UNIVERSITY OF
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Investigation of portfolio strategies by means of simulation

Master's thesis in Mathematical Statistics

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Abstract

Portfolio insurance strategies are constructed to limit an investors loss but still reward them when the market goes up. In this thesis we compare two portfolio insurance strategies, Constant proportion portfolio insurance (CPPI) and Option based portfolio insurance. This is done by simulating a stock pattern with two different models, Irrational fractional brownian motion and Constant elasticity of variance. We also simulate an interest curve for which we price a Zero coupon bond (ZCB). This is also done by using two different models, Ho-Lee and Black-Derman-Toy. The models are implemented in Matlab for which we then do several simulations and analyse what the result would have been if we invested in these stock and bond simulations according to the CPPI and OBPI portfolios.

We found that the OBPI portfolio is safer when the market goes down and usually the CPPI portfolio performs better in upward markets. But there are exceptions when the OBPI portfolio surprisingly performs better than the CPPI portfolio when we act as an aggressive investor even in upward markets. In general, however, the OBPI portfolio seems to be a better choice when the market conditions are uncertain while the CPPI portfolio might be good for a more risk taking investor or if the market is expected to rise.

Keywords: Constant proportion portfolio insurance, Option based portfolio insurance, Irrational fraction brownian motion, Constant elasticity of variance, Ho-Lee, Black-Derman-Toy

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1
Introduction

In this thesis we will study and compare two different portfolio strategies. Both these strategies allows an investor to allocate money between a risky asset and a risk-free asset, however, one strategy is dynamic while the other is a static strategy. The price path of the risky asset will be generated by using two different models. To obtain a risk-free asset, which will be a zero coupon bond (ZCB), we will first generate an interest rate curve with two different models. Then, given the dynamics used to generate the interest rate we will price a ZCB. Once we have the price path of these assets we will test the two portfolio strategies and see under which market environments, and underlying models, they produce the best and worst results.

1.1 Background

Portfolio insurance strategies are constructed in a way such that they limit an investors loss but still allow them to participate in upside markets. The two strategies we will compare in this thesis are the constant proportion portfolio insurance (CPPI) and the option based portfolio insurance (OBPI). Both portfolio strategies insures, that is guarantees, an investor a certain amount of money at the end of an investment cycle. The natural way to compare these strategies is to evaluate their performance, but also too look deeper into under what market circumstances they perform better and worse. To investigate these strategies we need tools to model a risky asset (usually a stock or a stock index) and a risk-free asset (usually a bond).

In this thesis we will study two models that models a stock index, the irrational fraction brownian motion model (IFBM) and the constant elasticity of variance model (CEV). These models will be set up in matlab and adjusted in order to do simulations based on a given data set which will be a part of the S&P500 index.

Similarly, we will study two models to model an interest rate, the Ho-Lee mdoel and Black-Derman-Toy model (BDT). This models will also be set up in matlab and adjusted to suit a data set which is taken from the 6-month libor rate and also the yield curve.

2

Theory

In this chapter we go trough the necessary theory needed to understand the models used to simulate stock prices and interest rate curves. Furthermore, we study how to price both European options and zero coupon bonds based on the dynamics of the underlying asset. Lastly, the two portfolio insurance strategies and their characteristics are explained.

2.1 Basics

Definition 2.1.1 *The logarithmic return X_t of a stock S_t in a finite time set is defined by*

$$X_t = \log\left(\frac{S_t}{S_{t-1}}\right).$$

The benefits of logarithmic returns will be used several times in this thesis. When we calculate historical returns on a data set and its mean and standard deviation it will be based on logarithmic returns.

Definition 2.1.2 *An European option is a contract that allows the owner of the contract to buy or sell an underlying asset at a pre-determined price at a pre-determined date.*

The pre-determined price is known as the Strike price and the pre-determined date is the maturity. If the owner has the right to buy the underlying asset the option is called a call option. If the owner has the right to sell the underlying asset the option is called a put option. One of the portfolio strategies we will study in this thesis require ownership of a put option written on the risky-asset. Therefore we must be able to determine a fair price of an European put option, given its underlying asset dynamics, throughout our investment period.

Definition 2.1.3 *A zero coupon bond with face value K is a contract that promises to pay K to its owner at maturity $T > 0$.*

The price of a ZCB will mainly depend on the underlying interest rate that it is based upon. We assume in this thesis that all ZCB's are similar and differ only by their face value and maturity. However, we set the face value to $K = 1$ since owning a ZCB with face value K is equivalent to owning K shares of ZCB with face value 1. The maturity of the ZCB's will be the same as the maturity of the European options.

2.2 Irrational fractional brownian motion model

The Irrational fractional brownian motion (IFBM) model was introduced 2016 by Dhesi et al [3]. IFBM is based on the geometric brownian motion but has an added term. The ordinary geometric brownian motion assumes that investors act rational, which in real world is not entirely true. The motivation, therefore, for the extra term in the IFBM model is to capture these irrational investors in the model. By modifying the stochastic differential equation (SDE) of a geometric brownian motion one can obtain the following time dicretization for the IFBM model

$$dS_t = \alpha dt + \sigma Z\sqrt{dt} + \alpha K f(Z) dt \quad \text{where} \quad \alpha = \mu + \frac{1}{2}\sigma^2 \tag{2.1}$$

in which S_t is the stock price, μ is the expected average return, σ is the volatility, $Z \sim N(0,1)$ is a random number from the standard normal distribution and K is a parameter. When $K = 0$ we recover the geometric brownian motion. The irrational term takes the form of the function $f(Z)$. Dhesi et al propose the following realisation of $f(Z)$

$$f(Z) = \arctan(Z)\Big(2e^{-c\frac{Z^2}{2}} - 1\Big) \tag{2.2}$$

where c is a parameter to estimate based on the underlying data and Z is, as mentioned earlier, a random number from the standard normal distribution. With an application of Ito's lemma one can find the following closed form solution to (2.1)

$$S_t = S_0 e^{\mu dt + \sigma Z\sqrt{dt} + \mu K f(Z) dt}. \tag{2.3}$$

A discrete version of (2.3) will be used to simulate stock paths. A thorough explanation on how the IFBM model is used to simulate stock price is provided in Chapter 3.

2.2.1 Option pricing with IFBM

We also need formulas to obtain the price of European options when the underlying asset follows the IFBM model. These values are obtained trough the Black-Scholes (BS) model. In the BS model it is assumed that the underlying asset follows a geometric brownian motion. As the IFBM model is based on the geometric brownian motion with an added term we will use the BS model to price European options. Let C_t and P_t denote the price for a call and put option at time t respectively. The BS model for a European option is then

$$C_t = S_t N(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$P_t = Ke^{-r(T-t)}N(-d_2) - S_t N(-d1)$$

with

$$d_1 = \frac{\ln(\frac{S_t}{K}) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \qquad d_2 = d_1 - \sigma\sqrt{T-t}$$

and

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{-y^2}{2}} dy$$

where K is the strike price, r is the risk-free interest rate and $T-t$ is time to maturity in years. Figure 2.1 and 2.2 shows a three year relationship between a stock and a put option written on the same stock when $K = 0.9$ and $S_1 = 1$.

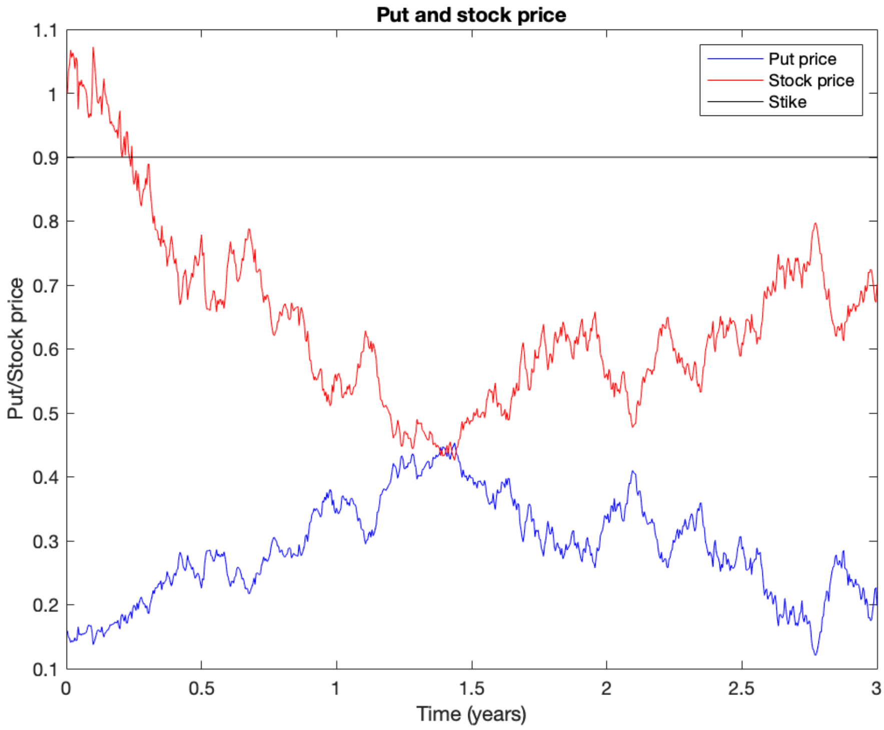


Figure 2.1: Stock and put option paths when the stock goes below K

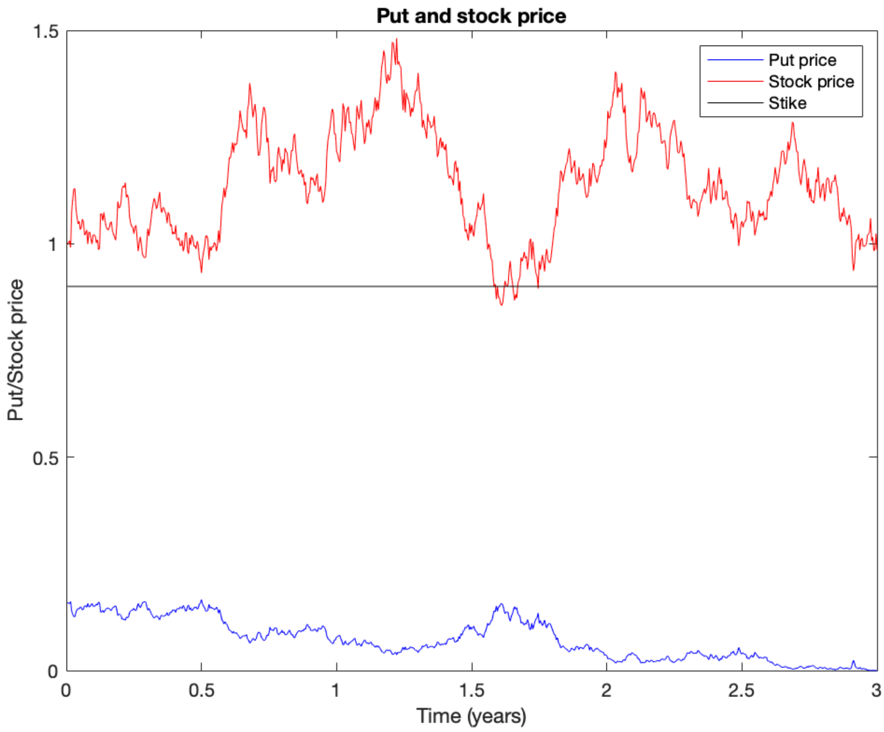


Figure 2.2: Stock and put option paths when the stock stays above K

In Figure 2.1 and 2.2 we see a three-year simulation of a stock with the corresponding put option with strike $K = 0.9$ and maturity $T = 3$. In Figure 2.1 we see how the put option acts in a down movement of the stock. In Figure 2.2 we see how the option price goes to price 0 as the stock stays above the strike and time reaches maturity.

2.3 Constant elasticity of variance model

The constant elasticity of variance (CEV) model is a local volatility model which was developed by John C. Cox and Stephen A. Ross in 1975 [2]. It captures the fact that the log returns may not have a constant volatility, which the IFBM model assumes. This fact might be quite important as there are often periods where the volatility of the time series have lower volatility and higher volatility. The volatility in the CEV model will be modelled as a function of S_t. The CEV model can be described with the following SDE

$$dS_t = (\mu - q)S_t dt + \sigma S_t^{\alpha} dW_t \tag{2.4}$$

in which S_t is the stock price, μ is the risk-free rate, q is the dividend paid by the stock, σ is the stocks volatility, $\alpha > 0$ is a shape parameter and dW_t is a brownian motion [5]. If $\alpha = 1$ the CEV model reduces to a geometric brownian motion. In this thesis we will assume that $\alpha \neq 1$ as we are interested in the CEV model. Moreover, we assume that are the stock S_t do not pay any dividend therefore $q = 0$. Unfortunately it does not exist a closed form solution to (2.4) when $\alpha \neq 1$. But there are still ways to simulate a stock price over time using the dynamics in (2.4) which will be discussed in the next chapter.

The volatility in the CEV model will behave differently depending on what value for α we use. If $\alpha < 1$, the volatility will decrease as the stock price increase, and volatility will increase if the stock price decrease. The last statement is shown as an example in Figure 2.3 and 2.4. If $\alpha > 1$, volatility will increase as the stock price increase, and volatility will decrease as the stock price decrease. Moreover, if $\alpha < 1$ we obtain a porbability distribution with a hevy left tail and less heavy right tail. If $\alpha > 1$ we get the reversed, a heavy right tail and less heavy left tail [5]. If the left tail is heavier we have a higher probability of getting negative returns. But if the right tail is heavier we have a higher probability of getting positive returns. In Figure 2.3 a three year simulation of a stock is shown and in Figure 2.4 we have the corresponding volatility change using the CEV model with $\alpha = 0.5$.

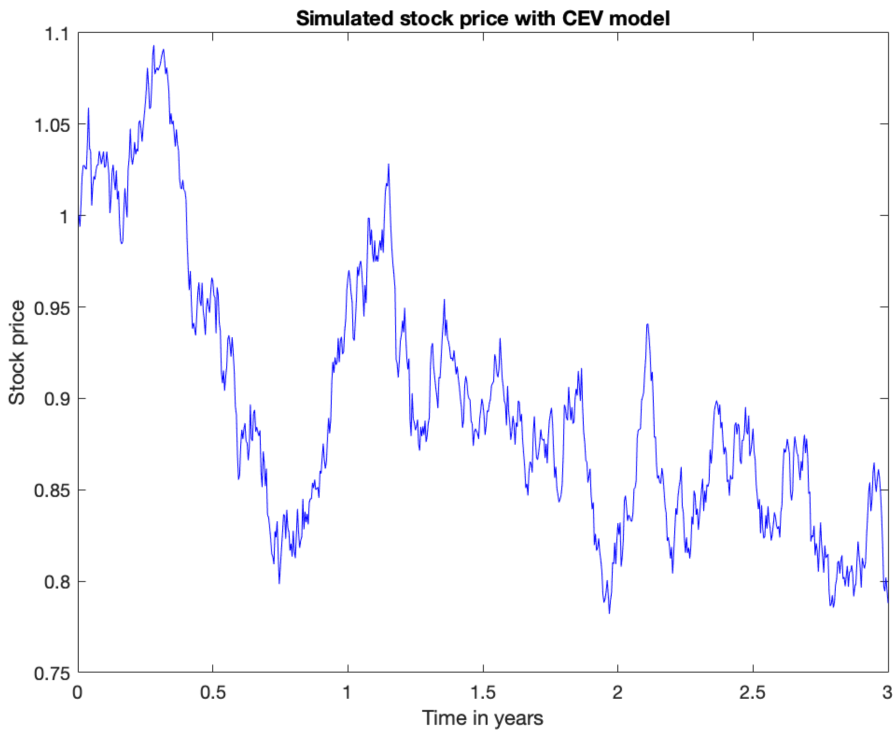


Figure 2.3: Stock price simulation using the CEV model with $S_1 = 1$ and $\alpha = 0.5$. The volatility corresponding to this specific stock path can be seen in Figure 2.4.

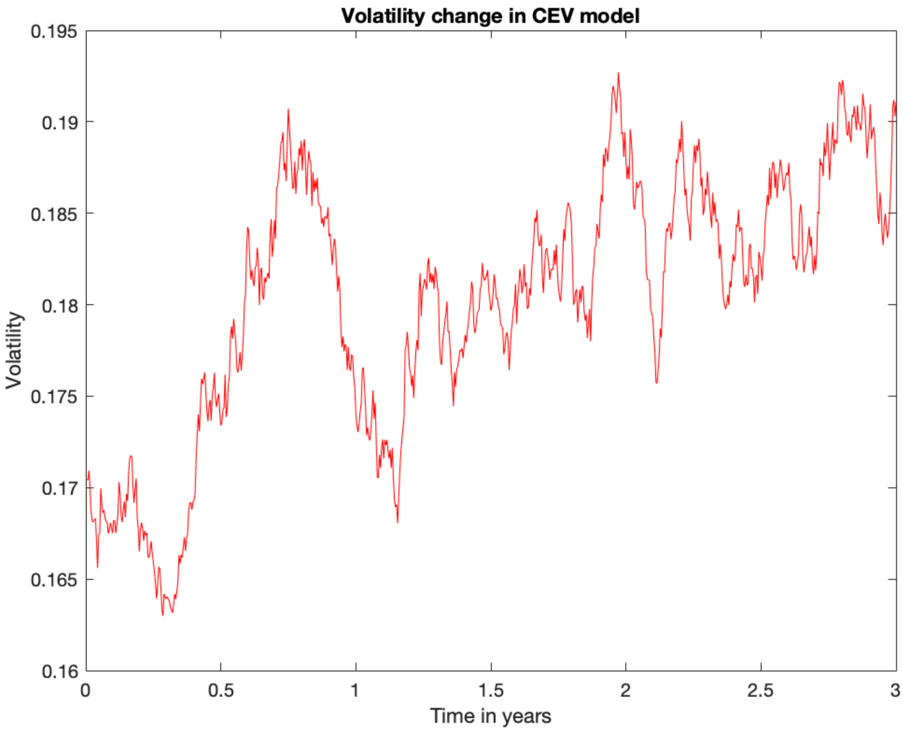


Figure 2.4: Volatility change corresponding to the stock price shown in Figure 2.3.

2.3.1 Option pricing with CEV

In the CEV model there exist closed form solutions to the price of European call and put options. These closed form solutions can be divided into three cases. However, as we assume $\alpha \neq 1$, we are only concerned about the situations when $0 < \alpha < 1$ and $\alpha > 1$. Let C and P denote the price of a call and put option respectively. We then have the following formulas to price European options using the CEV model [5].

If $0 < \alpha < 1$

$$C_t = S_t e^{-q(T-t)}[1 - \chi^2(a, b+2, c)] - Ke^{-r(T-t)}\chi^2(c, b, a)$$

$$P_t = Ke^{-r(T-t)}[1 - \chi^2(c, b, a)] - S_t e^{-q(T-t)}\chi^2(a, b+2, c).$$

If $\alpha > 1$

$$C_t = S_t e^{-q(T-t)}[1 - \chi^2(c, -b, a)] - Ke^{-r(T-t)}\chi^2(a, 2-b, c)$$
$$P_t = Ke^{-r(T-t)}[1 - \chi^2(a, 2-b, c)] - S_t e^{-q(T-t)}\chi^2(c, -b, a)$$

with

$$a = \frac{[Ke^{-(r-q)(T-t)}]^{2(1-\alpha)}}{(1-\alpha)^2 v} \qquad b = \frac{1}{(1-\alpha)} \qquad c = \frac{S_t^{2(1-\alpha)}}{(1-\alpha)^2 v}$$

where

$$v = \frac{\sigma^2}{2(r-q)(\alpha-1)}\left[e^{2(r-q)(\alpha-1)(T-t)} - 1\right].$$

As mentioned earlier, we assume that the underlying stock S_t do not pay any dividend therefore $q = 0$ in the equations above. If, however, $\alpha = 1$ we would recover a geometric brownian motion from the CEV model and we would use the Black-Scholes formula, which was explained in Section 2.2.1, to price European options. The relationship between a stock and an option written on the same stock is similar in the CEV model as the relationship shown in Figure 2.1 and 2.2 for the IFBM model.

2.4 Ho-Lee model

The Ho-Lee model was developed by Ho and Lee in 1986 [4]. The model is used to model interest rate evolution and can be expressed with the following SDE

$$dr = \theta(t)dt + \sigma dW_t \tag{2.5}$$

where $\theta(t)$ defines the average direction that r moves to at time t and σ is the volatility. It can be shown that

$$\theta(t) = F_t(0,t) + \sigma^2 t \tag{2.6}$$

where $F(0,t)$ is the instantaneous forward rate, and $F_t(0,t)$ is the partial derivative of the instantaneous forward rate with respect to t [5] [1]. To explain what forward rates are, imagine you have two bonds with maturity $t = 1$ and $t = 2$ years. The first bond pays 3% interest while the other pays 4%. If we make a 100 dollar investment the return of the bonds will be

$$100e^{0.03\cdot 1} = 103.0455 \quad \text{and} \quad 100e^{0.04\cdot 2} = 108.3287$$

respectively. Imagine now that we invested in the first bond which paid 3% after 1 year, but wanted to continue to invest for another year and make the same return as the second bond which paid 4% after two years, what interest rate is then required during the second year? The answer is 5% as

$$100e^{0.03\cdot 1}e^{0.05\cdot 1} = 108.3287.$$

We see that this return is the same as if we invested in the bond which paid 4% after two years, therefore the forward rate in this example is 5%. The forward rates can be derived with

$$F(0,t) = -\frac{\partial}{\partial t}\ln B(0,t) \quad \text{with} \quad B(0,t) = e^{-Rt}$$

where R is the observed rate from the yield curve for a specific maturity [5]. The yield curve are the rates for bonds with different maturities. When these rates are plotted on a graph they form what is known as the yield curve. The derived forward rates is then approximated with a polynomial where $F_t(0,t)$ in (2.6) is the derivative of this polynomial.

[1] For derivation of $\theta(t)$ see http://www-2.rotman.utoronto.ca/∼hull/technicalnotes/TechnicalNote31.pdf

There are two drawbacks worth to mention about the Ho-Lee model. The first one is that the Ho-Lee model does not account for mean reversion. Mean reversion means that the interest rate will tend to go back to its long-run average. The second drawback is that the model allows negative interest rate, that is $r_t < 0$ is a possibility for all $t \geq 0$. Although it is possible to have negative interest rates it is not very common. However, most simulations will actually stay positive and therefore this should not have a negative impact on our results.

To simulate interest rates we need to solve (2.5). Given (2.6), if we integrate (2.5) we obtain

$$r_t = F(0,t) + \frac{\sigma^2 t^2}{2} + \sigma \int_0^t dw_s. \tag{2.7}$$

A discrete version of (2.7) is used to simulate interest rate. More about this in Section 3.3.

In the portfolio strategies we need to price a Zero Coupon Bond (ZCB). The price of a ZCB with face value 1 at time t with maturity T (in years) is given by the following expectation

$$B(t,T) = \mathbb{E}[e^{-\int_t^T r_s ds}]. \tag{2.8}$$

Using the dynamics given in (2.5) it is possible to compute the expectation in (2.8) and the result looks like

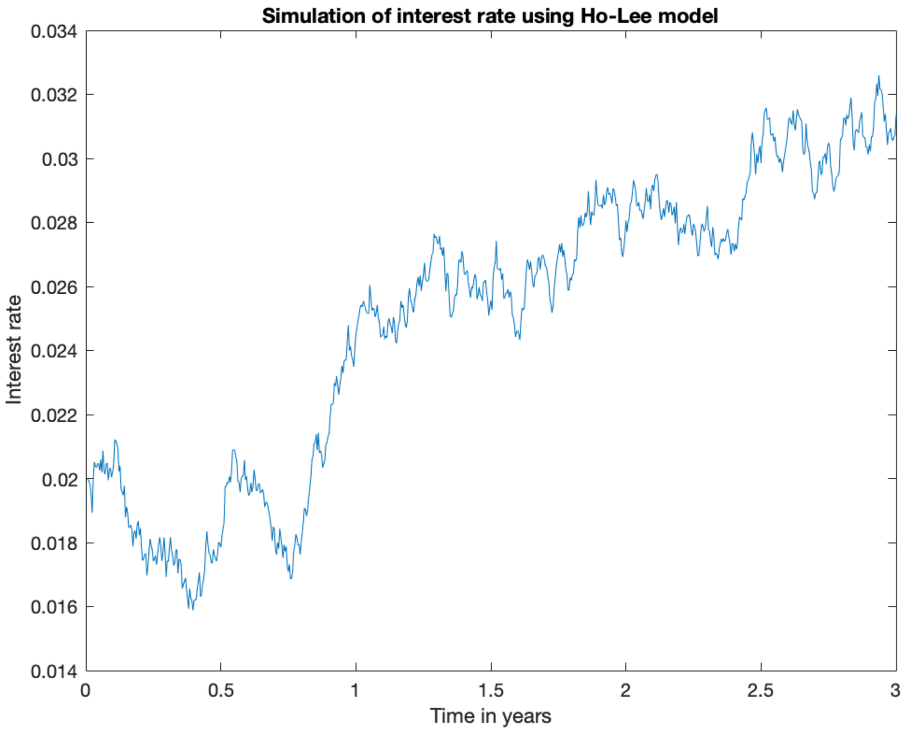
$$B(t,T) = A(t,T)e^{-(T-t)r(t)} \tag{2.9}$$

where

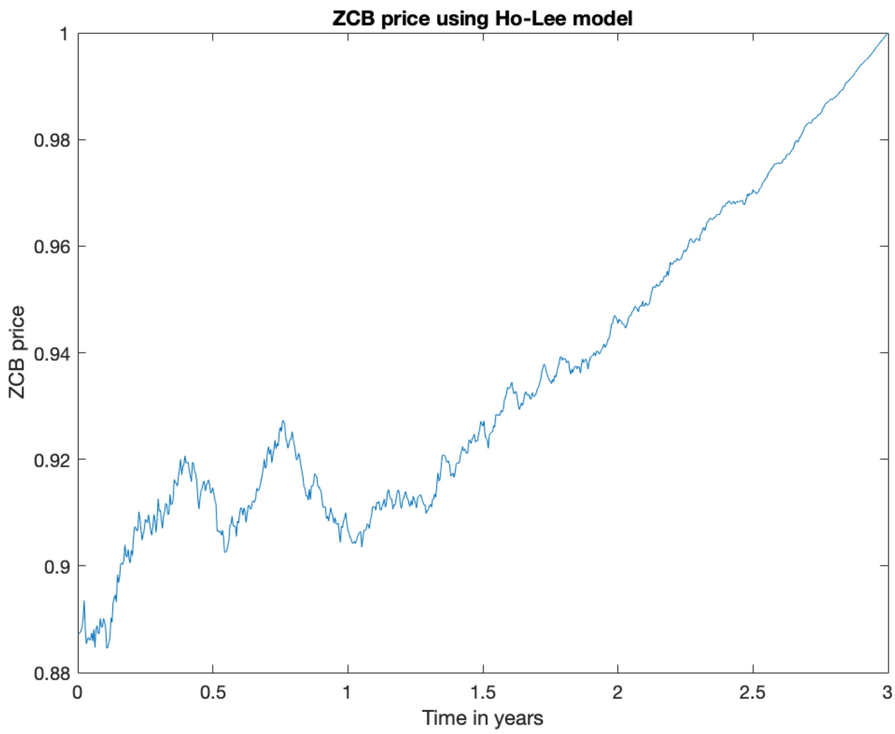
$$\ln A(t,T) = \ln\left(\frac{B(0,T)}{B(0,t)}\right) + (T-t)\frac{\partial \ln B(0,t)}{\partial t} - \frac{1}{2}\sigma^2 t(T-t)^2.$$

These formulas can be seen in [5]. (2.9) guarantees that the bond price at maturity is $B(T,T) = 1$ which makes sense as the bond at maturity will pay the face value to its owner which we said is 1. As mentioned in the Introduction we assume for simplicity all bonds we can invest in have a face value of 1 since owning K bonds with a face value of 1 is equivalent to owning 1 bond with face value K.

Figure 2.5 shows a three year simulation of an interest rate curve and its ZCB bond price using the Ho-Lee model. There are primarily two things that affects the price of a ZCB. The first is the interest rate. If the interest rate goes down the price of the ZCB will go up and if the interest rate goes up the price of the ZCB will go down. The second factor is time. As time goes to maturity the ZCB goes to its face value, which in this scenario is 1. At maturity the bond price will be equal to its face value which is guaranteed by (2.9). Therefore, time will have a bigger effect on the ZCB price than interest rate as $t \to T$. This relationship can also be seen in Figure 2.5.



(a) Interest rate



(b) ZCB price

Figure 2.5: A three year simulation of interest rate with the corresponding ZCB price evolution using the Ho-Lee model. Notice how the ZCB price continues to rise after 1.5 years despite an increasing interest rate.

2.5 Black-Derman-Toy model

In 1990 Fischer Black, Emanuel Derman and William Toy introduced the Black-Derman-Toy (BDT) model [1]. The model can be described with the following SDE [9]

$$d \ln r_t = [\theta(t) - a(t) \ln r]dt + \sigma(t)dW_t \quad \text{where} \quad a(t) = -\frac{\sigma'(t)}{\sigma(t)}. \tag{2.10}$$

If the BDT model have a constant volatility the model is reduced to a lognormal version of the Ho-Lee model. This can be seen in (2.10) as $\sigma'(t)$ becomes zero if σ is a constant and therefore $a(t)$ would become zero.

There are two advantages that the BDT model have over the Ho-Lee model. The first is that BDT has mean reversion. The second advantage is that the interest cannot become negative. Although it is possible to have negative interest rates, it is usually not something to strive for.

On the other hand, the BDT model cannot, unlike the Ho-Lee model, be solved analytically. Therefore we have to use the tool of a binomial tree. Figure 2.6 gives a visual representation of what a binomial tree looks like.

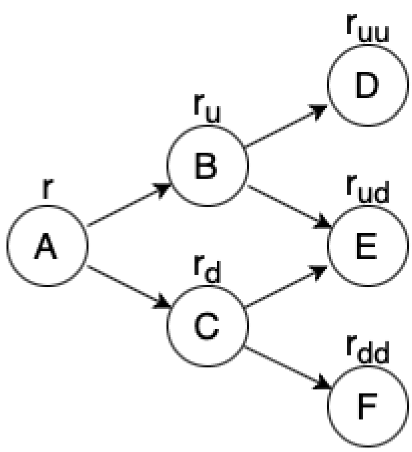


Figure 2.6: A visual representation of a binomial tree. The circles denotes the nodes A-F, r is the interest rate and u and d represents up or down movements respectively. Note that one can end up at node E in two ways, either by first going up and then do a down movement or by first going down and then do an up movement.

The idea behind the binomial tree is that at each node the interest rate has a probability p to move upward and probability $(1 - p)$ to move downward. In practice p is usually set to 0.5, meaning the interest rate r has an equal probability of moving upwards as to moving downward from any node. What we need to do is to determine what the interest rate is at each node and then simulate a path that goes trough the binomial tree. An algorithm on how to proceed with this will be outlined in the next chapter.

2.6 Portfolio insurance strategies

In this section we will go trough the two strategies that will be investigated in this thesis. They are the Option-based portfolio insurance (OBPI) strategy and the Constant proportion portfolio insurance (CPPI) strategy. Both these strategies are designed in a way such that an investor will limit their losses but still earn a profit by participating in an increasing market.

2.6.1 CPPI

The CPPI strategy was introduced in 1986 by Perold a. [8]. CPPI is a dynamic strategy which purpose is to contain its portfolio value above a floor set by the investor. There are two assets an investor can invest in, a risky asset and a risk-free asset (typically a stock and a bond). The investor begins by setting a floor F which is the lowest value they will allow their portfolio to reach. It is clear that F has to be lower than the initial value of the portfolio. Next the investor calculates the cushion

$$C_t = V_t^{CPPI} - F$$

where V_t^{CPPI} is the value of the portfolio at time t. The exposure to the risky asset is then given by

$$E_t = \min\{mC_t, V_t^{CPPI}\}$$

where m is a multiplier set by the investor. The multiplier m will reflect an investors willingness to take risk. A higher m will result to a higher exposure to the risky asset and lower exposure to the risk-free asset.

The multiplier m will have the characteristics that if it is a higher value an investor will obtain a higher profit in a market rise but, on the other hand, reach the floor faster in a market decline. If m has a lower value the investor will obtain a lower profit in a market rise but reach the floor slower in a market decline. Both the floor F and multiplier m are exogenous to the model and set by the investor. How these parameters are determined is discussed in the next chapter.

The remaining $V_t^{CPPI} - E_t$, which is not invested in the risky asset, is invested in the risk-free asset. The change in value of the portfolio is therefore given by

$$dV_t^{CPPI} = E_t \frac{dS_t}{S_t} + (V_t^{CPPI} - E_t)\frac{dB_t}{B_t}. \tag{2.11}$$

2.6.2 OBPI

The OBPI strategy was introduced in 1976 by Leland and Rubinstein [6]. OBPI, unlike CPPI, is a self-financing portfolio strategy. At time t_1 the investor buys q shares of a stock. The investor also purchases q shares of a put option written on the stock. The OBPI startegy also contains θ shares of a risk-free asset. The portfolio value is then given by

$$V_t^{OBPI} = q(S_t + P(S_t, K, T-t)) + \theta B_t \tag{2.12}$$

where P is the value of an European put option on the underlying asset S_t, with strike K and maturity T. At maturity the portfolio value is given by

$$V_T^{OBPI} = q(S_T + (K - S_T)_+) + \theta B_T$$

where $(K - S_T)_+ = \max\{K - S_T, 0\}$, or equivalently

$$(K - S_T)_+ = \begin{cases} K - S_T & \text{if } K > S_T \\ 0 & \text{if } K \leq S_T \end{cases}$$

is the pay-off function for the European put option. (2.12) promises the investor at least $qK + \theta$ at maturity. $qK + \theta$ will act as the floor in OBPI but has a different meaning than the floor in CPPI. In the CPPI, V_t^{CPPI} can never be lower than its floor. However, it is possible that V_t^{OBPI} is lower than its floor except on maturity when the investor is guaranteed at least the floor as a return on their investment.

3

Methods

In this chapter we describe how the models and investment strategies are implemented. We assume throughout this chapter that time is given in discrete form $\{t_i\}_{i=1}^n$ where $n = TN$ with $N = 252$ (assuming 252 trading days per year) and T is the number of years. Furthermore, let $dt = t_{i+1} - t_i$ denote a daily change in time.

3.1 IFBM Implementation

To simulate stock prices using the IFBM model we use a discrete form of (2.3). We get

$$S_{t_{i+1}} = S_{t_i} e^{\mu dt + \sigma Z \sqrt{dt} + \mu K f(Z) dt} \tag{3.1}$$

where $Z \sim N(0,1)$ and f is the function described in (2.2). The parameters μ and σ are based on historical data. To calculate μ and σ let $\tilde{S}_{t_i}$ denote the stock price at time t_i for a given data set with length n where $t_i = 1, \ldots, n$. The logarithmic return is given by

$$\mu_{t_i} = \ln\left(\frac{\tilde{S}_{t_{i+1}}}{\tilde{S}_{t_i}}\right).$$

Next we calculate its corresponding sample standard deviation

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\mu_{t_i} - \bar{\mu})^2} \quad \text{where} \quad \bar{\mu} = \frac{1}{n}\sum_{i=1}^{n}\mu_{t_i}$$

where n is the number of observations in the data set. We can then estimate σ and μ as

$$\sigma = \frac{s}{\sqrt{\frac{1}{n}}} \quad \text{and} \quad \mu = \bar{\mu}. \tag{3.2}$$

In (2.3) we can see that the model includes two other parameters that are unknown, K and c. To estimate these parameters we minimize the mean squared error between real data on a stock or index and a stock or index generated by the discretization in (3.1). The equation to minimize then becomes

$$\frac{1}{n}\sum_{i=1}^{n}\left(S_{t_i}^{IFBM} - S_{t_i}^{DATA}\right)^2$$

where n is the number of observations in the real data set. To solve this optimization problem the function *fmincon* in Matlab is applied. The IFBM model still includes a random number for each iteration. In order to obtain a robust answer this optimization will be done several times and the average value for K and c are then used in the real IFBM model.

3.2 CEV Implementation

As stated earlier, it is not possible to find a closed form solution to (2.4) when $\alpha \neq 1$. One approach to simulate stock prices using the dynamics in the CEV model is to use a discrete version of (2.4)[1]. However, we will use another approach to simulate stock prices using the Cev model as described in [7]. Let $\delta = \sigma_0 S_{t_1}^{\alpha-1}$ where σ_0 is the initial volatility. Now, set $S_{t_1} = 1$ and $\sigma_{cev,t_1} = \delta / S_{t_1}^{\alpha}$ then we have

$$S_{t_{i+1}} = S_{t_i} e^{(r-\frac{1}{2}\sigma_{cev,t_i})dt+\sigma_{cev,t_i}\sqrt{dt}\epsilon_i} \quad \text{with} \quad \sigma_{cev,t_{i+1}} = \delta / S_{t_i}^{\alpha} \tag{3.3}$$

and $\epsilon_i \sim N(0,1)$. As shown in (3.3) the parameter $\sigma_{cev,t_{i+1}}$ will be updated for every time unit t_i. However, the initial volatility σ_0 need to be estimated. This will be done the same way as we estimated volatility in the IFBM model, that is, with historical voliatilty, see (3.2).

To find a suitable value for α we minimize the sum of squares between a real data set and the values that our model generates based on the real data set. In practice this is done by letting α take the values $\{0.1, 0.2, 0.3, \ldots, 2\}$. Then for each α the mean sum of squares between the model and the data set are saved as shown below

$$\frac{1}{n}\sum_{i=1}^{n}\left(S_i^{CEV} - S_i^{DATA}\right)^2.$$

Whichever α, in the set $\{0.1, 0.2, \ldots, 2\}$, that consistently generates the lowest sum of squares trough multiple simulations is the one we will use.

For each S_{t_i}, i.e for each day, the price of a put option written on the stock generated by the CEV model is calculated using the equations described in Section 2.3.1.

3.3 Ho-Lee Implementation

To implement the Ho-Lee model and price a ZCB we use a discrete form of both (2.7) and (2.9) and set $r_{t_1} = 0.02$. The discrete version of (2.7) looks as follows

$$r_{t_{i+1}} = r_{t_i} + F(0, t_{i+1}) - F(0, t_i) + \frac{dt^2\sigma^2}{2} + \sigma\sqrt{dt}\epsilon_i$$

where $\epsilon_i \sim N(0,1)$. The discrete version of (2.9) for the ZCB price looks like

$$B(t_i, t_n) = A(t_i, t_n)e^{-(t_n - t_i)r_{t_i}}$$

[1]For more information see *Hedging with Liquidity Risk under CEV Diffusion* (Sang-Hyeon Park and Kiseop Lee)

with

$$\ln A(t_i, t_n) = \ln\left(\frac{B(0,t_n)}{B(0,t_i)}\right) + (t_n - t_i)\frac{\partial \ln B(0,t_i)}{\partial t} - \frac{1}{2}\sigma^2 t_i (t_n - t_i)^2.$$

The volatility σ will be estimated using historical data as we did in the IFBM model, see (3.2).

Next we estimate the forward rate as explained in Section 2.4. In Figure 3.1 and 3.2 the yield curve and the forward rates derived from the yield curve are shown.

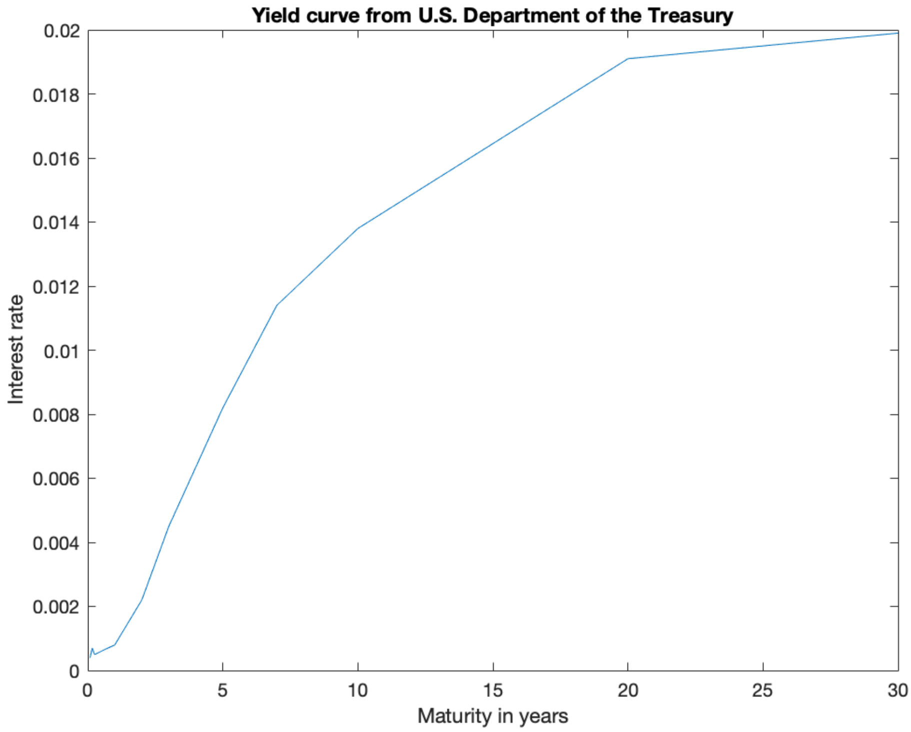


Figure 3.1: Yield curve as it was Sep 8 2021.

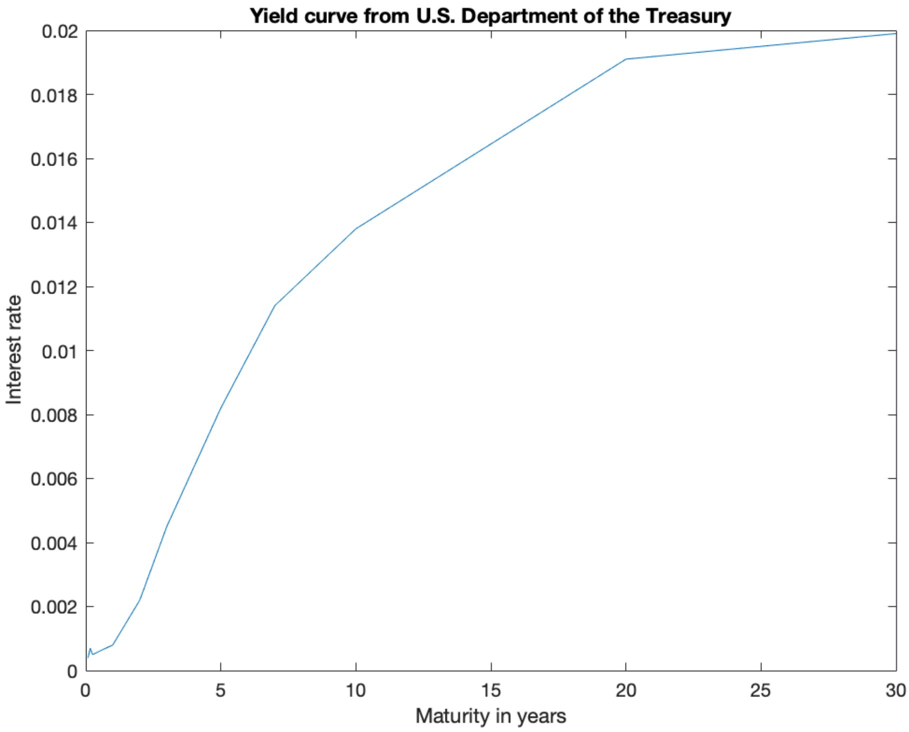


Figure 3.2: Derived forward rates from the yield curve in Figure 3.1

Now we fit a third degree polynomial to the derived forward rates in Figure 3.2 of the form

$$p(x) = p_1 t^3 + p_2 t^2 + p_3 t + p_4. \tag{3.4}$$

This can easily be done in Matlab using the function *fit*. When determining the best polynomial degree for the derived forward rates we take a couple of things into account. First we want to obtain a low residual mean squared error and a R^2 close to 1. However, we do not want to make the polynomial degree unnecessarily high making the polynomial too complicated. Therefore we test different degrees ranging from $1-5$ and see which one produce an optimal fit. It turns out that a polynomial of degree 3 produce the best fit, which is why the degree of the polynomial in (3.4) is 3. Result can be seen in Chapter 4.

3.4 Black-Derman-Toy Implementation

To implement the BDT we generate a binomial interest rate tree and a binomial bond tree in matrix form in matlab. We set $p = 0.5$ so we have an equal probability of going up or down for each time iteration. Down below an explanation of how the algorithm works is provided. To see details and further explanation of the theory behind the algorithm read [9].

The calculations are done by forward induction. The Arrow-Debreu securities (Arrow-Debreu securities means a state-price security) at time $i \cdot dt$ are therefore calculated by the already known Arrow-Debreu securities at time $(i-1)dt$. We define $Q_{i,j}$ as Arrow-Debreu securities. They can be seen as discounted probabilities and therefore by definition we have $Q_{0,0} = 1$. We can therefore price a discount bond maturing at time $(i+1)dt$ as

$$P(i+1) = \sum_j Q_{i,j} d_{i,j}$$

where $d_{i,j}$ is the time $i \cdot dt$, state j value of a discount bond maturing at time $(i+1)dt$ and hence $d_{i,j} = \frac{1}{1+r_{i,j}dt}$. The Arrow-Debreu prices are updated the following way

$$Q_{i,i} = \frac{1}{2} Q_{i-1,i-1} d_{i-1,i-1} \tag{3.5}$$

$$Q_{i,j} = \frac{1}{2} Q_{i-1,j-1} d_{i-1,j-1} + \frac{1}{2} Q_{i-1,j+1} d_{i-1,j+1} \tag{3.6}$$

$$Q_{i,-i} = \frac{1}{2} Q_{i-1,-i+1} d_{i-1,-i+1}. \tag{3.7}$$

The short term rate for each node (i,j) may be represented as

$$r_{i,j} = u(i) e^{\sigma(i) j \sqrt{dt}}. \tag{3.8}$$

Next we need to define the notation used to index the nodes of the tree. At time $i = 0$ there is a single state $j = 0$. At each time i there are $(i+1)$ possible states indexed as $j = -i, -i+2, \ldots, i-2, i$. For instance, if time is $i = 3$, we have $(i+1) = 4$ possible states with indices $\{-3, -1, 1, 3\}$ which corresponds to 3 down moves, 2 down and 1 up move (net -1 down move), 1 down and 2 up move (net 1 up move) and 3 up moves.

From the initial node $(0,0)$ we have a possible up or down move. We denote the up move $(1,1)$ as U and the down move $(1,-1)$ as D. At these nodes we define P_U^i and P_D^i as the price of a discount bond with maturity $i \cdot dt$, and R_U^i and R_D^i as the discount bond yields at node U and D corresponding to the discount bond prices. The following relationships must hold

$$P(i) = \frac{1}{1+r_{0,0}} \left(\frac{1}{2} P_U(i) + \frac{1}{2} P_D(i) \right) \qquad i = 2, \ldots, N$$

$$\sigma_R(i)\sqrt{dt} = \frac{1}{2} \ln \left(\frac{\ln P_U(i)}{\ln P_D(i)} \right) \qquad i = 2, \ldots, N.$$

These equations can be solved for $P_U(i)$ and $P_D(i)$ as

$$P_D(i)^{\exp(2\sigma_R(i)\sqrt{dt})} + P_D(i) = 2P(i)(1 + r_{0,0}dt) \tag{3.9}$$

and

$$P_U(i) = P_D(i)^{\exp(2\sigma_R(i)\sqrt{dt})}. \tag{3.10}$$

We also need to define state prices corresponding to nodes U and D. By definition we have that $Q_{U,1,1} = 1$ and $Q_{U,1,-1} = 1$, therefore the values of a discount bond maturing at time $(i+1)dt$ may be written as

$$P_U(i+1) = \sum_j Q_{U,i,j} d_{i,j} \qquad i = 1, \ldots, N-1 \tag{3.11}$$

$$P_D(i+1) = \sum_j Q_{D,i,j} d_{i,j} \qquad j \in \{-i, -i+2, \dots, i-2, i\} \tag{3.12}$$

where N is the number of time steps in the tree. The state prices may then be updated as in (3.5)-(3.7)

$$Q_{U,i,i} = \frac{1}{2} Q_{U,i-1,i-1} d_{i-1,i-1} \tag{3.13}$$

$$Q_{U,i,j} = \frac{1}{2} Q_{U,i-1,j-1} d_{i-1,j-1} + \frac{1}{2} Q_{U,i-1,j+1} d_{i-1,j+1} \tag{3.14}$$

$$Q_{U,i,-i+2} = \frac{1}{2} Q_{U,i-1,-i+3} d_{i-1,-i+3} \tag{3.15}$$

and

$$Q_{D,i,-i} = \frac{1}{2} Q_{D,i-1,-i+1} d_{i-1,-i+1} \tag{3.16}$$

$$Q_{D,i,j} = \frac{1}{2} Q_{D,i-1,j-1} d_{i-1,j-1} + \frac{1}{2} Q_{D,i-1,j+1} d_{i-1,j+1} \tag{3.17}$$

$$Q_{D,i,i-2} = \frac{1}{2} Q_{D,i-1,i-3} d_{i-1,i-3}. \tag{3.18}$$

With this knowledge it is possible to outline an algorithm in order to obtain the desired interest rate and bond price tree.

First, we need to define some initial values

$$\begin{aligned} r_{0,0} = u(0) &= \frac{e^{R(1)dt} - 1}{dt} \\ Q_{U,1,1} &= 1 \\ Q_{D,1,-1} &= 1 \\ \sigma(0) &= \sigma_R(1) \\ d_{0,0} &= \frac{1}{1 + r_{0,0} dt}. \end{aligned}$$

For each $i = 1, ..., N-1$ do the following steps.

1. By using a numerical method solve (3.9) for $P_D(i+1)$ and then solve (3.10) for $P_U(i+1)$.
2. Now that we have $P_D(i+1)$ and $P_U(i+1)$ we continue by solving $\sigma(i)$ and $u(i)$ in (3.11) and (3.12). This is done by doing the following substitution for $d_{i,j}$.

   $$d_{i,j} = \frac{1}{1 + u(i) \exp(\sigma(i) j \sqrt{dt})}$$

   and therefore

   $$P_U(i+1) = \sum_j \frac{Q_{U,i,j}}{1 + u(i) \exp(\sigma(i) j \sqrt{dt})} \qquad i = 1, ..., N-1$$

   $$P_D(i+1) = \sum_j \frac{Q_{D,i,j}}{1 + u(i) \exp(\sigma(i) j \sqrt{dt})} \qquad j \in \{-i, -i+2, \dots, i-2, i\}.$$

3. Now that we have the values for $u(i)$ and $\sigma(i)$ the one period interest rate can be found for each node $j = -i, ..., i$ by using (3.8) and the definition of the discount factor as $d_{i,j} = \frac{1}{1+r_{i,j}dt}$.
4. We can now update $Q_{U,i,j}$ and $Q_{D,i,j}$ according to (3.13)-(3.18).

3.5 Simulation of portfolio strategies

To compare the results between the two portfolio strategies, CPPI and OBPI, for each simulation both strategies will be based on the same stock and interest rate simulation. We will test all possible combination of a stock model and interest rate model among the four models we have discussed previously. This gives us 4 different combinations, IFBM with Ho-Lee, IFBM with BDT, CEV with Ho-Lee and finally CEV with BDT. All simulations will be done using maturity $T = 3$. Furthermore, at time t_1, that is the beginning of an investment period, we assume both strategies start with the same amount invested in the risky asset. If they have different exposure in the risky asset it would not be possible to do a fair comparison between the strategies. Now we will go trough how the portfolio strategies will be simulated.

3.5.1 CPPI

To implement the CPPI strategy we use a discrete version of (2.11). This gives us the following formula

$$V_{t_i}^{CPPI} = V_{t_{i-1}}^{CPPI} + E_{t_{i-1}} \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}} + \left(V_{t_{i-1}} - E_{t_{i-1}} \right) \frac{B(t_i, t_n) - B(t_{i-1}, t_n)}{B(t_{i-1}, t_n)}$$

where the exposure E_{t_i} to the risky asset has to be updated every time unit according to

$$E_{t_i} = \min\{mC_{t_i}, V_{t_{i-1}}^{CPPI}\} \quad \text{with} \quad C_{t_i} = V_{t_i}^{CPPI} - F.$$

We assume the initial value of the portfolio is $V_{t_1}^{CPPI} = 1$. The initial value of V_{t_i} does not affect a strategies performance as we are only interested in the percentage development. Its performance will, however, be easy to interpret if we set $V_{t_1}^{CPPI} = 1$. We set our floor $F = 0.9$, this give $c_{t_1} = 0.1$ as $V_{t_1}^{CPPI} = 1$. In our simulations the following values for m will be tested $m = \{1, 2.5, 4\}$. Therefore we will test portfolios where we start with an initial investment in the risky asset of 10%, 25% and 40%. This will give result for a passive investor and for an investor with a little higher risk appetite. The floor $F = 0.9$ ensures that $V_{t_i}^{CPPI} \geq 0.9$ holds true for all t_i. Meaning the portfolio value cannot be lower than the floor at any time during our investment period.

3.5.2 OBPI

To be able to compare the OBPI with CPPI we need the investment in risky asset in OBPI to equal the initial investment in the risky asset in CPPI. The risky investment in OBPI will be split between a stock and a put option written on the same stock. The remainder of our available amount to invest will be invested in a risk-free asset such as in the CPPI strategy. The evolution of the OBPI strategy can be described as a discrete form of (2.12) the following way

$$V_{t_i}^{OBPI} = q(S_{t_i} + P(S_{t_i}, K, T - t_i)) + \theta B(t_i, t_n) \tag{3.19}$$

where θ is the amount invested in the bond and, again, the initial value of the portfolio is $V_{t_1}^{OBPI} = 1$. The amount that the investor is guaranteed at maturity is now updated to

$$qK + \theta$$

which is known as the floor. The strike price K of the option will be set to 0.95. Remember that buying a put option means that we have the right, but not the obligation, to sell the underlying asset to the strike price, i.e 0.95. However, we will not do that in this strategy. We are simply interested in the change of value in the put option during our investment period.

The amount q that the investor invest in the risky asset will be the same as in CPPI. Therefore, we have $q = (V_{t_1}^{OBPI} - F)m = (1 - 0.9)m$ where m will take the same values as in the CPPI strategy discussed in the previous section. Note that q will change depending on what value for m we use. Therefore the floor will change accordingly. To obtain the amount to invest in the risk-free asset we rearrange the terms in (3.19) and perform the following calculation for time t_1

$$\theta = \frac{V_{t_1}^{OBPI} - q(S_{t_1} + P(S_{t_1}, K, T - t_1))}{B(t_1, t_n)}.$$

We will assume that we only have 1 (currency doesn't matter) to invest. Note that OBPI is a static strategy and therefore q and θ will stay the same throughout the investment period. Which means that the exposure to the riksy- and risk-free asset will stay the same for all t_i. CPPI, however, is a dynamic strategy meaning that the exposures to the assets are updated each time unit.

4

Result

4.1 Calibration

In this section we will go trough all parameter estimations for every model and every data set. The stock models are based on two different data sets which both contains stock price evolution for one year. The Ho-Lee model will be based on the 6-month Libor rate during a fixed time period. Finally, the BDT model will be based on the yield curve as of Sep 8 2021.

Each stock model will be paired with an interest rate model, as explained in previous chapter, and then 10000 simulations will be done to compare performance of the two portfolio strategies. This is done for all possible combinations of one stock model paired with one interest rate model, this gives us four unique pairings.

Both stock models, IFBM and CEV, are based on two different data sets. Both data sets are from the S&P500 index. One data set is from 2018 when the index had a return of -6% while the other is from 2020 with a return of 15%. In the tables below we will refer the data set from 2018 as S&P500- and the data set from 2020 as S&P500+. Ho-Lee and BDT are based on the 6-month LIBOR interest rate from Sep 8 2020 to Sep 7 2021. Furthermore, the yield curve are used in both interest rate models as it was seen on Sep 8 2021.

4.1.1 Stock models

For the IFBM model, we need to estimate the parameters K and c. This will be done, as explained in Section 3.1, by minimizing the mean squared error function. Results for the two data sets can be seen in Table 4.1.

Table 4.1: The table shows the approximated values for the parameters in the IFBM model for different data sets.

Parameter	Data set	Value
K	S&P500+	-15.4089
c	S&P500+	-0.0013
K	S&P500-	-16.3309
c	S&P500-	0.0001482

For the CEV model we only need to estimate the scaling parameter α. This was, as explained in previous chapter, done by minimizing sum of squares. The result is presented in Table 4.2 down below.

Table 4.2: The table shows the estimated value for the scaling parameter α in the CEV model for each data sets.

Parameter	Data set	Value
α	S&P500+	1.5
α	S&P500-	0.4

4.1.2 Interest rate models

In Figure 3.1 we saw the yield curve as it was Sep 8 2021. The values which the yield curve is based on can be seen in Table 4.3.

Table 4.3: The table shows the values in percentage which the yield curve in Figure 3.1 is based on

Maturity	Yield
$\frac{1}{12}$	0.04
$\frac{2}{12}$	0.08
$\frac{3}{12}$	0.05
$\frac{6}{12}$	0.06
1	0.08
2	0.22
3	0.45
5	0.82
7	1.14
10	1.38
20	1.91
30	1.99

In the Ho-Lee model we need to fit a polynomial to the derived forward rates, see Figure 3.2, from the yield curve. The optimal polynomial for this is a third degree polynomial as seen below

$$F(0,t) = p_1t^3 + p_2t^2 + p_3t + p_4.$$

The values of coefficients $p_1, \ldots, p_4$ can be found in Table 4.4.

Table 4.4: The coefficients for the polynomial approximation of the forward curve.

Coefficient	Value
p_1	$2.314e-06$
p_2	-0.0001666
p_3	0.003669
p_4	-0.001092

Coefficients $p_1, ..., p_4$ to the polynomial $F(0,t)$ gives the following approximation to the derived yield curve, see Figure 4.1.

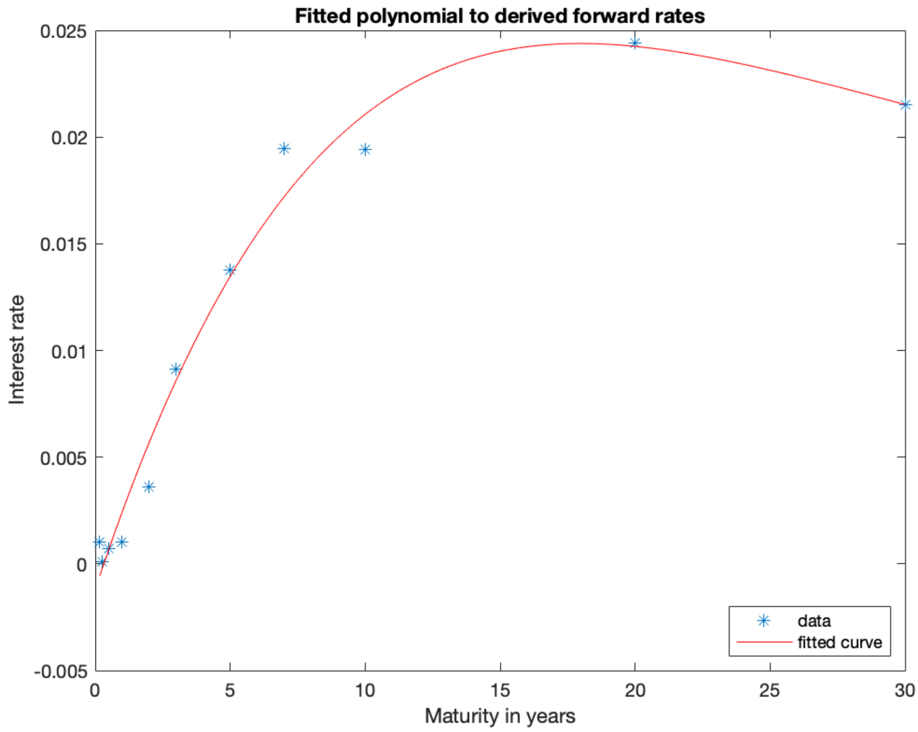


Figure 4.1: A third degree polynomial approximated to the derived forward rates from Sep 8 2021.

4.2 Portfolio results

Below is the average end value for each portfolio during different scenarios, such as different risk appetites, data set and underlying models for the risky- and risk-free asset. The floor for CPPI was set to 0.9. As mentioned previously OBPI's floor depends on the multiplier value m which will take the following values $\{1, 2.5, 4\}$. A strike price of 0.95 on the put option was used with a maturity of $T = 3$ years. This gives the following three different floor values for OBPI, floor $= \{0.99, 0.975, 0.96\}$ for $m = \{1, 2.5, 4\}$ respectively.

Table 4.5: Results on data generated by the IFBM and Ho-Lee models.

Strategy	Data set	T	m	$\bar{\sigma}$	Performance
CPPI	S&P500+	3	1	0.0477	1.257
OBPI	S&P500+	3	1	0.0407	1.142
CPPI	S&P500-	3	1	0.0193	1.063
OBPI	S&P500-	3	1	0.00923	1.009
CPPI	S&P500+	3	2.5	0.173	1.498
OBPI	S&P500+	3	2.5	0.102	1.353
CPPI	S&P500-	3	2.5	0.0402	0.998
OBPI	S&P500-	3	2.5	0.0250	1.026
CPPI	S&P500+	3	4	0.203	1.602
OBPI	S&P500+	3	4	0.168	1.577
CPPI	S&P500-	3	4	0.0462	0.965
OBPI	S&P500-	3	4	0.0386	1.040

Table 4.6: Results on data generated by the IFBM and BDT models.

Strategy	Data set	T	m	$\bar{\sigma}$	Performance
CPPI	S&P500+	3	1	0.0430	1.168
OBPI	S&P500+	3	1	0.0413	1.146
CPPI	S&P500-	3	1	0.0167	0.992
OBPI	S&P500-	3	1	0.00961	1.009
CPPI	S&P500+	3	2.5	0.178	1.457
OBPI	S&P500+	3	2.5	0.104	1.357
CPPI	S&P500-	3	2.5	0.0384	0.951
OBPI	S&P500-	3	2.5	0.0244	1.024
CPPI	S&P500+	3	4	0.203	1.560
OBPI	S&P500+	3	4	0.165	1.569
CPPI	S&P500-	3	4	0.0459	0.929
OBPI	S&P500-	3	4	0.0374	1.036

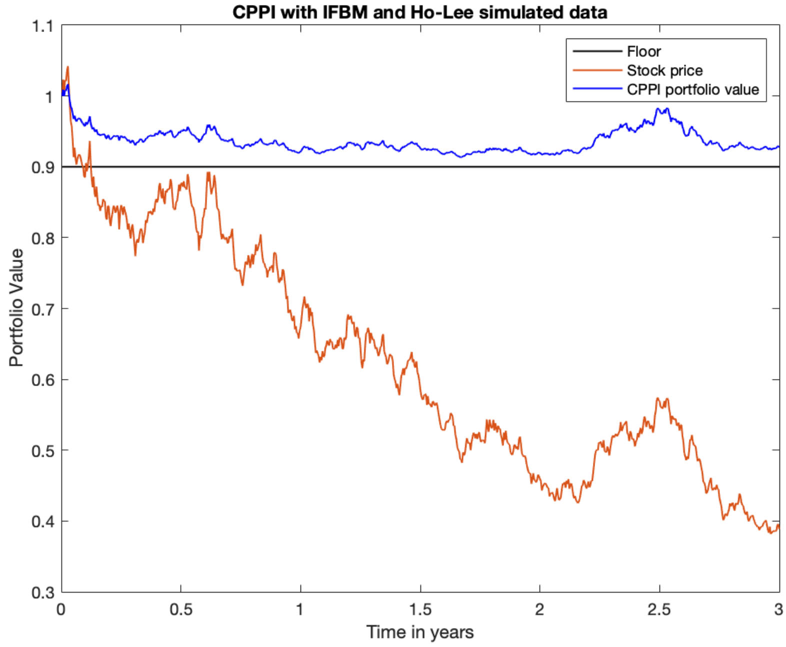
Table 4.7: Results on data generated by the CEV and Ho-Lee models.

Strategy	Data set	T	m	$\bar{\sigma}$	Performance
CPPI	S&P500+	3	1	0.0618	1.220
OBPI	S&P500+	3	1	0.0492	1.121
CPPI	S&P500-	3	1	0.0144	1.049
OBPI	S&P500-	3	1	0.00502	0.999
CPPI	S&P500+	3	2.5	0.200	1.360
OBPI	S&P500+	3	2.5	0.119	1.303
CPPI	S&P500-	3	2.5	0.0217	0.970
OBPI	S&P500-	3	2.5	0.0138	0.999
CPPI	S&P500+	3	4	0.239	1.418
OBPI	S&P500+	3	4	0.201	1.493
CPPI	S&P500-	3	4	0.0246	0.925
OBPI	S&P500-	3	4	0.0226	0.997

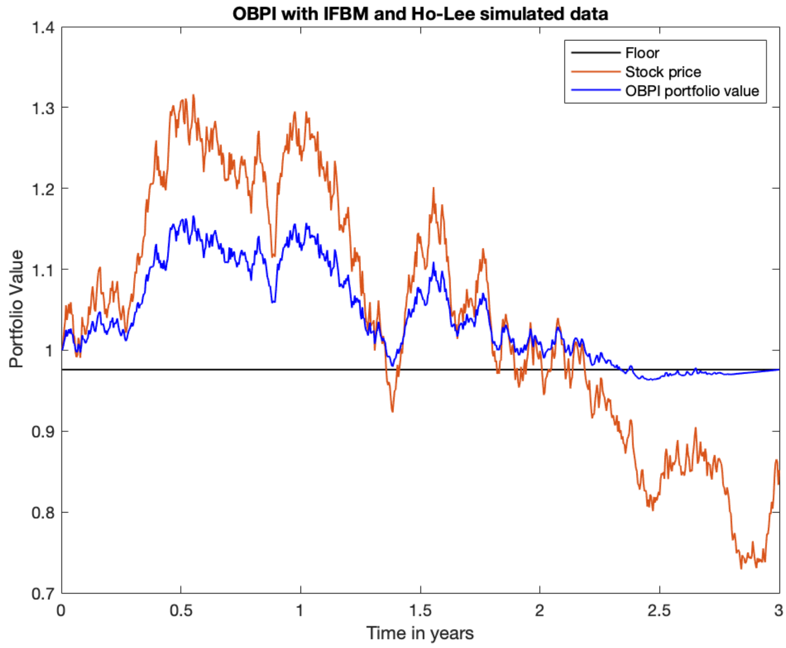
Table 4.8: Results on data generated by the CEV and BDT models.

Strategy	**Data set**	**T**	**m**	$\bar{\sigma}$	**Performance**
CPPI	S&P500+	3	1	0.0567	1.135
OBPI	S&P500+	3	1	0.0510	1.126
CPPI	S&P500-	3	1	0.0122	0.980
OBPI	S&P500-	3	1	0.00562	0.999
CPPI	S&P500+	3	2.5	0.215	1.328
OBPI	S&P500+	3	2.5	0.127	1.316
CPPI	S&P500-	3	2.5	0.0211	0.929
OBPI	S&P500-	3	2.5	0.0142	0.999
CPPI	S&P500+	3	4	0.246	1.386
OBPI	S&P500+	3	4	0.204	1.505
CPPI	S&P500-	3	4	0.0256	0.900
OBPI	S&P500-	3	4	0.0226	0.999

In Figure 4.2 an example of how both strategies behaves when the stock declines is shown.



(a) CPPI in a bear market.



(b) OBPI in a bear market.

Figure 4.2: In Figure (a) we see how the CPPI portfolio behave in an extreme case of negative stock path. The value of the portfolio will stay above its floor even for extreme downward movement in the stock. In Figure (b), however, we see that the OBPI portfolio is allowed to go beneath its floor but, will approach and, stay at least equal to the floor as maturity is reached.

5

Discussion

The purpose of this thesis was mainly to study the performance of two portfolio strategies, CPPI and OBPI, under different market environments. We have tested the strategies under different models, data and as both an aggressive and passive investor.

In general the result from Table 4.5-4.8 tell us that CPPI perform better than OBPI in a positive market while OBPI perform better than CPPI in a negative market. A positive market is expected when the models are based on the data set S&P500+ because it has a mean return of 15% but when the models are based on the S&P500- data set, which has a mean return of -6%, it should create a negative market. The CPPI strategy has the advantage that it is a dynamic strategy meaning it can increase or decrease its exposure in assets depending on how the stock moves. However, this leads to higher volatility compared to OBPI which can be seen in all simulation results in Table 4.5-4.8. Higher volatility means the risk is higher as well. It should provide a greater return in positive markets but a greater loss in negative markets compared to strategies with less volatility. Table 4.5-4.8 shows us that under the negative data set, and especially when the multiplier factor m is higher, that OBPI performs better. An investor could set a higher floor on CPPI, however, it lead to a greater exposure to the risk-free asset.

A somewhat unexpected result is that the OBPI portfolio performs better than the CPPI portfolio for Table 4.7 and 4.8 under the data set S&P500+ when $m = 4$. In this situation it was expected that CPPI should perform better as in a positive market it will increase its exposure to the stock asset as it should provide a higher return. Perhaps this result could be explained by the fact that these simulations have a slightly higher volatility compared to the corresponding simulations in Table 4.5 and 4.6, meaning that for the occasions when the stock goes down it causes too much damage for the CPPI portfolio. A simple example that illustrates this is, if a stock today costs 100 dollars and decrease 50% to 50 dollars it has to increase 100% to break even. The point is that it takes a greater effort to repair the damage taken by a loss. In general, however, when the volatility is higher OBPI should be seen a safer strategy but CPPI should reward with a higher return if the market is positive.

The asset that affect the performance and volatility most on the strategies is the stock. The bond asset, which represent a risk-free asset, is included to enable the possibility of having a floor on the strategies. Therefore we can understand from Table 4.5-4.8 that the CEV model provides more volatility than the IFBM model. Therefore, if we want to model something that probably contains more risk the CEV model might be a better option.

References

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