,\hat{t}]}\bigcup_{\mathbb{Q}\ni r<s}\{S<r\}\cap\{r<T<s\}\in\mathfrak{F}_{\hat{t}}\downarrow\mathfrak{F}_t \qquad \text{as} \quad \hat{t}\downarrow t.$$

To show that also $\{S<T\}\in\mathfrak{F}_S$, it is enough to check that $\{S\ge T\}\in\mathfrak{F}_S$, which we get from

$$\{T\le S\}\cap\{S\le t\}=\bigcap_{n=1}^{\infty}\bigcap_{\mathbb{Q}\ni s\in(t,\hat{t}]}\bigcup_{\mathbb{Q}\ni r<s}\{T+\tfrac{1}{n}<r\}\cap\{r<S<s\}\in\mathfrak{F}_{\hat{t}}\downarrow\mathfrak{F}_t \qquad \text{as} \quad \hat{t}\downarrow t. \quad \Box$$

Theorem 18, with proof. *Proof:* By Theorem 13, it is sufficient to prove that $X_{t\wedge T}=\mathbf{E}\{X(T)|\mathfrak{F}_t\}$. This we get (recalling that $\mathfrak{F}_{t\wedge T}\subseteq\mathfrak{F}_t$ by Exercise 92), provided that

$$X(T)\mathbf{1}_{\{T<t\}} \text{ is } \mathfrak{F}_{t\wedge T}\text{-measurable} \qquad \text{and} \qquad \mathbf{E}\{X(T)\mathbf{1}_{\{T\ge t\}}|\mathfrak{F}_{t\wedge T}\}=\mathbf{E}\{X(T)\mathbf{1}_{\{T\ge t\}}|\mathfrak{F}_t\},$$

from the fact that, by Theorem 16,

$$X_{t\wedge T}=\mathbf{E}\{X(T)|\mathfrak{F}_{t\wedge T}\}=X(T)\mathbf{1}_{\{T<t\}}+\mathbf{E}\{X(T)\mathbf{1}_{\{T\ge t\}}|\mathfrak{F}_t\}=\mathbf{E}\{X(T)|\mathfrak{F}_t\}.$$

The rv $X(T)\mathbf{1}_{\{T<t\}}$ is $\mathfrak{F}_{t\wedge T}$-measurable, since

$$\begin{aligned}\{X(T)\in B\}\cap\{T<t\}\cap\{t\wedge T\le s\}&=\{X(t\wedge T)\in B\}\cap\{t\wedge T<t\}\cap\{t\wedge T\le s\}\\&=\begin{cases}\{X(t\wedge T)\in B\}\cap\{t\wedge T<t\} & \text{for } s\ge t\\ \{X(t\wedge T)\in B\}\cap\{t\wedge T\le s\} & \text{for } s<t\end{cases}\in\mathfrak{F}_s \quad \text{for} \quad B\in\mathcal{B}(\mathbb{R}),\end{aligned}$$

by definition of stopping σ-algebras, since $X(t\wedge T)$ is $\mathfrak{F}_{t\wedge T}$-measurable by Theorem 6 (where $t\wedge T$ is a stopping time by Exercise 98), and since $\{t\wedge T<t\}=\bigcup_{\mathbb{Q}\ni u<t}\{t\wedge T\le u\}\in\mathfrak{F}_t\subseteq\mathfrak{F}_s$ for $s\ge t$.

Since $\mathbf{1}_{\{T\ge t\}}$ is $\mathfrak{F}_{t\wedge T}$-measurable, by Lemma 8.1, $\mathbf{E}\{X(T)\mathbf{1}_{\{T\ge t\}}|\mathfrak{F}_{t\wedge T}\}=\mathbf{E}\{X(T)|\mathfrak{F}_{t\wedge T}\}\mathbf{1}_{\{T\ge t\}}$.

For $H\in\mathfrak{F}_t$, $H\cap\{T\ge t\}\in\mathfrak{F}_{t\wedge T}=\mathfrak{F}_t\cap\mathfrak{F}_T$ (cf. Exercise 92), since trivially in $\mathfrak{F}_t$, and in $\mathfrak{F}_T$ by

$$H\cap\{T\ge t\}\cap\{T\le s\}=\begin{cases}\{H\cap\{T<t\}^c\cap\{T\le s\}\in\mathfrak{F}_s & \text{for } s\ge t\\ \emptyset & \text{for } s<t\end{cases}\in\mathfrak{F}_s.$$

Hence we get $\mathbf{E}\{X(T)\mathbf{1}_{\{T\ge t\}}|\mathfrak{F}_{t\wedge T}\}=\mathbf{E}\{X(T)\mathbf{1}_{\{T\ge t\}}|\mathfrak{F}_t\}$, using Lemma 8.1, since for $H\in\mathfrak{F}_t$,

$$\begin{aligned}\mathbf{E}\{\mathbf{E}\{X(T)\mathbf{1}_{\{T\ge t\}}|\mathfrak{F}_{t\wedge T}\}\mathbf{1}_H\}=\mathbf{E}\{\mathbf{E}\{X(T)|\mathfrak{F}_{t\wedge T}\}\mathbf{1}_{\{T\ge t\}}\mathbf{1}_H\}&=\mathbf{E}\{\mathbf{E}\{X(T)\mathbf{1}_{\{T\ge t\}}\mathbf{1}_H|\mathfrak{F}_{t\wedge T}\}\}\\&=\mathbf{E}\{X(T)\mathbf{1}_{\{T\ge t\}}\mathbf{1}_H\}\\&=\mathbf{E}\{\mathbf{E}\{X(T)\mathbf{1}_{\{T\ge t\}}|\mathfrak{F}_t\}\mathbf{1}_H\}. \quad \Box\end{aligned}$$

EXERCISE 99 Why is it enough to check that $\{S<T\}\in\mathfrak{F}_S,\mathfrak{F}_T$ in Lemma 8.1?

Corollary to Theorem 18. *Proof:* By Lemma 8.1, $\mathbf{1}_{\{T\leq S\}}$ is $\mathfrak{F}_{S\wedge T}$-measurable, and thus $\mathfrak{F}_T$-measurable, so that $\mathbf{E}\{Y\mathbf{1}_{\{T\leq S\}}|\mathfrak{F}_T\}$ is $\mathfrak{F}_{S\wedge T}$-measurable, since

$$\mathbf{E}\{Y\mathbf{1}_{\{T\leq S\}}|\mathfrak{F}_T\}\mathbf{1}_{\{S\wedge T\leq t\}}=\mathbf{E}\{Y|\mathfrak{F}_T\}\mathbf{1}_{\{T\leq S\}}\mathbf{1}_{\{T\leq t\}}$$

is $\mathfrak{F}_t$-measurable. This gives

$$\mathbf{E}\{\mathbf{E}\{Y\mathbf{1}_{\{T\leq S\}}|\mathfrak{F}_T\}|\mathfrak{F}_S\}=\mathbf{E}\{\mathbf{E}\{Y\mathbf{1}_{\{T\leq S\}}|\mathfrak{F}_T\}|\mathfrak{F}_{S\wedge T}\}=\mathbf{E}\{Y\mathbf{1}_{\{T\leq S\}}|\mathfrak{F}_{S\wedge T}\}.$$

Similarly, $\mathbf{E}\{\mathbf{E}\{Y\mathbf{1}_{\{T>S\}}|\mathfrak{F}_T\}|\mathfrak{F}_S\}$ is $\mathfrak{F}_{S\wedge T}$-measurable, and thus $\mathfrak{F}_S$-measurable, since

$$\mathbf{E}\{\mathbf{E}\{Y|\mathfrak{F}_T\}|\mathfrak{F}_S\}\mathbf{1}_{\{T>S\}}\mathbf{1}_{\{S\wedge T\leq t\}}=\mathbf{E}\{\mathbf{E}\{Y|\mathfrak{F}_T\}|\mathfrak{F}_S\}\mathbf{1}_{\{T>S\}}\mathbf{1}_{\{S\leq t\}}$$

is $\mathfrak{F}_t$-measurable (by Lemma 8.1), which gives

$$\mathbf{E}\{\mathbf{E}\{Y\mathbf{1}_{\{T>S\}}|\mathfrak{F}_T\}|\mathfrak{F}_S\}=\mathbf{E}\{\mathbf{E}\{Y\mathbf{1}_{\{T>S\}}|\mathfrak{F}_T\}|\mathfrak{F}_{S\wedge T}\}=\mathbf{E}\{Y\mathbf{1}_{\{T>S\}}|\mathfrak{F}_{S\wedge T}\}.$$

Adding things up, we conclude that $\mathbf{E}\{\mathbf{E}\{Y|\mathfrak{F}_T\}|\mathfrak{F}_S\}=\mathbf{E}\{Y|\mathfrak{F}_{S\wedge T}\}$. □

Theorem 19.

Corollary 1 to Theorem 19.

EXERCISE 100 Prove Corollary 1 to Theorem 19.

Corollary 2 to Theorem 19.

EXERCISE 101 Prove Corollary 2 to Theorem 19.

Theorem 20. This is really very important!

Theorem 21, with proof.

8.3 Poisson Process, Brownian Motion: Section I.3 in Protter

EXERCISE 102 Show that $\{X(t)–\mathbf{E}\{X(1)\}\cdot t\}_{t\geq 0}$ and $\{[X(t)–\mathbf{E}\{X(1)\}\cdot t]^2–\mathbf{Var}\ \{X(1)\}\cdot t\}_{t\geq 0}$ are martingales for a suitably integrable $\mathbb{R}$-valued Lévy process X.

Theorem 24. This follows directly from Exercise 102.

Definition of natural filtration.

Definition of n-dimensional Brownian motion (`Bm`) and standard `Bm`. Stochastic calculus uses a different definition of `BM` than is usual: The news are that `BM` is an adpated process, with independent increments defined relative the σ-algebra of "the past", that nothing is said about the value at zero, and that the variance matrix (at 1 say), may have non-zero off-diagonal elements.

Theorem 27. This follows directly from Exercise 102.

Theorem 28. This is well-known from basic stochastic calculus.

Theorem 29. This is well-known from basic stochastic calculus.

8.4 Levy Processes: Section I.4 in Protter

Definition of Lévy process. The definition corresponds to Definition 1.6 of Lévy process in law. The stochastic calculus Lévy process definition differs from the usual one in that the process is adapted, with independent increments defined relative the σ-algebra of the past. A stochastic calculus Lévy process (in law) is always a "ususal" Lévy process in law (see Exercise 103 below).

EXERCISE 103 Why is a stochastic calculus Lévy process a Lévy process in law?

Convention: We always assume that Lévy processes are cádlág!

Theorem 31 (possibly with an idea of the proof).

EXERCISE 104 Show that $\{e^{i\theta X(t)}/\mathbf{E}\{e^{i\theta X(t)}\}\}_{t\geq 0}$ is a $\mathbb{C}$-valued martingale, and has a cádlág modification, for X a Lévy process and $\theta\in\mathbb{R}$.

Theorem 32, with proof. The proof exemplifies that Optional Sampling often cannot be used directly, since requiring closedness or bounded stopping times, so that one has to proceed e.g., by limits of bounded stopping times, or by other technical tricks.

Corollary to Theorem 33. This is well-known from basic stochastic calculus.

Theorem 34.

EXERCISE 105 Give an alternative proof of Theorem 34.

Theorem 40.

EXERCISE 106 Give an alternative proof of Theorem 40.

8.5 Local Martingales: Section I.5 in Protter

Definition of local martingale.

EXERCISE 107 Show that martingales are local martingales.

Definition of reducing stopping time.

Theorem 44, with proof. *Help to Proof:* In (a), by Theorem 17 for the `ui` martingale $M(T\wedge\cdot)$, first with stopping times $S\wedge t\leq t$, and then with $S\wedge s\leq t$, Corollary to Theorem 18 gives

$$\mathbf{E}\{M(S\wedge t)|\mathfrak{F}_s\} = \mathbf{E}\{\mathbf{E}\{M(T\wedge t)|\mathfrak{F}_{S\wedge t}\}|\mathfrak{F}_s\} = \mathbf{E}\{M(T\wedge t)|\mathfrak{F}_{S\wedge s}\} = M(S\wedge s) \qquad \text{for} \quad s<t.$$

Hence $M(S\wedge\cdot)$ is a martingale, which is `ui` by Theorems 13 and 16 (see Exercise 108 below).

In (c), skip the details with M_0, to get an equally good proof, that is immediate from (a). □

EXERCISE 108 Show that $M(S\wedge\cdot)$ is `ui` in the above proof.

Corollary to Theorem 44. This is immediate from Theorem 44.b.

Definition of <u>local</u> property.

Theorem 45.

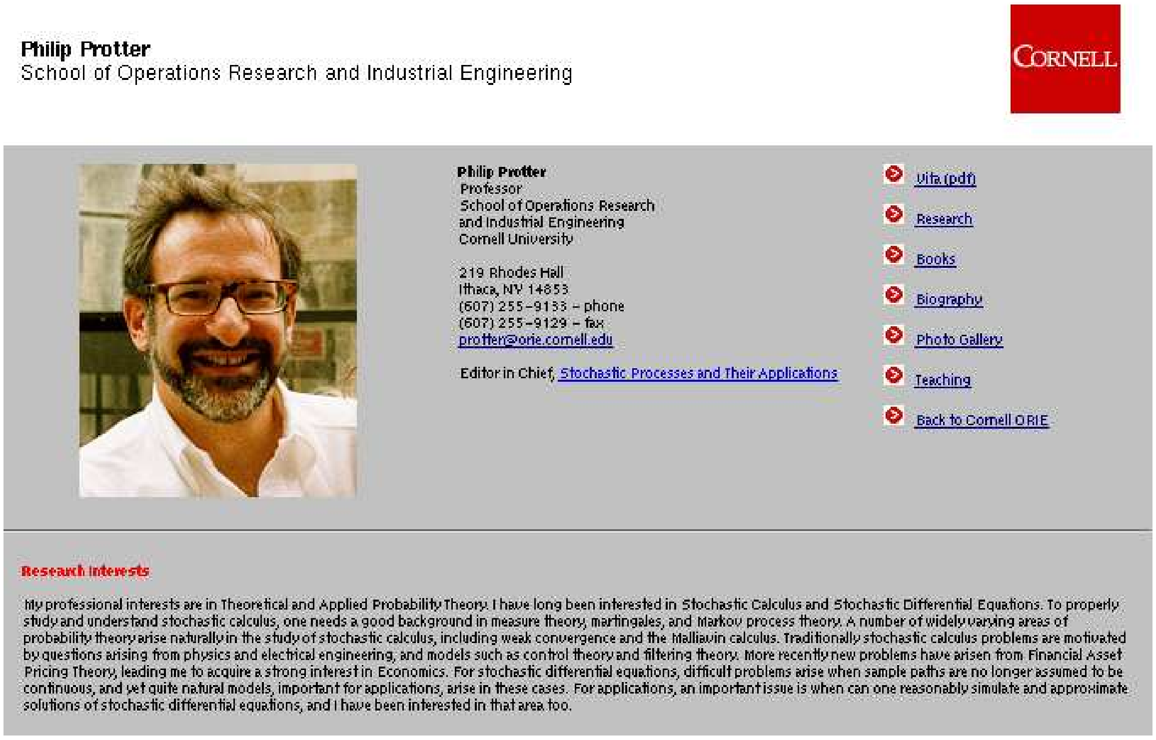
EXERCISE 109 Prove Theorem 45.

Theorem 46.

EXERCISE 110 Prove Theorem 46.

Theorem 47, with proof. [It is not terribly difficult to prove that a local martingale X is a martingale iff. it is of Dirichlet class DL, i.e., iff. $\{X(T):T\leq t \text{ stopping time}\}$ is `ui` for each $t>0$.]

On `http://www.orie.cornell.edu/~protter/` we found



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Editor in Chief, Stochastic Processes and Their Applications

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Research Interests

My professional interests are in Theoretical and Applied Probability Theory. I have long been interested in Stochastic Calculus and Stochastic Differential Equations. To properly study and understand stochastic calculus, one needs a good background in measure theory, martingales, and Markov process theory. A number of widely varying areas of probability theory arise naturally in the study of stochastic calculus, including weak convergence and the Malliavin calculus. Traditionally stochastic calculus problems are motivated by questions arising from physics and electrical engineering, and models such as control theory and filtering theory. More recently new problems have arisen from Financial Asset Pricing Theory, leading me to acquire a strong interest in Economics. For stochastic differential equations, difficult problems arise when sample paths are no longer assumed to be continuous, and yet quite natural models, important for applications, arise in these cases. For applications, an important issue is when can one reasonably simulate and approximate solutions of stochastic differential equations, and I have been interested in that area too.

8.8 Några tidiga tankar om detta med Bioinformatik (likn.) ...

ur "Falstaff Fakir (1895): *Ny och nyttig lärobok i zoologi.* Förtjänstfull avhandling för att erhålla en bra plats, med åtskilliga illustrationer efter iaktagelser i naturen.":

"Då ...... en bra plats såsom direktör för zoologien nu är ledig, gör jag vad jag länge tänkt. Jag ger ut

mitt vetande om djuren,

och gör det med rätta.

*

Djuren

indelar jag naturligtvis i två slag:

I. Djur.
II. Odjur.

Som de sistnämda äro äldst, bjuder mig redan vördnaden för ålderdomen att först beskriva dem.

II. Odjur.

Odjuren skilja sig från djuren dels genom sitt *o*, dels genom sin *skapnad*. De flesta av odjuren äro hur som helst, medan de mesta av djuren följa vissa reglor i avseende på sitt skelett, sin färg, sin själ, sitt kynne, sina födoämnen m.m. dylikt.

Det äldsta och vackraste odjuret är

Leviathan.

Job omtalar detta odjur med en bestämd hätskhet, och andra förf. i samma ämne äro även genomträngda av en viss aversion ...... Numera har den spelat ut sin roll och förekommer endast sporadiskt vid läsaremöten i ...... Afrika, Stockholm m.fl. svårtillgängliga trakter.

Fågeln Phoenix

är ett hundraårsodjur. När detta vackra flygfä uppnått 100 års ålder, ställer det på sin födelsedag till en mycket besynnerlig fest, genom att tända eld på sig själv, ...

Vad dessa fåglar göra under mellantiderna, har alltjämt varit höljt i dunkel. Till och med rockemottagaren på Hôtel Phoenix i Stockholm vet int därom. Frid över hans tioöring i alla fall.

Att sagde rockavdragare skulle hava goda kunskaper om

Fågeln Rock

skulle man gärna antaga. Men så är icke fallet. Sagda fågel levde dock säkert förr i världen, ... Darwin påstår helt rått, att det var en ko med en rock på, men detta torde vara ett av de många obevisade hugskott, som på sin tid så ofta förmörkade den eljest ganska klarsynte vetenskapsgubbens själ.

Gripen står i heraldiskt hänseende mycket nära de s.k. *statsdjuren*, om vilka mera ......

Draken ...... levde mest av jungfrur samt av kristet blod. Syresättningen i dess eget blod var så intensiv, att dess andedräkt var eld. ......

Endast en drakhona är med visshet känd, den s.k. *drakan i Babel*, ...

Bibeln har sin *Behemot* att uppvisa, och Jämtland sitt unika *Storsjö*-odjur. ......

Helhästen är, såsom namnet angiver, en hel häst, men dessutom ett spöke, ...

Fen-ris-ulven och *Cerberus* likna varandra däruti, att de båda äro utdöda, men äro varandra olika genom antalet huvuden. ......

Sfinxen levde helst i Egypten, vanligen i sällskap med ......

Om vi se närmare på *Fågel Blå*, finna vi densamma ofta antaga formen av en bok, ...

Stadsodjuren.

Av dessa finnes ett i varje respektabel stat; det underhålles på allmänn bekostnad och dyrkas ......

Schweiziska huvudstadens Berns statsodjur är en björn, och även dess läte fruktansvärt, såsom du nog ofta själv erfarit, och ännu oftare skall erfara, så vitt jag misstänker dig rätt. Låt denna delikata vink förbättra ditt sinne, så kan du med ljuvare samvetsfrid höra, vad jag i det följande skall sia för dig och de dina om de ännu levande

Djuren.

I. Djur.

Om det är svårt, ja, rent av omöjligt uppställa någon indelning av de resp. odjuren emellan, ställer sig detta dock jämförelsevis lätt när det gäller de för oss alla mer än välkända djuren. Härvidlag måste jag dock bestämt protestera och avvika mot och från gamla, slentrianmässiga uppdelningen i däggdjur, fåglar, fiskar, krypdjur, kräk, o.s.v. - av skäl, som jag strax skall blotta.

Förhållandet är nämligen, att det är på hög tid att även inom zoologien införa ett s.k.,

Naturligt system

såsom ju redan fallet är i den vetenskapliga växtvärlden eller botanismen. Då du alltså ser på en krokodil med högra öga och på en luktviol med ditt vänstra öga, observerar du genast, att växten sitter tyst och stilla, medan djuret kvittrar, kväker, morrar m.m. - med andra ord: *rör sig*.

Denna huvudskillnad mellan djuren och växter tvingar oss uppställa *den* såsom huvudprincip vid de förras indelning i klasser och stånd. Då emellertid all rörelse i allmännhet försigår medelst fötterna, uppdela vi djuren i två bjärt från varandra skilda grupper:

Djur med fötter och *Djur utan fötter.*

Sedan jag sålunda fotat mitt system på fötterna, vill jag endast i förbigående nämna, att en man med pedantiska böjelser lätt ur den allmänna principen kan uppställa underklasser, såsom djur med en fot (vakant), d:o med två, d:o med tre fötter, med trettiotre fötter etc.

A. Djur med fötter.

Det högsta och ädlaste djuret i zoologisk mening är naturligtvis *tusenfotingen*, alldenstund han uteslutande består av fötter, såsom framgår av den fotogravyr jag i ett obevakat ögonblick tog av honom. Dessa fötter äro jämt 1000, vilket glädjande förmått mången entusiastisk djurolog att glädjestrålande skrika: "Det var tusan!"

Över denna tusenfotings framstående ställning i systemet känna sig dock de andra djuren icke avundsjuka, härvid följande den gyllene och egyptiska vishetsregeln:

O b s a l v e r a mycket noga:
med din lott, du får, dig foga!

Av dylika vishetsregler och vetenskapsdefinitioner *vimlar* i själva verket den egyptiska vitterheten; och jag skall i det följande draga fram några av dem ur deras dunkel, i den mån de kunna upp- och belysa vetenskapen. Över en pharaonisk munkskänks sakrofag i Mempispyramiden står tex. inristat:

A n j o f i s k e n i sin ask
passar präktigt efter gask.

(Förstår du en egyptisk pik, träffade läsare? Rodna gärna! Bättra dig och bliv såsom *bältan:*

B ä l t a n utan skryt och skrävlan
låter världen ha sin ävlan.)

Ehuru tusenfotingen står högst hos vetenskapsmännen, är dock *lejonet* sedan gammalt djurens konung, men åtnjuter intet regelbundet apanage. Det är lika gult som grymt, lever helst i Afrika och av niggrar. Dessa senare bära kring fotknölarna amuletter av koppar, vilka lejonet aldrig förtär, varuti man vill spåra en gärd åt religionen.

A propos lejon ......

B. Djuren utan fötter

vilka äro de förra underlägsna i de flesta hänseenden. Så sakna de t.ex. fötter och tydlig själ. Somliga av dem kunna nöja sig med att helt simpelt kräla, andra åter måste för sin existens på ett eller annat sätt hava lärt sig simma. Därför är t.ex. den fisk, som icke är simkunnig, fullkomligt värnlös, om han faller i vattnet.

Åter andra varken simma eller kräla, utan flyta menlöst omkring på Oceanen - s.k. urdjur - eller sitta still på ett eller annat värdelöst föremål. T.ex. *korallen*, ...

10 Lecture 10 15/10 - 02: Sections I.6-7 and II.1-3 in Protter

10.1 Stieltjes Integration: Section I.6 in Protter

Definition of increasing process and finite variation process (FV). Notice that cádlág is baked into the definition.

Remark. In my view, unlike Protter, it is best to also bake "adapted" into the definition of increasing (FV) processes, since this is what we meet later, and the additional generality is not needed.

Definition of total variation process $\{|\cdot|_t\}_{t\geq 0}$.

EXERCISE 111 Show that a process has finite variation iff. it is the difference between two increasing processes.

Definition 10.1 A stochastic process $\{X(t)\}_{t\geq 0} = \{X(\omega;t)\}_{t\geq 0}$ is measurable if

$$X : \Omega\times[0,\infty)\rightarrow\mathbb{R} \qquad \text{is} \quad \mathcal{F}\times\mathcal{B}([0,\infty))\text{-measurable.}$$

This is what Protter calls "jointly measurable".

EXERCISE 112 Show that a process is measurable if right- or left-continuous.

Notation $F\cdot A$.

EXERCISE 113 Calculate $N\cdot N$ for a Pp N.

Theorem 48, with proof. The only non-immediate thing in the proof is the technicalities of the measurability issue. These may well be skipped.

Corollary to Theorem 48, with proof. I prefer $d|A|_t$ to Protter's notation $|dA_t|$ (which is inconsistent, at best).

Theorem 49.

EXERCISE 114 Outline a proof of Theorem 49.

Theorem 50, with proof.

Corollary to Theorem 50.

EXERCISE 115 Prove Corollary to Theorem 50.

10.2 Naive Stochastic Integration: Section I.7 in Protter

Naive Stochastic Integration is Impossible!

10.3 Introduction to Semimartingales: Section II.1 in Protter

Definition of simple predictable process. Notice their left-continuity.

Notation $\mathbb{S}$ and $\mathbb{S}_u$.

Notation $\mathbb{L}^0$. Exercise 76 characterizes this topology.

Notation $I_X(H)$. Please, look at this carefully!

Definition of total semimartingale.

Definition of semimartingale.

10.4 Properties of Semimartingales: Section II.2 in Protter

Theorem 1, with proof. This result is immediate (in the meaning "immediate").

Theorem 2, with proof. This result is immediate.

EXERCISE 116 Give the full details of the continuity argument in proof of Theorem 2 (i.e., that convergence in **Q**-probability follows from that in **P**-probability).

Theorem 3, with proof. This result is immediate.

Theorem 4, with proof. This result is immediate.

Theorem 5. This result is simple, but slightly technical to prove anyway.

Corollary to Theorem 5, with proof.

Theorem 6, with proof.

Corollary to Theorem 6. This is immediate from Theorem 6.

10.5 Examples of Semimartingales: Section II.3 in Protter

Theorem 7, with proof. In the theorem, "finite total variation" means finite variation over $[0,\infty)$. In the proof, $\|\cdot\|_u$ is the sup- (uniform) norm over $\Omega\times[0,\infty)$.

Theorem 10.2 (OPTIONAL SAMPLING) Let X be a submartingale and S and T stopping times. If T is bounded, or if X is `ui`, then

$$\mathbf{E}\{X(T)|\mathfrak{F}_S\} \geq X(S\wedge T).$$

Theorem 8, with proof. In the theorem, "square-integrable" means $\sup_{t\geq 0} \mathbf{E}\{X(t)^2\} < \infty$.
Proof: Pick a $t \geq 0$. Since H_i is $\mathfrak{F}_{T_i}$-adapted, the Optional Sampling Theorem 10.1 (which applies by Theorem I.13, since X is `ui` by Exercise 97), give

$$\mathbf{E}\{(X(T_{i+1}\wedge t)-X(T_i\wedge t))^2|\mathfrak{F}_{T_i}\} = \mathbf{E}\{X(T_{i+1}\wedge t)^2-X(T_i\wedge t)^2|\mathfrak{F}_{T_i}\}-2\mathbf{E}\{X(T_{i+1}\wedge t)-X(T_i\wedge t)|\mathfrak{F}_{T_i}\}X(T_i\wedge t),$$

with the second term on the right zero. Hence it follows that

$$\mathbf{E}\{H_i^2(X(T_{i+1}\wedge t)-X(T_i\wedge t))^2\} \leq \|H_i\|_u^2 \mathbf{E}\{X(T_{i+1}\wedge t)^2-X(T_i\wedge t)^2\}.$$

In a similar fashion, we see that

$$\begin{aligned}
&\mathbf{E}\{H_iH_j(X(T_{i+1}\wedge t)-X(T_i\wedge t))(X(T_{j+1}\wedge t)-X(T_j\wedge t))\} \\
&\qquad = \mathbf{E}\{\mathbf{E}\{X(T_{j+1}\wedge t)-X(T_j\wedge t)|\mathfrak{F}_{T_j}\}\, H_jH_i(X(T_{i+1}\wedge t)-X(T_i\wedge t))\} \\
&\qquad = \mathbf{E}\{[X(T_j\wedge t)-X(T_j\wedge t)]\, H_jH_i[X(T_{i+1}\wedge t)-X(T_i\wedge t)]\} = 0 \qquad \text{for} \quad j\geq i.
\end{aligned}$$

Furthermore, X^2 is a submartingale (by Corollary 1 to Theorem I.19), so that

$$\mathbf{E}\{X(T\wedge t)^2\} \leq \mathbf{E}\{\mathbf{E}\{X(t)^2|\mathfrak{F}_T\}\} = \mathbf{E}\{X(t)^2\}$$

for any stopping time T, by the Optional Sampling Theorem 10.1. We may assume that $X(0) = 0$, since X is a semimartingale iff. $X-X(0)$ is. Putting things together, we get $\mathbb{L}^2$-continuity

$$\begin{aligned}
\mathbf{E}\{I_{X(t\wedge\cdot)}(H)^2\} &= \mathbf{E}\{[\textstyle\sum_{i=0}^{n-1} H_i(X(T_{i+1}\wedge t)-X(T_i\wedge t))]^2\} \\
&= \mathbf{E}\{\textstyle\sum_{i=0}^{n-1} H_i^2(X(T_{i+1}\wedge t)-X(T_i\wedge t))^2\} \\
&\leq \|H\|_u^2 \textstyle\sum_{i=0}^{n-1} \mathbf{E}\{X(T_{i+1}\wedge t)^2-X(T_i\wedge t)^2\} \\
&\leq \|H\|_u^2\mathbf{E}\{X(T_n\wedge t)^2\} \leq \|H\|_u^2\mathbf{E}\{X(t)^2\} \to 0 \qquad \text{as} \quad H \to_{\mathbb{S}_u} 0. \quad \square
\end{aligned}$$

Corollary 1 to Theorem 8, with proof. The proof also uses Theorem I.44.a.

Corollary 2 to Theorem 8. Protter's proof is not good in my opinion.

EXERCISE 117 Give a better proof of Corollary 2 to Theorem 8.

Definition of decomposable process.

Theorem 9, with proof.

Corollary to Theorem 9.

EXERCISE 118 Prove Corollary to Theorem 9.

10.6 Vi tror i alla fall på "the General Theory" ...

11 Lecture 11 18/10 - 02: Sections II.4-6 in Protter

11.1 Stochastic Integrals: Section II.4 in Protter

Definition of $\underline{\mathbb{D}}$, $\underline{\mathbb{L}}$ and $\underline{\mathbf{b}\mathbb{L}}$.

Definition of convergence uniformly on compacts in probability ucp.

Notation $\underline{X_t^\star}$.

Notation $\underline{\mathbb{D}_{\mathtt{ucp}}}$, $\underline{\mathbb{L}_{\mathtt{ucp}}}$ and $\underline{\mathbf{b}\mathbb{L}_{\mathtt{ucp}}}$. Notice that $\mathbb{D}_{\mathtt{ucp}}$ is complete metrizable.

EXERCISE 119 Show that ucp-limits of cádlág processes are cádlág.

EXERCISE 120 Show that $\mathbb{D}_{\mathtt{ucp}}$ is complete. Is it an Orlicz space?

Theorem 10.

Definition of $\underline{J_X : \mathbb{S}_{\mathtt{ucp}} \to \mathbb{D}_{\mathtt{ucp}}}$.

Notation $\underline{J_X(H)}$, $\underline{\int H\,dX}$ and $\underline{H\cdot X}$.

Theorem 11, with proof.

Definition of $\underline{J_X : \mathbb{L}_{\mathtt{ucp}} \to \mathbb{D}_{\mathtt{ucp}}}$. The continuity argument for extension used here is standard.

EXERCISE 121 Show in detail how continuity follows from that at 0 for I_X and J_X.

11.2 Properties of Stochastic Integrals: Section II.5 in Protter

Theorem 12. This is simple for H simple predictable.

EXERCISE 122 Prove Theorem 12 in general, from the simple predictable case.

Theorem 13. This is simple for H simple predictable.

EXERCISE 123 Prove Theorem 13 in general, from the simple predictable case.

Protter's **Notation** $\underline{H_P \cdot X}$ [which I find a bit unusual (isn't $(H\cdot X)_P$ better?)].

Theorem 14. This is trivial for H simple predictable.

EXERCISE 124 Prove Theorem 14 in general, from the simple predictable case.

Corollary to Theorem 15.

EXERCISE 125 Prove Corollary to Theorem 15 [e.g., using the probability $(Q+P)/2$].

Theorem 16. This is trivial for H $\mathfrak{F}$- and $\mathfrak{G}$-simple predictable.

EXERCISE 126 Prove Theorem 16 in general, from the simple predictable case.

Theorem 17, with proof. This is trivial for H simple predictable.

Theorem 18, with proof. This follows from Theorem 14, and from Theorems 14 and 17, respectively.

Corollary to Theorem 18, with proof. (One should not try to think too deep here, but instead buy the proof as cheaply as possible.)

Theorem 19. *Proof.* It is a useful exercise to show associativity $I_{H\cdot X(t\wedge\cdot)}(G) = J_X(GH)(t)$ for $G\in\mathbb{S}$ and $H\in\mathbb{L}$. Picking $t>0$, $H\in\mathbb{L}$, and $G_n\to_{\mathbb{S}_u}0$, we have $G_nH\to_{\mathbb{L}_{\mathbf{ucp}}}0$, so that

$$I_{H\cdot X(t\wedge\cdot)}(G_n) =_{\text{associativity}} I_{X(t\wedge\cdot)}(G_nH) = J_X(G_nH)(t) \to_{\mathbf{P}} 0 \qquad \text{since} \qquad J_X(G_nH) \to_{\mathbb{D}_{\mathbf{ucp}}} 0. \quad \square$$

EXERCISE 127 Prove the above claimed associativity for $G\in\mathbb{S}$ and $H\in\mathbb{L}$.

EXERCISE 128 Prove associativity in general, from the case $G\in\mathbb{S}$ and $H\in\mathbb{L}$.

EXERCISE 129 Why is the difficult middle part of Protter's proof needed? (Beets me!).

Theorem 20.

EXERCISE 130 Elaborate on the details in the proof of Theorem 20. (The proof of Theorem 8 may be useful as guidance.)

Definition of random partition $\sigma = \{T_0,\ldots,T_k\}$.

Definition of sequence of random partitions tending to the identity.

Notation Y^σ for $Y\in\mathbb{D}\cup\mathbb{L}$ and σ a random partition..

Theorem 21. This should be regarded as immediate from the continuity of J_X.

11.3 Det är viktigt att börja tidigt med "the General Theory"!

Därför och på allmänn begäran förevisar jag nu för första gången någonsin tre privata fotografiska bilder av mig själv, och det i tre olika stadier av teoretisk genialitet:

Jag vill se bevis av precis allting, varenda litet δpsilon, jag menar ϵpsilon. Det är det som är det svåra och viktiga, och som håller i längden. Annars blir allt ett luftslott utan någon ordentlig grund!

Det bästa är att bevisa utvalda resultat, och låta bli bevisa andra. Då kommer man någon vart, allt känns roligare och mera menigsfullt. Chansen ökar att man får verklig nytta av det man lärt!

Bevis är inte alls särskilt viktiga (tänker du försöka hitta fel i dem, eller vad?), *utan det är själva tolkningen och ev. tillämpbarhet som kräver eftertanke. Det är ju faktiskt en verklig värld vi lever i!*

12 Lecture 12 22/10 - 02: Sections II.6-8 in Protter

12.1 Quadratic Variation of Semimartingales: Section II.6 in Protter

Definition of quadratic variation process.

Definition of quadratic covariation (bracket) process.

Exercise 131 Prove the polarization identity in Protter and $[X,Y] = \frac{1}{4}([X+Y,X+Y]-[X-Y,X-Y])$.

Theorem 22.

Corollary 1 to Theorem 22, with proof.

Corollary 2 to Theorem 22, with proof.

Theorem 23.

Corollary.

Exercise 132 Prove Corollary to Theorem 25.

Definition of $[X,X]^c$.

Definition of quadratic pure jump process.

Theorem 26, with proof.

Exercise 133 The idea in the proof of Theorem 26 is to show that $[X,X] = \Delta X \cdot X$. Why does this give the theorem?

Theorem 27.

Corollary 1 to Theorem 27, with proof.

Corollary 2 to Theorem 27, with proof.

Corollary 3 to Theorem 27.

Corollary 4 to Theorem 27.

Exercise 134 For a local martingale M with $\mathbf{E}\{[M,M](\infty)\} < \infty$, show that $\sup_{t\geq 0} \mathbf{E}\{M(t)^2\} = \mathbf{E}\{[M,M](\infty)\}$.

Corollary 5 to Theorem 27.

Theorem 28, with proof.

EXERCISE 135 Why does $[X,X]^c=0 \Rightarrow [X,Y]^c=0$ in the proof of Theorem 28?

Theorem 29.

Theorem 30, with proof. (It is really something to be able to prove this so easy!)

12.2 Itô's Formula: Section II.7 in Protter

Theorem 32.

Theorem 31 (regarded as a Corollary to Theorem 32), with proof.

EXERCISE 136 Explain how Theorem 31 follows from Theorem 32 for $f \in \mathbb{C}^2$.

Corollary to Theorem 32, with proof. This is "the usual" Itô's Formula.

EXERCISE 137 Show that for `Bm` $(B \cdot B)(t) = \frac{1}{2}B^2 - \frac{1}{2}t$.

EXERCISE 138 Prove Corollary to Theorem 32.

Theorem 33.

Definition of <u>(Fisk-) Stratonovich integral $\int Y \circ dX$</u>.

Theorem 34.

12.3 Application of Itô's Formula: Section II.8 in Protter

Theorem 36. Obviously, this is extremly important!

Definition of <u>stochastic (Doléans-Dade) exponential</u>.

Special cases of continuous semimartingale, and of `Bm` (called geometric `Bm`).

Theorem 37, with proof.

Corollary to Theorem 37, with proof.

EXERCISE 139 In the proof of Corollary to Theorem 37, why is $[X,X] + [X,-X] + [X,[X,X]] = 0$?

Theorems 38-39 and 41. These are fundamental in `Bm` stochastic calculus, and proofs belong there, albeit simple. Theorem 41 is the famous DAMBIS-DUBINS-SCHWARZ THEOREM.

12.4 Är Göteborgsvitsen död? Döm själv ...

mha. det utvalt roligaste (vitsigaste?)*dödsroligaste?* Sättarens anm. från GP en fredag och konsekutiv lördag. OBS: Inslaget skall ej uppfattas som uttryckande åsikt, smak, eller saknad av endera eller bådadera hos Lecture Notes utsände.*Hos GP's skribenter då?* Korrekturläsarens anm.

DAGENS ROS

Tack till dig busschaufför som körde linje 58 onsdagen den 16 oktober kl 7.30 mot Marklandsgatan. Du gav oss passagerare små uppmuntrande ord på vägen och önskade oss en trevlig dag fast vädret var grått och trist. Hela den dagen kändes mycket lättare med en gång.

En gladare passagerare

VÅGAT. Lachapelle-Photographs visas på The Barbican Art Gallery 10 oktober till 23 december.

Döda kor värmer bostäder

STOCKHOLM: Från början av november kommer Karlskogabornas bostäder att värmas upp av självdöda kor och grisar. Kraftvärmeverket i Björkborn blir först i landet att använda den nya kovärmen i kommersiell skala.

Älgko åt ihjäl sig på äpplen

ALINGSÅS: Jaktlaget var övertygat om att en tjuvskytt hade varit framme när en död älgko hittades i en villaträdgård i Lo mellan Alingsås och Sollebrunn. Polis tillkallades för att säkra spår. Ut ryckte Örjan Eliasson. Han arbetar normalt som inre befäl. Men eftersom han även är jägare ansågs han vara mest lämpad för uppdraget. Och tur var väl det. Det vill säga att han var insatt i hur det här med jakt fungerar:

– Jag tvättade av älgen och konstaterade att det inte fanns några kulhål. Älgen hade helt enkelt ätit ihjäl sig.

Kon hade tillsammans med sin årskalv varit synlig kring villan redan dagen före det att hon hittades död. Förmodligen lockad av dignande äppelträd och all fallfrukt i trädgården.När sedan villaägaren väl släckt ljuset och slumrat in började hon av allt att döma kalasa på frukten. Kanske blev hon berusad av de jästa äpplena och hade svårt att sluta festandet. Bara ett litet äpple till och så vidare. Till sist sa kroppen stopp.

Vikten kontrolleras på finska löntagare

HELSINGFORS: Parterna på den finländska arbetsmarknaden är överens om ett tungt mål, bokstavligt talat. Arbetsgivarna och löntagarna ska genomföra en miljonkampanj som har som mål att 200 000 finländare ska banta en miljon kilogram fram till våren. Det blir fem kilo per person.

Viktkontroll ska bli en naturlig del av den finländska arbetsmiljön är det tänkt, mot bakgrund av att övervikt och fetma har blivit betydligt vanligare i Finland de två senaste årtiondena och att övervikt är kopplat till många folksjukdomar. (TT-FNB)

Dela klasserna på gympan

Vi är tre tjejer som tycker att man ska dela upp klassen i kill- och tjejgrupper i skolgympan. Då blir man inte nedskjuten i till exempel fotboll. Vissa kanske vågar mera, till exempel att våga försöka lägga mål.

MOTION

Stavgång med Korpen. Sami Lansmansgårdens vårdcentral, kl 10.30.

Före vitlöken

70-talet, ett gastronomiskt svart hål

Mivitotal (222 kronor) passar alla, står det i en broschyr: "gammal eller ung, elitidrottare eller 'soffliggare'. Till dig som stressar eller du som slarvar lite med maten". Det innehåller Q10, bioflavonoider, alfaalfa och fibrer.

RESEFRÅGAN

Jag skall åka mellan Los Angeles och San Francisco, men vill inte hyra bil. Går det tåg eller buss den sträckan?

Louise

Lång önskelista på järnvägsprojekt

"Många bra projekt kommer att behöva stå tillbaka för att de allra viktigaste ska kunna förverkligas", säger Bo Bylund, Banverkets generaldirektör enligt ett pressmeddelande.

FÖRRESTEN såg vi att Madonna oroade sig för att maken Guy Ritchie var lite homosexuell efter att han betett sig underligt tillsammans med andra män i sitt filmteam. "Just det. Det gäller att vara uppmärksam. Själv har jag en kompis som misstänkte att hans fru var lite gravid", avslöjar Förrestfabs securitychef.

Vad har Ikea emot de hemlösa?

– Äter du förresten fiskolja? Det gör cellerna mjuka och vi behöver ju det fettet för vi äter alldeles för litet fisk. Särskilt de som bantar måste vara uppmärksamma på detta.

"Svenskar är ju generellt dumma"

JAG SITTER i Danmark och beklagar mig inför en kollega. Jag är irriterad på ett svenskt företag som jag tycker har tagit ett felaktigt beslut.

Min danske kollega lutar sig fram över bordet, tittar på mig och säger lugnt.

– Ja, det var dumt gjort. Men svenskar är ju generellt dumma.

Min danske kollega brukar skryta med att det går åt minst tolv armenier för att lura en dansk. Men minsta barnunge kan antagligen kolla bort en svensk.

Inget besked om torsken

Till fjälls som singel

Singel i Salto. Det är ett av Svenska turistföreningens ungefär åttio resepaket i nya vinterkatalogen. Resan går till Saltoluokta fjällstation och riktar sig till alla som är nyfikna på fjällen och dessutom singlar.

En riktigt svinig historia

GOLF: Österåkers golfklubb har fått oväntat besök i form av en flock vildsvin. Och de har knappast gått spårlöst förbi, snarare uppträtt riktigt svinaktigt. Man lägger ner hela sin själ på golfbanan och så förstörs den så här lätt, säger Åke Cajstedt, vd vid Österåkers golfklubb till internetsidan golf.se.

Efter att vildsvinen gått till attack mot det sjätte golfhålet, valde klubben att sätta upp elstängsel, men då valde vildsvinen i stället golfbanans andra hål.

Blair vill att IRA upplöses

Iran långt från världseliten

GOLF: Iran gör debut i amatör-VM i golf för damer, som spelas just nu i Kuala Lumpur i Malaysia. Och visar sig ha en hel del att lära.

Efter två dagars spel ligger den slöjbeklädda iranska trion 39:a och sist på 173 slag över par, att jämföras med Thailand som leder på åtta under par.

Zohreh Kasrai har det riktigt tungt. Hon har gått 61 respektive 65 slag över banans par (totalt 134 respektive 138 slag per rond). Mina Varzi är lagets bästa. Hennes bästa rond var 110 slag, 37 slag över par.

Regeringen vill stoppa barnäktenskap

Dödskamp för hopkrokade renar

BODEN: Med kronorna ihoptrasslade satt de båda rentjurarna ihop i två veckor innan båda var döda. Älgjägaren Stefan Tuvebäck fick ta del av ett makabert skådespel under lördagens älgjakt utanför Boden, skriver Norrländska Socialdemokraten.

Två renar med jättelika kronor hade gjort upp i en kamp som slutade med döden för dem båda. Den ena rentjuren dog i armarna på Stefan efter att i flera veckor ha släpat runt på kadavret efter sin rival.

Institution läggs ner efter bråk

Bråken på Statistiska institutionen vid Göteborgs universitet väntas nu leda till att institutionen läggs ner.

– Kvalitetsmässigt för utbildningen ser jag inga problem. Sedan är det sorgligt att det ska behöva gå så här långt, säger Carl Fredrik Nilsson i studentkåren.

13 **Lecture 13 24/10 - 02:** Sections III.1-3 in Protter

This chapter develops technology needed to extend the stochastic integral $J_X : \mathbb{L} \to \mathbb{D}$ from $\mathbb{L}$ to the larger class of predictable processes (see Section III.8).

13.1 Introduction: Section III.1 in Protter

(Recall) **Definition** of decomposable process.

Definition of classical semimartingale.

Now follows a list of the main results of Chapter III:

Theorem 1. This is the main result!

Definition of (locally) natural process.

The natural processes are closely related to predictable processes.

EXERCISE 140 Explain all details of the argument that $A(T \wedge \cdot\,)$ is natural for A natural and T a stopping time.

Theorem 2, with proof.

EXERCISE 141 Why have continuous FV processes locally integrable variation?

Theorem 3.

13.2 The Doob-Meyer Decompositions: Section III.2 in Protter

Definition of potential.

EXERCISE 142 Why exist $\lim_{t\to\infty} \mathbf{E}\{X(t)\}$ for non-negative supermartingales?

Theorem 4, with proof. This DOOB DECOMPOSITION is the discrete analogue of the continuous time DOOB-MEYER DECOMPOSITION, that in turn is crucial for us.

EXERCISE 143 Give all details of the existence part of proof of Theorem 4.

Theorem 5, with proof. **Notice:** Protter has forgotten to say that Z is `UI` in the theorem (which is needed to prove Corollary to Theorem 6 below).

EXERCISE 144 Explain why Z is non-negative in the proof of Theorem 5.

Theorem 6.

Corollary to Theorem 6, with proof.

Theorem 7.

13.3 Quasimartingales: Section III.3 in Protter

Definition of partition of $[0, \infty]$.

Definition of variation of X along a partition.

Definition of variation of X.

Definition of quasimartingale.

EXERCISE 145 Which quasimartingales are martingales?

Theorem 8.

EXERCISE 146 In one direction Theorem 8 is easy: Prove that part.

Theorem 9, with proof.

EXERCISE 147 Explain how uniqueness in Theorem 9 follows from Lemma on page 92 in Protter's book (or in some other way).

EXERCISE 148 Why are locally integrable variation processes local quasimartingales?

Definition of compensator.

Fact. Compensators of increasing processes with (locally) integrable variation are increasing with (locally) integrable variation. With proof!

EXERCISE 149 Why are decreasing processes with adequate integrability properties (local) supermartingales?

13.4 Theoretical Background for Chapters III-V in Protter's Book

Following the introductory Chapters I-II, the main treatment of stochastic integrals and differential equations is in Chapters III-V. To take in that material properly, a bit more is needed in terms of theoretical background, than for the introduction. Most participants have much of that background already. However, there will certainly be topics to "fresh up" for everyone, so that not too much time is spent later on things of background character. For convenience, I have assembled a list of what one ideally should know, to do well with Chapters III-V. Please notice that the listed material is fundamental also in many other areas of applied probability, so that it is always well spent time to work with it, if not well-known previously.

Section III.2. The Doob-Meyer Decompositions. K.M. Rao's proof is much more simple and intiutive than previous ones, and uses basic potential theory. See e.g.,

Doob, J.L. (1983). *Classical Potential Theory and Its Probabilistic Counterpart.* Springer. *Part 1.*

Chapter IV. General Stochastic Integration. Here the Musielak-Orlicz space machinery outlined in Lectures 6-7 comes into play. See e.g.,

Kwapień, S. and Woyczyński, W. (1992). *Random Series and Stochastic Integrals.* Birkhäuser. *Chapters 0 and 7-9.*

Musielak, J. (1983). *Orlicz Spaces and Modular Spaces.* Springer. *Chapters I-II.*

Rao, M.M. and Ren, Z.D. (1991). *Theory of Orlicz Spaces.* Dekker.

Chapter V. SDE. In general, this chapter requires distribution theory together with elliptic PDE, and variational calculus, deterministic and stochastic (i.e., Malliavin calculus). Standard texts are e.g.,

Federer, H. (1969). *Geometric Measure Theory.* Springer. *(This one takes time, but is worth it!)*

Hörmander, L. (1983-4). *The Analysis of Linear Partial Differential Operators.* Springer. *Part I and Part III Chapters XVII-XIX.*

Nualart, D. (1995). *The Malliavin calculus and Related Topics.* Springer.

Section V.2. H^p-norms for Semimartingales. This requires Hardy spaces. See e.g.,

Douglas, R.G. (1972). *Banach Algebra Techniques in Operator Theory.* Academic. *Chapter 6.*

Section V.6. Markov Nature of Solutions. This requires Markov Processes. (Riktiga såna!) See e.g.,

Blumenthal, R.M. and Getoor, R.K. (1968). *Markov Processes and Potential Theory.* Academic. *Chapters 0-2.*

Sections V.7-8. Flows of SDE and flows as Diffeomorhisms. This requires basic knowledge of infinite dimensional differential geometry (Hilbert space generality), and of stochastic flows. See e.g.,

Kunita, H. (1990). *Stochastic flows and stochastic differential equations.* Cambridge. *Chapters 4-5.*

Okubo, T. (1987). *Differential geometry.* Dekker.

Chapters III-V. Rigorous treatment of stochastic differentials requires stochastic nonstandard analysis. Elementary aspects of complex analysis in several variables is used on multidimensional complex martingales, as is functors (Banach operator ideals), and probability in Banach spaces. See e.g.,

Hörmander, L. (1988). *Complex Analysis in Several Variables.* North-Holland. *Chapter 2.*

Ledoux, M. and Talagrand, M. (1980). *Probability in Banach Spaces.* Springer.

Michnor, P.W. (1978). *Functors and Categories of Banach Spaces.* Springer.

Stoyan, K.D. and Bayod, J.M. (1986). *Infinitesimal Stochastic Analysis.* North-Holland.

14 Lecture 14 6/11 - 02 8 am: Sections III.4-5 in Protter

14.1 The Fundamental Theorem: Section III.4 in Protter

Definition of predictable stopping time.

EXERCISE 150 Show that non-random times are predictable.

Definition of announcing a predictable stopping time.

EXERCISE 151 Explain why $T_n = \inf\{t > 0 : |X(t)| \geq c - \frac{1}{n}\}$ (n large enough) announces $T = \inf\{t > 0 : |X(t)| \geq c\}$ for $c > 0$ a constant and X an adapted continuous process with $X(0) = 0$.

Definition of accessible stopping time.

EXERCISE 152 Explain why the time to the first jump for a Poisson process is not accessible.

Definition of an envelop of an accessible stopping time.

EXERCISE 153 Find an envelop of a stopping time whose set of possible values is countable.

Definition of totally inaccessible stopping time.

EXERCISE 154 Is the time to the first jump for a Poisson process totally inaccessible?

Notation T_Λ for $\Lambda \in \mathfrak{F}_T$ and T stopping time.

EXERCISE 155 Prove that T_Λ is a stopping time.

Theorem 10, with proof.

Theorem 12.

EXERCISE 156 Motivate Theorem 12.

Theorem 13.

Corollary to Theorem 13.

14.2 Classical Semimartingales: Section III.5 in Protter

Theorem 14, with proof.

EXERCISE 157 Show that classical semimartingales are decomposable.

EXERCISE 158 Show that, for a classical semimartingale X, we really have $X = M + A$, as in the proof of Theorem 14.

Corollary to Theorem 14, with proof.

Theorem 15, with proof.

Theorem 16, with proof.

EXERCISE 159 In the proof of Theorem 16, why is $X(t \wedge \cdot)$ a supermartingale?

EXERCISE 160 Show that a submartingale is a semimartingale.

Theorem 17, with proof (albeit a bit longish).

EXERCISE 161 Explain the details of the inequalities $\mathbf{E}\{\int_0^{t\wedge T} d|A|\} \leq m + \beta + \mathbf{E}\{|(\Delta M)(T)|\} < \infty$, in the proof of Theorem 17.

EXERCISE 162 In the last computation in Theorem 17, motivate thorougly (not just repeating Protter's words) the exchange of order of limits $\mathbf{E}\{\lim ...\} = \lim \mathbf{E}\{...\}$.

EXERCISE 163 In the last computation in Theorem 17, prove the last equality.

Definition of <u>special semimartingale</u>.

Theorem 18.

EXERCISE 164 Check the proof of Theorem 18.

Definition of <u>canonical decomposition</u>.

Theorem 19, with proof.

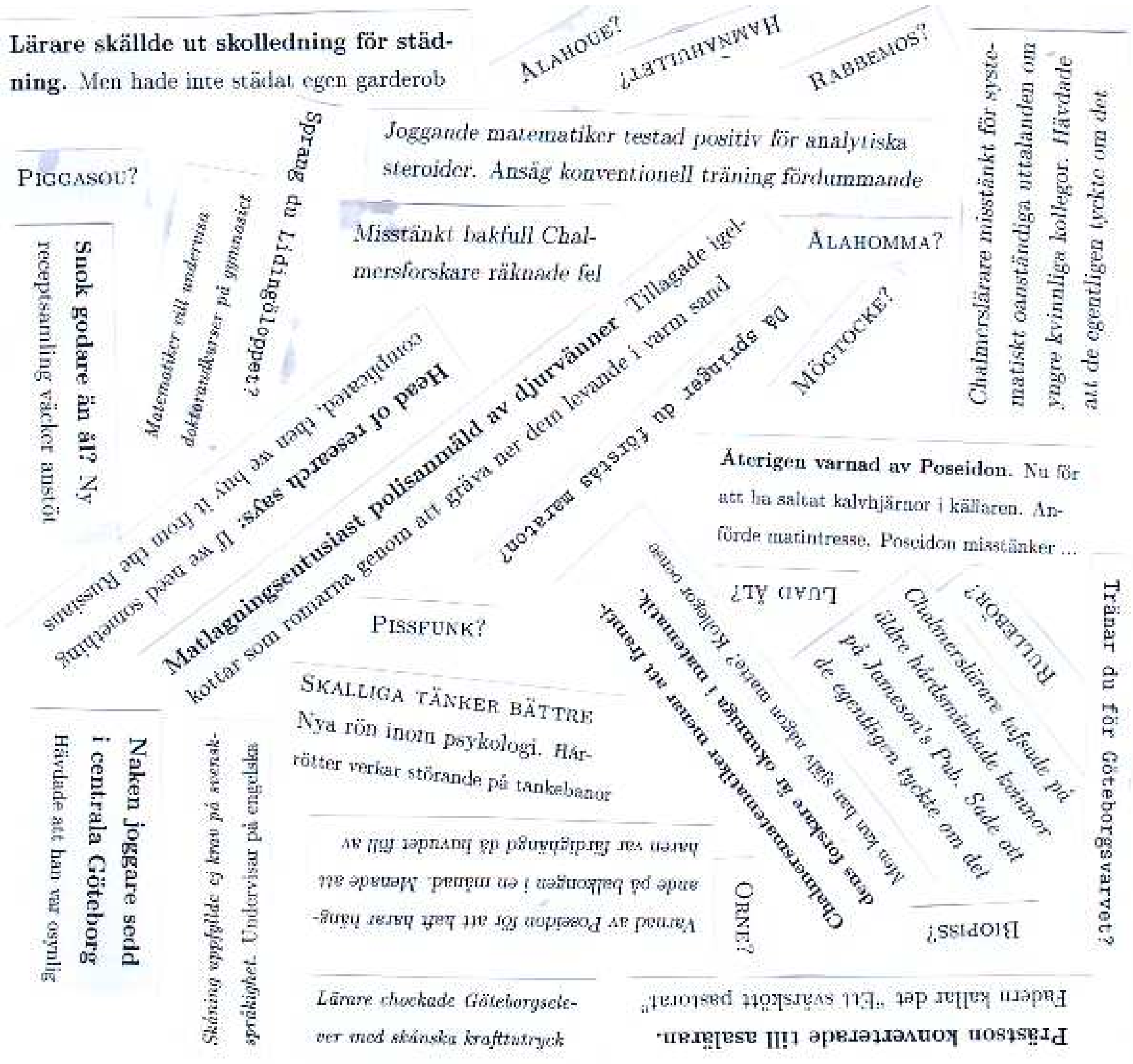
EXERCISE 165 Explain how Theorem 13 gives $X = X(0) + N + B$, in the proof of Theorem 19.

EXERCISE 166 Explain, using e.g., Exercise 148, why B is a local quasimartingale, in the proof of Theorem 19.

Corollary to Theorem 19. (The proof of the Corollary is harder than that of the theorem!)

EXERCISE 167 Show that bounded jump Lévy processes are special semimartingales (there it came!), and that Lévy processes with $\int_1^\infty x\, d\nu(x) = \infty$ are not.

14.3 Inte ens Föreläsaren är Perfekt ...



15 **Lecture 15 8/11 - 02:** Sections III.6-8 in Protter

15.1 Girsanov's Theorem: Section III.6 in Protter

Definition of <u>equivalent</u> probability measures.

Notation $P \sim Q$.

Lemma before Theorem 20.

EXERCISE 168 prove Lemma.

Theorem 20, with proof.

EXERCISE 169 Verify that $Z^{-1} = \mathbf{E}_{\mathbf{Q}}\{\frac{d\mathbf{P}}{d\mathbf{Q}}|\mathfrak{F}_t\}$ in the proof of Theorem 20.

Theorem 21, with proof. (This is "usual" Girsanov's Theorem.)

15.2 Bichteler-Dellacherie Theorem: Section III.7 in Protter

Theorem 22.

15.3 Natural and Predictable Processes: Section III.8 in Protter

Definition of <u>predictable</u> σ-algebra.

Theorem 23.

EXERCISE 170 Check the details of the proof of Theorem 23.

Corollary 1 to Theorem 23, with proof. **Notice:** The conclusion should be that A is <u>locally</u> natural (i.e., Protter has forgotten "locally").

Corollary 2 to Theorem 23, possibly with proof.

Theorem 26.

Theorem 27.

15.4 Lite Julpyssel.

Först en vinpalör som ger tillräckligt kunnande i standardårgångar för typviner, för att lura nästan varje sommelier [men kräver att man har med den stora portmonän (med ett undantag)]: Para alltså ihop vin med år. (Minnesluckor bör kunna hjälpas upp mha. `www`.)

Annus mirabilis, förutom Douro	1921, 1985
Annus mirabilis i Douro	1931
Cheval Blanc	1945, 1963, 1994
Cru Beaujolais	1947, (1982?)
DRC	1961, (1982?)
Haut Brion	1961, 1990
Hermitage la Chapelle	1967, 1983, 1988-1990
Latour	1971, 1982, 1998
Leoville las Cases, Pichon Comtesse de Lalande	1975
Marquis Alexandre de Lur-Saluce	1976, 1983, 1988-1990
la Mission haut Brion	1978, 1985, 1988-1990
Montrachet	1982
Montrose	1982
Mouton	1982, 1986
MSR	1982, 1985, 1988-1990
Palmer	1983
Petrus	1989
Le Pin	1990
Quinta do Noval Nacional	1990
Romanee Conti	1991
Syndicate Grande Marque	1994
Taylor	1995

Dessutom något om regioner, för att testa din sommelier. (Tappa ej initiativet!)

Baron Lacoste Lynchad av Comtessa efter Malmcitat	Region ...... ?
Gevre Charmade LatRik Kamrer	Region ...... ?
Petrus Evangelium Pinade Gay Pastor	Region ...... ?

Goda lösningar till dessa grundläggande vinövningar räknas som en "Exercise", i syfte att uppnå rimlig salongsfäighet. (Detta kan även doktorander ha nytta utav.) Dessutom, den som lämnar den bästa lösningen vinner en låda `2620 Oltina La Revedere`.

EXERCISE 54 The easy part of Theorem 5.1 is that for `id` $\{Y_n\}_{n=1}^\infty$ and Y with triplets (A_n,ν_n,γ_n) and (A,ν,γ), respectively, $Y_n \to_d Y$ if

(1) $\lim_{n\to\infty}\int_{|y|>\varepsilon} f(y)\,d\nu_n(y) = \int_{|y|>\varepsilon} f(y)\,d\nu(y)$ for bounded $f\in\mathbb{C}(\mathbb{R}^d)$ and $\varepsilon>0$,

(2) $\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty} |\langle\theta,(A_n-A)\theta\rangle + \int_{|y|\le\varepsilon}\langle\theta,y\rangle^2\,d\nu_n(y)| = 0$ for $\theta\in\mathbb{R}^d$, and

(3) $\lim_{n\to\infty}\gamma_n = \gamma$.

Proof. Define measures ρ_n on $\mathbb{R}^d$ by $d\rho_n(y) = 1\wedge\langle\theta,y\rangle^2\,d\nu_n(y)$. By (2), for $\theta\in\mathbb{R}^d$,

$$\int_{|y|\le\varepsilon} d\rho_n(y) \le \int_{|y|\le\varepsilon}\langle\theta,y\rangle^2\,d\nu_n(y) \le \int_{|y|\le\varepsilon}\langle\theta,y\rangle^2\,d\nu_n(y) + \langle\theta,A_n\theta\rangle < K_1$$

for $n\ge\hat{n}$, for some constants $K_1>0$ and $\hat{n}\in\mathbb{N}$. Thus this holds for all $n\in\mathbb{N}$, possibly with a greater value for K_1. Hence Dominated Convergence gives

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}\int_{|y|\le\varepsilon}[e^{i\langle\theta,y\rangle}-1-i\langle\theta,y\rangle+\tfrac{1}{2}\langle\theta,y\rangle^2]\,d\nu_n(y) = 0,$$

since $|e^{i\langle\theta,y\rangle}-1-i\langle\theta,y\rangle+\frac{1}{2}\langle\theta,y\rangle^2| \le K\langle\theta,y\rangle^2$ for a constant $K>0$. Further, (1) gives

$$\lim_{n\to\infty}\int\limits_{|y|>\varepsilon} e^{i\langle\theta,y\rangle}-1-i\langle\theta,y\rangle\,d\nu_n(y) = \int\limits_{|y|>\varepsilon} e^{i\langle\theta,y\rangle}-1-i\langle\theta,y\rangle\,d\nu(y) \to \int\limits_{\mathbb{R}^d} e^{i\langle\theta,y\rangle}-1-i\langle\theta,y\rangle\,d\nu(y)$$

as $\varepsilon\downarrow 0$, using Dominated Convergence and that ν is a Lévy measure at the end. Now we get just by inspection the desired conclusion that, as $n\to\infty$,

$$i\langle\theta,\gamma_n\rangle-\tfrac{1}{2}\langle\theta,A_n\theta\rangle+\int\limits_{\mathbb{R}^d} e^{i\langle\theta,y\rangle}-1-i\langle\theta,y\rangle\,d\nu_n(y) \to i\langle\theta,\gamma\rangle-\tfrac{1}{2}\langle\theta,A\theta\rangle+\int\limits_{\mathbb{R}^d} e^{i\langle\theta,y\rangle}-1-i\langle\theta,y\rangle\,d\nu(y).$$

Albeit correct in idea, there is an error in the proof in Sato's book (in the direction carried out above), and his definition of ρ_n has to be changed to the above one. #

EXERCISE 91 Protter's proof is not good in my opinion, and makes the problem harder than it is. We give a quicker and more transparent argument below. First it should be clerified that, if $\mathbf{P}\{T=\infty\}>0$, then it is only processes X with a well-defined (measurable) value $X(\infty)$ that feature in the theorem.

It suffices check that $\{X(T)\in G\}\cap\{T\le t\}\in\mathfrak{F}_t$ for open $G\subseteq\mathbb{R}$, because then

$$\{X(T)\in G\} = \{X(\infty)\in G\}\cap\{T=\infty\}\cup\bigcup_{n=1}^\infty\{X(T)\in G\}\cap\{T\le n\}\in\mathfrak{F}.$$

But the required membership follows from cádláguity and adaptedness of X, since

$$\mathbf{1}_{\{X(T)\in G\}}\mathbf{1}_{\{T\le t\}} = \lim_{n\to\infty}\sum_{k=0}^{2^n-1}\mathbf{1}_{\{X(\frac{k+1}{2^n}t)\in G\}}\mathbf{1}_{\{\frac{k}{2^n}t\le T<\frac{k+1}{2^n}t\}} + \mathbf{1}_{\{X(t)\in G\}}\mathbf{1}_{\{T=t\}} \quad \#$$