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MASTER'S THESIS

A Relative Value Trading Strategy in Swaps and Principal Component Analysis

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Executive Summary

This master thesis describes a trading strategy in fixed income derivatives called swap butterfly. We apply principal component analysis (PCA) to swap rate time-series, and show how PCA can be used as a tool for finding relative value on the swap rate curve. We show that PCA reduces the dimensionality of factors describing the variation of related swap rates and that principal components can be seen as factors describing shifts in level, slope and curvature of the swap rate curve. We also show that PCA can be used to weight the components of a swap butterfly strategy in order to profit from unusual curve patterns of the swap rate curve, and stay hedged against shifts in the overall level and slope of the curve.

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Chapter 1

Introduction

1.1 Fixed income relative value trading

The purpose of fixed income relative value trading is to profit from perceived dislocations of fundamental variables such as interest rates. The belief is that interest rates revert back to fair levels, or historical mean levels over time, something that can be exploited and very profitable. The perspective for reversion varies from days to several months, depending on historical trends and the dynamics of the market variables in question. There are a broad set of fixed-income and fixed-income derivative instruments that are related in one way or another. When one maturity type or single security is miss-priced with respect to others, there are many ways to set up trading strategies with relatively low risk in order to capture the opportunity.

In this master thesis we will look at one of these relative value strategies in detail, the butterfly strategy in swaps. The potential percentage gain of such trade is often low, but by considering markets with relatively high volume, the trader can earn considerable profit. Statistical arbitrage is a technique to trade with favorable probabilities. In this thesis we will identify trading opportunities for a butterfly strategy using statistical arguments. Just like owners of casinos that win slightly over 50 percent of the games, statistical arbitrage can identify trades with favorable probabilities.

In Chapter 1 we define the swap instrument, look at its pricing, its risk characteristics and how the butterfly strategy is constructed to profit from unusual curve patterns in the swap curve. Chapter 2 describes principal component analysis (PCA) and how it is applied to the butterfly strategy in several ways. In Chapter 3 we apply PCA to a real world situation with real market data.

Chapter 2

Theory

2.1 Swaps

2.1.1 The swap instrument

A swap is a contractual agreement in which two parties agree to make periodic payments to each other. Two parties can enter to swap payments of nearly anything, and the concept of a swap is not limited to financial markets. However, in this thesis we concentrate on a "plain vanilla swap", the interest rate swap, or more precisely the fixed-for-floating interest rate swap. In this swap agreement, one party will agree to make floating rate payments based on LIBOR[1] to the other party, which in turn make fixed interest rate payments to the former party of the swap agreement. Consider the following simple example: Company A has a $100 million debt on floating interest rate. They believe that the floating rate will increase in the future and would like to hedge this interest risk exposure. They enter a swap agreement with a swaps dealer to pay the fixed rate, and receive the floating rate on a $100 million notional amount. See Exhibit one below.

[1] The London Interbank Borrowing Rate. An interest rate offered between banks with high credit rate.

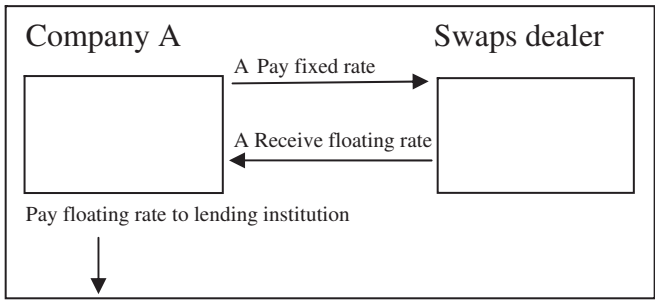

Exhibit 1: Flow of funds for a swap agreement.

The swap will therefore have two "legs", the fixed and floating leg, referring to the the fixed interest payment stream and the floating interest payment stream, respectively. A typical payment schedule is displayed in Figure 1 below.

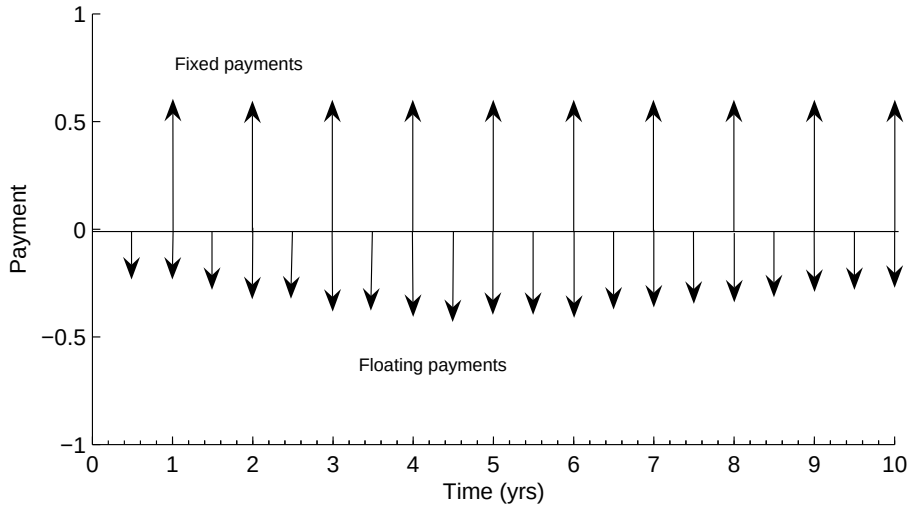

Figure 1: Payment schedule for a swap.

Figure 1 shows the payment schedule for a ten-year swap which receives fixed payments on annual basis and pays floating payments on semi-annual basis.

The notional principal:

Fixed and floating payments are calculated as if they were payments of interest on an amount of money borrowed or lent. This amount is referred

to as the notional principal N. It is called principal because this amount is actually never exchanged, as it only functions as a part of the specification of the fixed and floating amounts to be paid or received.

The fixed rate:

This is the rate applied to the notional principal to calculate the fixed amounts. From day to day, market participants quote the price at which they are prepared to execute a particular swap by quoting the fixed rate.

Dates of payments:

Fixed rate payments are usually paid either annually or every six months. The last payment date is known as the termination date or the maturity date of the swap. Two other relevant dates are the trade date, on which the two parties agree to do the swap, and the effective date, when the first fixed and floating payments start to accrue. If a payment date falls on a weekend or bank holiday, and the next working day is in the same calendar month, then the payment is made on this next working day. Otherwise, payment is made on the latest preceding day at which banks are open. This way of specifying payment days are called *modified following business day*, and is the common standard for swaps in the US.

The fixed rate payments:

Collectively, the fixed rate payments on a swap are known as the *fixed leg*. Each fixed payment amount FA is determined by the notional amount N, the fixed rate R_f, and the "fixed rate day-count fraction" $FRDF$, according to the formula

$$FA = N \times R_f \times FRDF.$$

The "fixed rate day-count fraction" is the fraction of a year since the previous payment (or since the effective date of the swap). Suppose a swap has a fixed payment due on date $D_2(\mathrm{day}_2, \mathrm{month}_2, \mathrm{year}_2)$, the previous fixed payment date (or effective date if it is the first payment) was $D_1(\mathrm{day}_1, \mathrm{month}_1, \mathrm{year}_1)$. In the US the common standard for calculating fixed payments is the (30/360) basis. In this case

$$FRDF = \frac{\Delta_{360}(D_1, D_2)}{360},$$

where $\Delta_{360}(D_1, D_2)$ denotes the number of days from D_1 until D_2, calculated assuming that all months have 30 days. In other words, we have

$$\begin{aligned}\Delta_{360}(D_1, D_2) &= \max(0, 30 - \text{day}_1) + \min(30, \text{day}_2) \\ &\quad +360 \times (\text{year}_2 - \text{year}_1) \\ &\quad +30 \times (\text{month}_2 - \text{month}_1 - 1).\end{aligned}$$

Unfortunately, the swap market is unable to decide on one single standard to calculate the fixed time-fraction between dates. There are several possible standards for doing this, for example actual/360, actual/365 or as in our example (30/360). Further details on how to calculate the day-count fraction can be found in [MIR] and [HUL].

The floating rate payments:

Known collectively as the *floating leg*, standard floating rate payments are said to be "set in advance, paid in arrears". Each floating rate payment has three dates associated with it: D_s, the setting date; D_1, the date when interest start to accrue, and D_2, the payment date. The setting date is usually two business days prior to the previous floating rate payment (for the first floating payment, it is the trade date of the swap). In our application, the floating rate is the six month LIBOR rate. Each floating payment amount FLA is determined by the notional amount N, the floating rate R_{fl}, and by the "floating rate day-count fraction" $FLRDF$, according to the formula

$$FLA = N \times R_{\text{fl}} \times FLRDF.$$

Here D_1 is the effective date for the rate R_{fl}, and D_2 the floating payment date. In the US, the floating rate day-count fraction is calculated on an (actual/360) basis, which means that

$$FLRDF = \frac{D_2 - D_1}{360}.$$

At initiation of the swap, the first floating payment is known since the rate, three-month or six-month LIBOR rate depending on the floating payment frequency, is directly observable in the LIBOR market. For subsequent periods, the actual LIBOR rate is unknown and has to be estimated using implied LIBOR forward rates.

2.1.2 Pricing fixed-for-float interest rate swap

The present value of a cash flow:

The idea of discounted cashflows will be required in valuing a swap. It should be clear that \$1 received today is worth more than \$1 received after one year[2]. This is because \$1 today can immediately be invested at prevailing rates and thus earn interest. Suppose that \$1 today can be invested for n years at an interest rate of R (expressed as a decimal). Then the value of \$1 after n years is $\$1(1+R)^n$. Or putting it in another way: \$1 after n years corresponds to $\$1 \times 1/(1+R)^n$ today. The factor $D_n = 1/(1+R)^n$ is called the "discount factor" for cashflows occurring in n years time.

The idea of discount factors can be extended to periodic interest rates. Investing \$1 for half a year at the rate R gives an amount $\$1(1+R/2)$ after six months. Reinvesting this for another six months at rate R gives $\$1(1+R/2)^2$ at maturity. This is *not* the same amount that would be received investing \$1 for one year at R, reflecting the difference between semi-annual and annual compounding.

One can derive a rate S at which \$1 invested semi-annually for one year is equivalent to \$1 invested for one year at the *annual* rate R. This requires that

$$\left(1+\frac{S}{2}\right)^2 = (1+R) \quad \Rightarrow \quad S = 2[\sqrt{1+R}-1].$$

The idea generalizes so that if R_n and R_m are rates with respectively n and m compounding periods per year, then the two are equivalent when

$$R_m = m\left[\left(1+\frac{R_n}{n}\right)^{n/m} - 1\right].$$

In addition to being affected by the compounding frequency, a quoted interest rate R is also affected on what basis it is measured. For example, a one year rate quoted on (actual/360) basis will be slightly different from a one year rate quoted on (actual/365) or (30/360) basis. One can convert rates between the different bases using the following equations [MIR]

$$\begin{aligned}
R_{(\text{actual}/360)} &= \frac{360}{365} \times R_{(\text{actual}/365)}, \\
R_{(30/365)} &= \frac{D_2 - D_1}{\Delta_{360}(D_1, D_2)} \times R_{(\text{actual}/360)}, \\
R_{(30/365)} &= \frac{D_2 - D_1}{\Delta_{360}(D_1, D_2)} \times \frac{360}{365} \times R_{(\text{actual}/365)}. \qquad (2.1)
\end{aligned}$$

[2] In an economy with positive, non-zero interest rates.

In order to valuate the future cash flows for the swap, it is necessary to have discount factors F_t for future dates t. A money market discount factor, such as a factor based upon LIBOR rates, is defined as

$$F_t = \frac{1}{1 + R_{(\text{actual}/360)} \times ((D_2 - D_1)/360)}.$$

Here $(D_2 - D_1)/360$ is the day-count fraction appropriate to money market rates.

Money market rates are observable out to one year, and in order to discount cash flows beyond that date, other rates are needed. Market participants use money market rates to discount cash flows in the near future, futures rate from the Eurodollar futures market is bootstrapped for the intermediate horizon, and quoted swap rates from the swap market is used for the longer horizon. By using interpolation techniques, any discount factor can be derived for dates up to the longest observed interest rate. The actual procedure to derive discount factors from market data is outside the scope of this thesis, but can be found in [MIR].

Assuming that we are equipped with discount factors for any future date, present values PV of any future cash flow C_t at time t is calculated by discounting the cash flow back to time zero, by multiplying the cash flow C_t by the discount factor F_t, applicable to the day corresponding to t,

$$PV(C_t) = F_t \times C_t.$$

Valuing the fixed cashflows from a swap:

The fixed leg is the the simplest cash flow stream to evaluate, since the fixed cash flows are known with certainty when the swap is initiated. The evaluation is therefore just a question of discounting them back to the evaluation date, using appropriate discounting factors F_t,

$$PV(\text{fixed leg}) = \sum_t F_t \times FA_t = N \sum_t F_t \times R_f \times FRDF_t. \tag{2.2}$$

Here FA_t is the fixed cash flow amounts occurring on dates D_t, R_f the fixed rate, and $FRDF_t$ the fixed rate day count fraction applicable to D_t.

Valuing the floating cashflows from a swap:

Consider a single LIBOR cashflow C due to have the rate set at a date D_1 and paid on D_2 six months later. We know the discount factor F_1 applicable

to D_1, but we do not know what what the six-month LIBOR rate will be at D_1. This information is encapsulated in the forward rate Φ_{D_1,D_2}, which by definition is the rate one uses to discount a cashflow from D_2 to D_1. From Equation 2.1, the present value of C is then given by

$$PV(C) = F_1 \times \frac{C}{1 + (\Phi_{D_1,D_2}/100) \times ((D_2 - D_1)/360)}. \tag{2.3}$$

This present value can be re-expressed as

$$PV(C) = F_2 C, \tag{2.4}$$

since the cashflow C occurs at D_2. Equating Equations 2.3 and 2.4, and solving for Φ, gives

$$\Phi_{D_1,D_2} = \left[\frac{F_1}{F_2} - 1\right] \times \frac{36000}{(D_2 - D_1)} = \left[\frac{F_1}{F_2} - 1\right] \times \frac{1}{\alpha_{D_1,D_2}},$$

where α_{D_1,D_2} is the number of days between D_1 and D_2, divided by the number of days in the year according to the right convention, divided by 100. By knowing discount factors for any two days we can calculate the implied forward LIBOR rate for any period using Equation 2.4.

Under the assumption that market participants are able to both invest and borrow at LIBOR[3], the market expectation for the value of LIBOR, calculated on a notional amount N from D_1 to D_2 is Φ_{D_1,D_2}. Thus, the floating payment FLA implied from this becomes

$$FLA_{D_2} = N \times \alpha_{D_1,D_2} \times \Phi_{D_1,D_2} = N \times \left[\frac{F_1}{F_2} - 1\right]. \tag{2.5}$$

Since this floating payment is received or paid at time D_2, the present value of this cashflow is the right hand side of Equation 2.5, discounted by F_2,

$$PV(FLA_{D_2}) = F_2 \times N \times \left[\frac{F_1}{F_2} - 1\right] = N \times (F_1 - F_2).$$

This value can be represented as a cashflow of \$N at time D_1 and a cashflow of $-\$N$ at time D_2, see Figure 2 below.

[3]This assumption is not true. However, the economic effect of this is insignificant, due to the fact that within a standard fixed-for-float interest swap portfolio, an approximately equal principal of swaps are paying LIBOR as are receiving LIBOR.

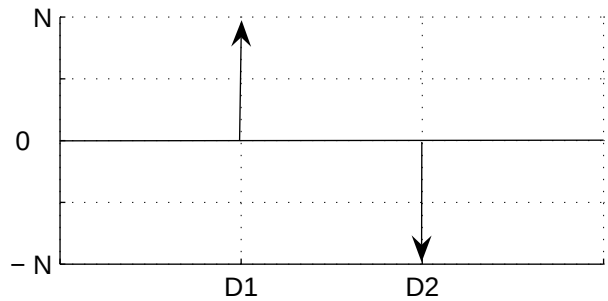

Figure 2: Single LIBOR payment representation.

Using the same methodology as above, for a two year swap with LIBOR paid semi-annually, the floating payments can be represented as shown in Figure 3 below.

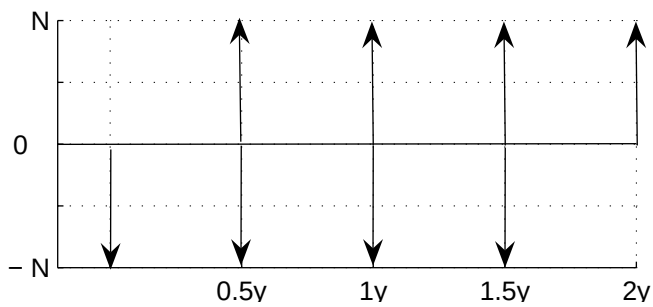

Figure 3: LIBOR payment representation for 2-year swap.

So, except at the beginning and end of a swap, the principals cancel out, leaving just two cashflows. Thus, the value of the floating LIBOR leg for this swap is just

$$FV(\text{floating leg}) = N \times (F_s - F_T), \tag{2.6}$$

where N is the notional amount, F_s is the discount factor at the start date of the swap, and F_T is the discount factor at the two year point.

Total value of the swap

The total value of the swap is the net present value NPV of the floating leg subtracted from the fixed leg. Using Equations 2.2 and 2.6, we get

$$\begin{aligned} NPV &= PV(\text{fixed leg}) - PV(\text{floating leg}) \\ &= N\left(R_f \textstyle\sum_t F_t \times FRDF_t - (F_s - F_T)\right), \end{aligned}$$

where N is the notional amount, R_f the fixed rate, $FRDF_t$ is the fixed rate day-count fraction, F_s the discount factor for the start date, and F_T the end date discount factor. The index t runs over all floating payments which may be quarterly or semi-annual. In many trading situations, it is not the present value itself which is of most relevance, but the fixed rate at which the trader will deal. When a swap is initiated, the fixed rate is set such that the NPV is zero. Using Equation 2.6, this implies that

$$R_f = \frac{F_s - F_T}{\sum_t F_t \times FRDF_t}.$$

Here R_f is the fixed rate that value the T-year swap to zero or par, or, in other words, the T-year swap par rate.

2.1.3 Interest rate exposure

A trader that is holding a swap position will most likely be concerned with two things: The value of the swap and how that value is likely to change. While it was explained in some detail in Section 2.1.2 how to calculate the value of a swap, we will here explain the swap values sensitivity to movements in interest rates.

Suppose that a trader entered into a swap agreement to pay fixed the five-year swap rate and receive float on some notional amount. The swap value at initiation was zero. A few days after the trade was entered, swap rates including the five year rate fell, and the NPV of the swap position fell. The reason for this is that a new swap position to pay fixed at a lower rate would have an advantage compared to the original swap, which is reflected in a new lower NPV of the original swap.

Suppose instead that the three, four and seven-year rates were to fall, but the five-year rate remain unchanged. What would now be the value of the swap? The answer is that it would be unchanged at zero! This is because the swap still is a par swap, and still being at the prevailing five year market rate. This is not to claim that present values of future cashflows remain constant. They do change, but taken together, the changes cancel out.

The vector Δ with \$-sensitivities for swaps is called the delta-vector. It specifies the \$-sensitivity of the value of swaps with respect to a one basis point 0.01% change in swap par-rates. The elements in the Δ-vector δ is derived by taking first derivative of the value of a par swap with tenor n years with respect to swap par-rates R. The delta for a n-year par swap is

given by [MIR]

$$\delta_n = -N \sum_{i=6m}^{nY} \alpha_i F_i.$$

A delta is not static in any way: It changes as time goes by, when both accrual-factors α_i and discount factors F_i change. A comprehensive investigation of risk measures for swaps such as the delta can be found in [MIR]. The swap delta will be used in the sequel as a tool for structuring the butterfly strategy in swaps.

2.1.4 The butterfly strategy in swaps

The market for swaps is very liquid but irregularities occasionally do occur. Traders are very quick to exploit such opportunities and have developed strategies to isolate the misaligned factor without taking more risk than necessary. One of these strategies is called the butterfly.

In the fixed income market, the typical butterfly trade consists of taking a long (short) position in a medium maturity security and a short (long) position in a shorter maturity security together with another short (long) position in a longer maturity security [MER]. For example, if a trader believes that the 10-year bond is relatively cheap compared with the 5-year note and the 30-year bond, the trader will buy the 10-year and sell short the 5 and 30-year bonds. By choosing the respective weights correctly, the trader can hedge against changes in interest rate level and slope risks and stay exposed to changes in curvature. The assumption is that the cheap 10-year bond will revert to a higher normal value relatively to the 5- and 30-year bond.

An equivalent butterfly trade consisting of fixed-for-floating interest rate swap instruments would be to pay (receive) fixed in the middle tenor and receive (pay) fixed in the shorter and longer tenor. For example, a trader believes that the 7-year par swap-rate is relatively low compared to the 2-year and 10-year par swap-rates. The trader can then enter into three swap contracts and pay fixed (go short) in the 7-year contract, and receive fixed (go long) in the 2-year and 10-year swap contracts. What the trader hopes for is that the 7-year swap rate will revert to its higher fair level.

By carefully deciding proportions of how much delta-risk to take in each swap position, the value of the butterfly will stay unaffected by both changes in the level and slope of the curve [MER]. The butterfly will only be affected by curvature changes that correspond to changes in the relative level between the 2-year, 7-year, and 10-year swap rate. A crucial question is how to get

the "right" proportions. Traders have developed several weighting schemes to achieve this. For example by splitting delta-risk equal between the two wings and the middle, resulting in hedges unaffected by parallel shifts in the curve, but still exposed to shifts in slope and curvature.

The common base for all the weighting techniques above is that they are based upon historical data. In other words, they assume that interest rate patterns will repeat in the future. Sometimes, they do not, and then the underlying assumptions break down. Another problem that traders face is how to identify what sectors to buy or sell when they want to establish a butterfly position. The fundamental issue here is the ability to analyze and identify in which sectors to pay or receive fixed. Traditionally, traders have compared market data to some more or less realistic model, and identified rich/cheap sectors when marked data deviates from model stipulated levels. As we will see in the sequel, PCA can be used as a tool to identify abnormal curve patterns as well.

2.2 Principal component analysis

The central idea of PCA is to reduce the dimensionality of a data set consisting of a large number of interrelated variables while retaining as much variation of the original data set as possible. The underlying data in our application will be time series of swap rates, and PCA will be applied to the covariance matrix of this time series data.

PCA is accomplished by transforming the covariance matrix of the original data into a new set of variables, called principal components. PCA carries out this decomposition of the covariance matrix by conducting a matrix diagonalization or eigen-decomposition of the original covariance matrix of rates into eigenvalues and eigenvectors. As a result of construction, the principal components are uncorrelated, in contrast to the original variables, and can be ordered in importance such that the first few components explain the majority of the original variation. Definitions in the following subsections can be found in [HOL] and [WOL].

2.2.1 Principal components

Let $\boldsymbol{R}$ be a set of swap rate observations, and let $r_i^t \in \boldsymbol{R}$ denote a single swap rate observation at time $t \in \{1, \ldots, T\}$ for tenor $i \in \{1, \ldots, n\}$. We define the matrix of centered swap rate observations $\tilde{\boldsymbol{R}}$ as the matrix with elements

$$\tilde{r}_i^t = r_i^t - E\left[\boldsymbol{r_i}\right], \tag{2.7}$$

where the expectation is taken over the times $t \in \{1, \ldots, T\}$.

Let $\boldsymbol{C}$ be the covariance matrix of $\tilde{\boldsymbol{R}}$, with elements given by

$$c_{i,j} = E\left[\tilde{\boldsymbol{r}}_{\boldsymbol{i}} \tilde{\boldsymbol{r}}_{\boldsymbol{j}}\right] - E\left[\tilde{\boldsymbol{r}}_{\boldsymbol{i}}\right] E\left[\tilde{\boldsymbol{r}}_{\boldsymbol{j}}\right].$$

Assume that $\boldsymbol{C}$ has non-degenerate eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding linearly independent eigenvectors $\boldsymbol{X_1}, \ldots, \boldsymbol{X_n}$, so that

$$\boldsymbol{C}\boldsymbol{X_i} = \lambda_i \boldsymbol{X_i}.$$

We form the matrices

$$\boldsymbol{P} = [\boldsymbol{X_1}, \boldsymbol{X_2}, \ldots, \boldsymbol{X_n}] = \begin{bmatrix} x_{1,1} & x_{1,2} & \ldots & x_{1,n} \\ x_{2,1} & x_{2,2} & \ldots & x_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n,1} & x_{n,2} & \ldots & x_{n,n} \end{bmatrix}$$

of eigenvectors or principal components of $\boldsymbol{C}$, and the diagonal matrix

$$\boldsymbol{D} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

with the corresponding eigenvalues. Then we have

$$\boldsymbol{C} = \boldsymbol{P}\boldsymbol{D}\boldsymbol{P}^{-1}.$$

The elements $x_{i,j}$ of $\boldsymbol{P}$ are called factor loadings and quantifies the sensitivity of each variable $\tilde{\boldsymbol{r}}_{\boldsymbol{i}}$ to changes in each principal component. By construction, the eigenvector matrix is orthogonal, so that the eigenvectors form an orthogonal base. The eigenvalues hold information about how much variation of the underlying data can be explained by each eigenvector. By ordering the eigenvectors in the order of descending eigenvalues (largest first), we can create an ordered orthogonal basis with the first eigenvector having the direction of largest variation of the data. In this way, we find directions in which the data set has the most significant amounts of variation. The amount of explanatory power can be derived by relating an eigenvalue to the sum of all eigenvalues as

$$\text{Variance explained}_i = \frac{\lambda_i}{\sum_j \lambda_j}. \tag{2.8}$$

2.2.2 Fair swap curve

When carrying out a PCA-analysis, we always get the same number of eigenvectors (or principal components) and eigenvalues as the number of tenors included in the covariance matrix. With the new orthogonal basis derived in Section 2.2.1, we can express the time-series data $\tilde{\boldsymbol{r}}_{\boldsymbol{i}}$ as coordinate vectors $\boldsymbol{y}_{\boldsymbol{i}}$, in the new orthogonal basis defined by the eigen-vector matrix $\boldsymbol{P}$, as

$$\boldsymbol{y}_{\boldsymbol{i}} = \boldsymbol{P}^T \tilde{\boldsymbol{r}}_{\boldsymbol{i}}. \tag{2.9}$$

We can reconstruct the original data vector $\tilde{\boldsymbol{r}}_{\boldsymbol{i}}$ from $\boldsymbol{y}_{\boldsymbol{i}}$ by

$$\tilde{\boldsymbol{r}}_{\boldsymbol{i}} = \boldsymbol{P}\boldsymbol{y}_{\boldsymbol{i}}. \tag{2.10}$$

Instead of using all the eigenvectors of the covariance matrix, we may represent the data in terms of only a few basis vectors of the orthogonal basis. Remember that we have the eigenvectors ordered by the magnitude of the eigenvalue. If we denote the matrix having the k first eigenvectors as rows by $\boldsymbol{P}_{\boldsymbol{k}}^T$, we can create a transformation similar to that in Equations 2.9 and 2.10, as

$$\boldsymbol{y}_{\boldsymbol{i}} = \boldsymbol{P}_{\boldsymbol{k}}^T \tilde{\boldsymbol{r}}_{\boldsymbol{i}}, \tag{2.11}$$

$$\tilde{\boldsymbol{r}}_{\boldsymbol{i}} = \boldsymbol{P}_{\boldsymbol{k}} \boldsymbol{y}_{\boldsymbol{i}}. \tag{2.12}$$

As we will see in the results of our investigation, most of the variation of the original data can be explained by a relatively small number of principal components. By assuming that the "relevant variation" of the data can be explained by only two principal components, we can recover what we call "fair" centered observations $\tilde{\boldsymbol{r}}_{\boldsymbol{i}}^{\boldsymbol{f}}$ for each tenor i, using Equation 2.12 with $k = 2$. If we also add the mean of $\boldsymbol{r}_{\boldsymbol{i}}$, we get the fair observation as

$$\boldsymbol{r}_{\boldsymbol{i}}^{\boldsymbol{f}} = \boldsymbol{P}_{\boldsymbol{k}} \boldsymbol{y}_{\boldsymbol{i}} + E\left[\boldsymbol{r}_{\boldsymbol{i}}\right],$$

which we can compare to the original observed rates.

We define the residual vector for tenor i by

$$\boldsymbol{RES}_{\boldsymbol{i}} = \boldsymbol{r}_{\boldsymbol{i}} - \boldsymbol{r}_{\boldsymbol{i}}^{\boldsymbol{f}}. \tag{2.13}$$

If we calculate the residual each day, then we get a measure of how much a swap-rate deviates from its fair level. Assuming that such deviations tend to be mean reverting, we can use the residual to find sectors of the swap curve with abnormally high deviations and put on strategies that profit from such reversions. This technique will become clear in Chapter 3.

2.2.3 The 3rd factor as relative value indicator

Assume that we do a principal component analysis on a covariance matrix where the underlying data is time-series data for three tenors only. We will get three eigen-vectors with corresponding eigenvalues. We order the eigenvectors by importance, so that the first principal component account for the majority of the variation of the underlying data. Since the eigenvectors are perpendicular, the second principal component describe variation that cannot be accounted for by the first factor, and the third factor describe the variation that cannot be accounted for by the first and second factor.

Taken together, the three components pertain to explain all variation of the data. Using Equation 2.9, we can derive coordinates for the data in the orthogonal basis defined by the three eigenvectors on each day. Assuming we are able hedge the exposure of a butterfly strategy to coordinate changes in the first and second factor, changes in the third factor coordinate is the only source of risk in the strategy. If the third factor coordinate also have mean reverting properties, it can be used as an indicator on relative value. In the Chapter 3 we will use real market data and check if this also holds in a real-world application.

2.3 Weighting methodology

To compare the fair swap-curve to the actual, and track the third factor coordinate, are both potential tools to identify sectors/tenors to invest when constructing a butterfly. The next step is to decide how to weight the butterfly. That is to decide how much notional capital to invest, or risk to take in each of the three components of the butterfly. The purpose of the strategy is to profit from when the curve reverts from extreme to normal levels of curvature. In order to achieve this, the strategy needs to be hedged against changes in first and second factor coordinates, and stay exposed to changes in third factor coordinates. In order to specify this trade we will need the information displayed in Table 1 below.

	Notional	**Delta**	**Factor 1**	**Factor 2**
Wing 1	N_{w1}	δ_{w1}	$x_{w1,1}$	$x_{w1,2}$
Middle	N_m	δ_m	$x_{m,1}$	$x_{m,2}$
Wing 2	N_{w2}	δ_{w2}	$x_{w2,1}$	$x_{w2,2}$

Table 1: Data required for weighting butterflies.

In Table 1, "Wing 1", "Middle" and "Wing 2" are the tenors of the swap, while "Notional" is the notional dollar-amount in each swap, and "Delta" its dollar sensitivity to a one basis point change in the corresponding swap rate. The last two columns list the factor loadings derived from principal component analysis on tome series for three tenors.

As explained earlier, the factor loadings are also measures of sensitivity for each corresponding swap-rate to changes in the first and second factor. In order to weight the butterfly strategy we set up the following two equations, that ensure that the butterfly is hedged against the first and second factor,

$$N_{w1}\,\delta_{w1}\,x_{w1,1} + N_{w2}\,\delta_{w2}\,x_{w2,1} \;=\; N_m\,\delta_m\,x_{m,1}, \tag{2.14}$$

$$N_{w1}\,\Delta_{w1}\,x_{w1,2} + N_{w2}\,\delta_{w2}\,x_{w2,2} \;=\; N_m\,\delta_m\,x_{m,2}. \tag{2.15}$$

Here Equation 2.14 gives the total sensitivity of the wings to the middle for the first factor, and Equation 2.15 the total sensitivity of the wings to the middle for the second factor. Notice that we have three unknown N_{w1}, N_{w2} and N_m, but only two equations.

If we transform Equations 2.14 and 2.15 by dividing the left and right-hand side by $N_m\,\delta_m$, we get the set of equations

$$\tilde{w}_{w1}\,x_{w1,1} + \tilde{w}_{w2}\,x_{w2,1} \;=\; x_{m,1}, \tag{2.16}$$

$$\tilde{w}_{w1}\,x_{w1,2} + \tilde{w}_{w2}\,x_{w2,2} \;=\; x_{m,2}, \tag{2.17}$$

where $\tilde{w}_{w1}$ and $\tilde{w}_{w2}$ are the risk-weights, defined as

$$\tilde{w}_{w1} = \frac{N_{w1}\,\delta_{w1}}{N_m\,\delta_m} \quad \text{and} \quad \tilde{w}_{w2} = \frac{N_{w2}\,\delta_{w2}}{N_m\,\delta_m}.$$

In the transformed system of Equations 2.16 and 2.17, we have only two unknown variables $\tilde{w}_{w1}$ and $\tilde{w}_{w2}$. The solution is

$$\tilde{w}_1 = \frac{x_{m,1}\,x_{w2,2} - x_{m,2}\,x_{w2,1}}{x_{w1,1}\,x_{w2,2} - x_{w2,1}\,x_{w1,2}}, \tag{2.18}$$

$$\tilde{w}_2 = \frac{x_{m,2}\,x_{w1,1} - x_{m,1}\,x_{w1,2}}{x_{w2,2}\,x_{w1,1} - x_{w2,1}\,x_{w1,2}}. \tag{2.19}$$

2.4 Tracking the butterfly trade

The butterfly spread S, which is the risk-weighted dispersion of the three swap-rates that constitute the butterfly, is defined as [MER]

$$S = r_{w1}\,\tilde{w}_{w1} + r_{w2}\,\tilde{w}_{w2} - r_m. \tag{2.20}$$

The butterfly spread can be used as a measure of performance for the butterfly. For example, if the trade is set to profit from narrower swap spread, that a drop in the spread S is directly related to the profit/loss of the strategy.

An approximate profit/loss is calculated by multiplying the change in spread Δ_S to the total delta-sensitivity assumed in the middle swap position for the butterfly [MER], giving

$$\text{profit/loss} = \Delta_S \, N_m \, \delta_m.$$

The current butterfly spread can also be related to historical spreads of the same weighting scheme by observing the deviation of the current spread to its historical average. By transforming the spread observation at time t into a standardized normal variable Z, we define the Z-score as

$$Z\text{-score} = \frac{S_t - E\left[S\right]}{\sigma_S}.$$

2.5 Data

The underlying data set we will use for our investigation comes from one of the major investment banks on Wall Street. We have daily swap-rate observations for the 2-year up to 30-year tenor with yearly intervals. The time series starts Dec 1, 1995 and ends June 22, 2004.

Chapter 3

Results

In this chapter, we will aim to provide the reader with a practical example how PCA can be applied to butterfly strategies. Analysis is based upon swap rate data between Dec 1, 1995 to July 11, 2004, and specifically we will analyze the butterfly consisting of the 2-year, 5-year and 10-year swaps.

3.1 The delta-vector

The swap delta calculation can be done in Matlab using the fixed income derivatives toolbox, but there are similar packages in FinCad that give comparable results. The results in Table 2 below assume $10,000 notional.

Tenor (yrs)	**Delta**
2	$1.928
3	$2.828
4	$3.708
5	$4.543
6	$5.337
7	$6.094
8	$7.809
9	$7.486
10	$8.125
12	$9.302
15	$10.822
20	$12.820
30	$15.445

Table 2: Selected swap sensitivities as of July 11, 2003.

3.2 Eigenvectors as principal components

First we calculate the centered swap rates for each day between Jan 1, 1995 and July 11, 2003 for the 2-year, 5-year and 10-year swap rates using Equation 2.7. The results are displayed in Figure 4 below.

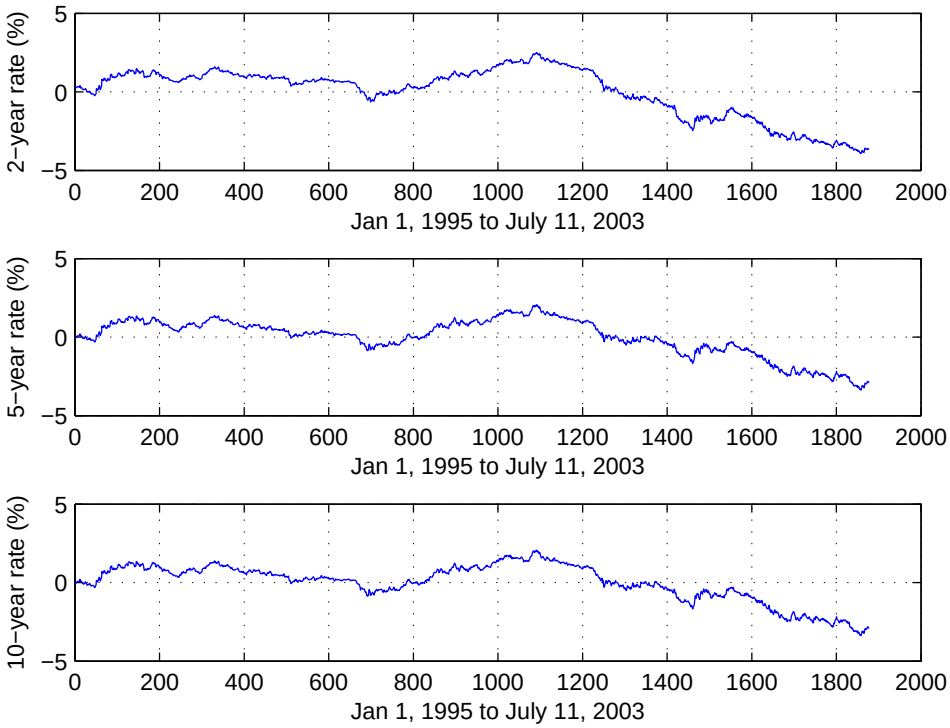

Figure 4: Daily centered swap rates for the 2, 5 and 10-year swap rates.

In order to see how these three variables relate to each other, we calculate the covariance matrix according to Equation 2.7, the result of which is given in Table 3 below.

	2-year	**5-year**	**10-year**
2-year	2.533	1.855	1.392
5-year	1.855	1.409	1.084
10-year	1.392	1.084	0.851

Table 3: Covariance matrix for the 2, 5 and 10-year swap rate.

The diagonal elements of Table 3 give the variance for each swap rate, while the off-diagonal elements describe how the swap rate movements relate to each other. We see that there is a stronger relation between the 2-year and 5-year rate, compared to the 5-year and 10-year, as well as between the 2-year and 10-year rate. By applying the command `pcacov(C)` in Matlab, we do a principal component analysis on the covariance matrix above. We get $\boldsymbol{P}$, the eigenvector matrix, and $\boldsymbol{D}$ the diagonal matrix with corresponding eigenvalues, λ_1, λ_2 and λ_3, see Table 4 below.

Tenor	$\boldsymbol{X}_1$	$\boldsymbol{X}_2$	$\boldsymbol{X}_3$
2-year factor loadings	-0.728	0.639	0.247
5-year factor loadings	-0.545	-0.321	-0.774
10-year factor loadings	-0.415	-0.699	0.582
Eigenvalue	4.725	0.773	0.0024
Variance explained	98.35%	1.61%	0.04%

Table 4: PCA results for the 2-5-10 covariance matrix.

Each factor loading in Table 4 describes how the corresponding rate would change if we had a one-unit change in respective factor coordinate. The factor loadings for the first eigenvector are all negative and relatively equal in magnitude. Therefore, if we had a 10 basis point increase in the first factor coordinate, the 2-year rate would decrease by approximately 7.3 basis points, the five-year by 5.5 basis points, and the 10-year rate by 4.2 basis points. The implication roughly correspond to a parallel downward shift in the curve. The second factor imply the curve to flatten, since the 2-year rate would increase and the 5-year and 10-year rate decrease. In other words the second factor describe how the slope changes. The third factor show rates going in opposite directions where the 2-year and 10-year rate would increase and the 5-year decrease. In other words, the third factor describe changes in curvature of the swap curve. In the last two rows of Table 4, we see the eigenvalues as well as the percentage variance explained,

calculated according to Equation 2.8. We see that a major portion 98.35% of the variation can be explained by roughly parallel shifts of the curve, 1.61% is attributable is slope changes and only 0.04% to curvature.

3.3 The third factor

We derived the formula for the coordinate of the third factor in Equation 2.8 and suggested that it can be used as a relative value indicator to identify events of abnormal curvature (extreme level of factor three coordinate). Figure 5 below depicts the third factor coordinate on daily basis from Jan 1, 1995 to July 11, 2003.

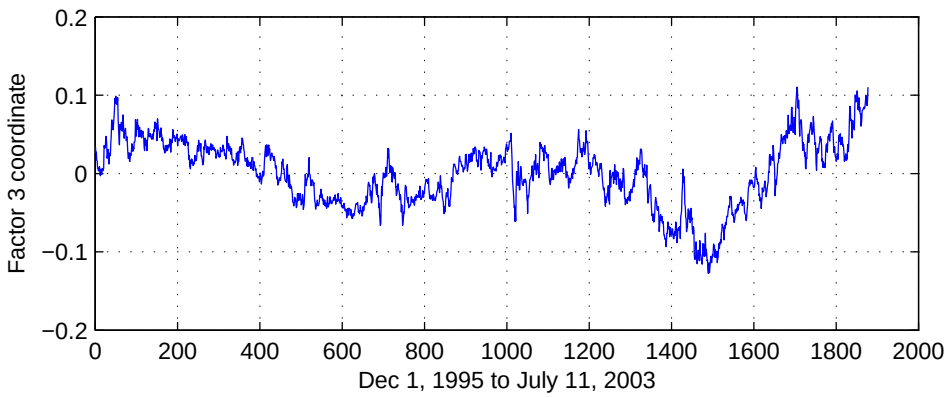

Figure 5: Daily observations of the third factor coordinate.

In Figure 5 we see that coordinate seem fairly centered around zero. If we calculate the mean of the observations above we indeed get a number very close to zero, ($\approx 10^{-16}$). The mean reverting property of the third factor can be used to find a profitable butterfly strategy. When the coordinate of factor three reaches a particular high or low level for a set of three swaps, it indicates that the curve has unusual curvature given the level and steepness of the curve. The purpose of a butterfly strategy is to exploit such curve patterns. The problem is then to weight the butterfly such that its value stays unaffected by changes in the first and second factor coordinate but gain when the third factor coordinate revert to zero.

3.4 Implied fair swap curve

Instead of applying PCA on a set of three swap rates, we can do the same analysis on a larger covariance matrix including the 2 to 10-year with annual intervals, 12, 15, 20, 25 and 30-year tenors. We get the same number of principal components as number of tenors included in the covariance matrix. As we observed when we applied PCA to the three-rate example, the first two or three principal components often capture most part ($\approx 99.9\%$) of the variance. Instead of presenting all the factor loadings in a table, we visualize the first three principal components in Figure 6 below. The analysis is based upon time series between Jan 1, 1995 to July 11, 2003.

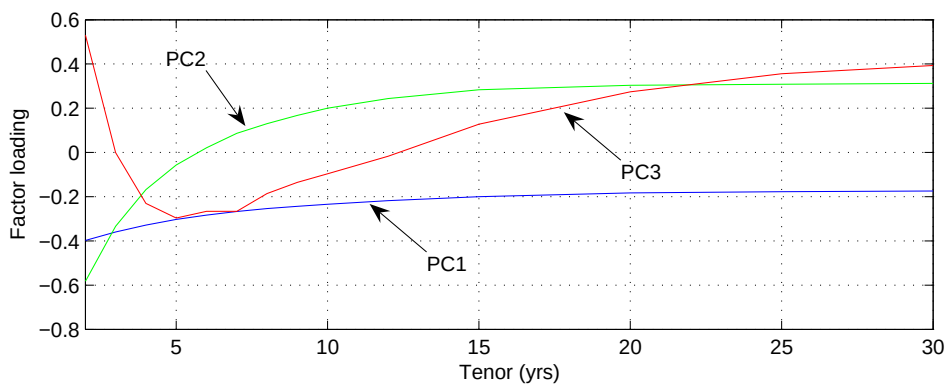

Figure 6: Factor loadings for 14-node case.

Factor loadings for the first principal component imply roughly a parallel shift in the curve, the second factor a change in the slope, and the third factor a change in curvature. The variance of the underlying data explained by the first three principal components are as shown in Table 5 below.

	$\boldsymbol{X}_1$	$\boldsymbol{X}_2$	$\boldsymbol{X}_3$
Eigenvalue	15.310	0.299	0.015
Variance explained	97.98%	1.91%	0.10%
Cumulative explained	97.98%	99.89%	99.99%

Table 5: PCA results for 14-node case.

Even though we have 14 principal components in total, we only need two of them to explain 99.89% of the variation of the underlying data. This is

a powerful feature of PCA, because it reduces the dimensionality of factors needed to explain the variation without loosing much information about the underlying data.

As we explained in Chapter 2, the principal components can be used to recover the actual swap rate observations at each day, by Equations 2.9 and 2.10. We are interested in identifying sectors of the swap curve that are higher or lower than what we defined as the fair curve (Equations 2.11 and 2.12). We have defined the fair curve as the curve recovered by only two principal components. Figure 7 below displays the actual and fair curve, as well as the residual (Equation 2.13), as of July 11, 2003.

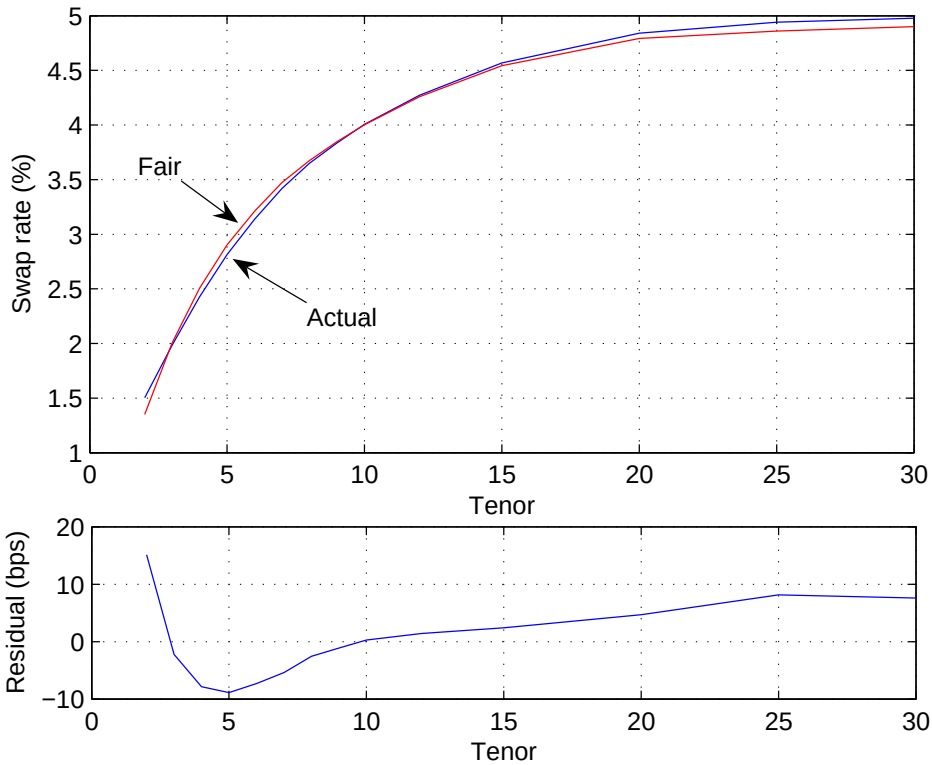

Figure 7: Fair curve and residuals for 14-node PCA.

In the upper plot in Figure 7, both the actual and fair swap curve are plotted. We see that the fair curve closely follows the actual curve. In the lower plot we have the residual, and we see that the 2-year swap rate is higher than the fair level. Intermediate tenors such as the 3-7 year are below, while the

long end of the curve are higher than fair.

According to this analysis, it would be beneficial to enter a butterfly where we receive fixed and pay the floating rate in the 2-year and 10-year swap, while paying fixed and receive float in the 5-year swap. What we expect is that the 2-year and 10-year swap rate will decrease to its fair level. We would then receive the high fixed rate, and pay floating on declining rates. We also expect the 5-year rate to pick up from its relatively low level, gradually increase our received floating rate payments.

We earlier plotted the third factor coordinate for the same butterfly and that result is also related to Figure 7. A historically high observation of the third factor translates into a larger gap between the actual and fair swap curve or in other words, a larger residual for those tenors included in the swap butterfly. An extremely high level of factor three coordinate for a specific day and butterfly indicates that it ought to be advantageous to pay fixed and receive float the middle tenor, and receive fixed and pay float on the wings of the butterfly. An abnormally low third factor coordinate indicate that it is advantageous to receive fixed and pay float in the middle tenor, while paying fixed and receive float on the wings of the butterfly.

3.5 PCA weighting scheme

Now when we have two methods of finding values on the swap curve, comparing the actual to the fair swap curve, and track the third factor coordinate for various butterfly structures, we need to calculate appropriate weights for the butterfly. As shown in Chapter 2, we solve the system of Equations 2.18 and 2.19 to get the weights such that the butterfly stay hedged against changes in the first and second factor coordinates. The important thing is to use factor loadings derived from PCA on three nodes. Factor loadings derived from PCA on an entire curve such as in the fair curve calculation, will be slightly different from PCA on three nodes. This is due to the fact that PCA on a larger set of data consider the dynamics of more variables (nodes), and that will be reflected in factor loadings as well.

Table 6 below shows the results for calculating risk-weights, spread and Z-score for the 2-5-10 butterfly. The underlying covariance matrix is calculated on daily swap rates from Jan 1, 1995 to July 11, 2003.

	2-year	**5-year**	**10-year**	**Spread**	**Z-score**
Risk weight	31.95%	100%	75.21%	67.83	2.63

Table 6: Risk-weights, Spread and Z-score for the 2-5-10 butterfly.

In Table 6 we see that in order for the butterfly to stay hedged for coordinate changes in factor one and two, we need to have approximately 32% of the middle delta-risk invested into the 2-year swap, and 75.21% of the risk in the 10-year swap. We also see that the current spread is equal to 67.83, and if we put this spread in relation to historical spreads for the same butterfly and weighting scheme, we get a Z-score equal to 2.63. Before we make any conclusions based on the Z-score, we have to check that the spreads are normal distributed. One way of doing this is to plot quantiles of a normal distribution against quantiles of the sample of spreads S_t, as is done in Figure 8 below. If the distribution of S is normal, then this plot should be linear.

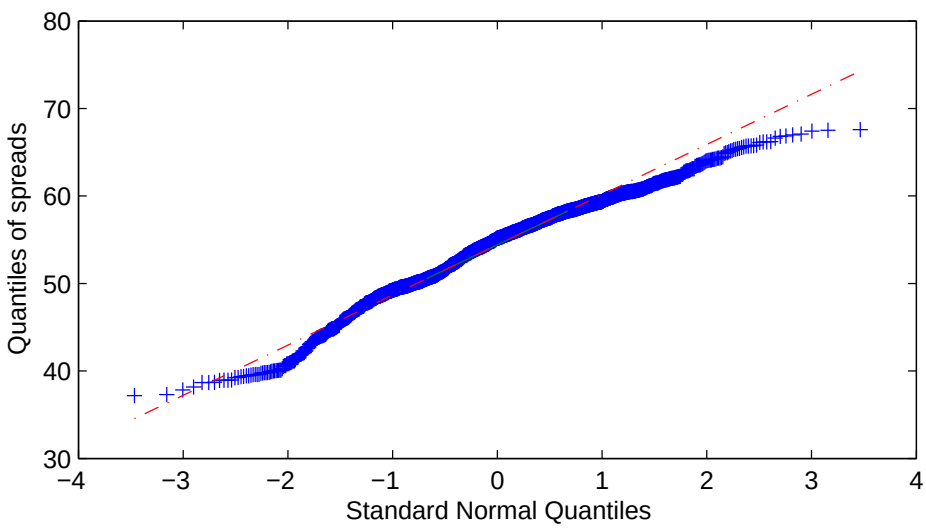

Figure 8: Normal quantile-quantile plot for 2-5-10 butterfly spreads.

In Figure 8 we see that spread observations are close to linear, and we conclude that our Z-score of 2.63 is a relatively rare observation.

3.6 Tracking the butterfly strategy

Assume that we entered into the swap butterfly consisting of going long the 2-year and 10-year swaps, and short the 5-year swap on July 11, 2003. Further, assume that the components of the butterfly are weighted according

to the results in Section 3.5. As we described in Chapter 2, there are several ways to track the performance of the butterfly after the trade is done. Ways to track the butterfly is by measuring the butterfly spread, its Z-score and the third factor coordinate on each day following the trade date of the butterfly. The most important however is to track the unrealized profit/loss of the strategy. In Figure 9 below, we have tracked the Z-score, butterfly spread and third factor coordinate, before and after the trade date. The break between the two periods is shown as a solid vertical line, dividing the plots into two sections.

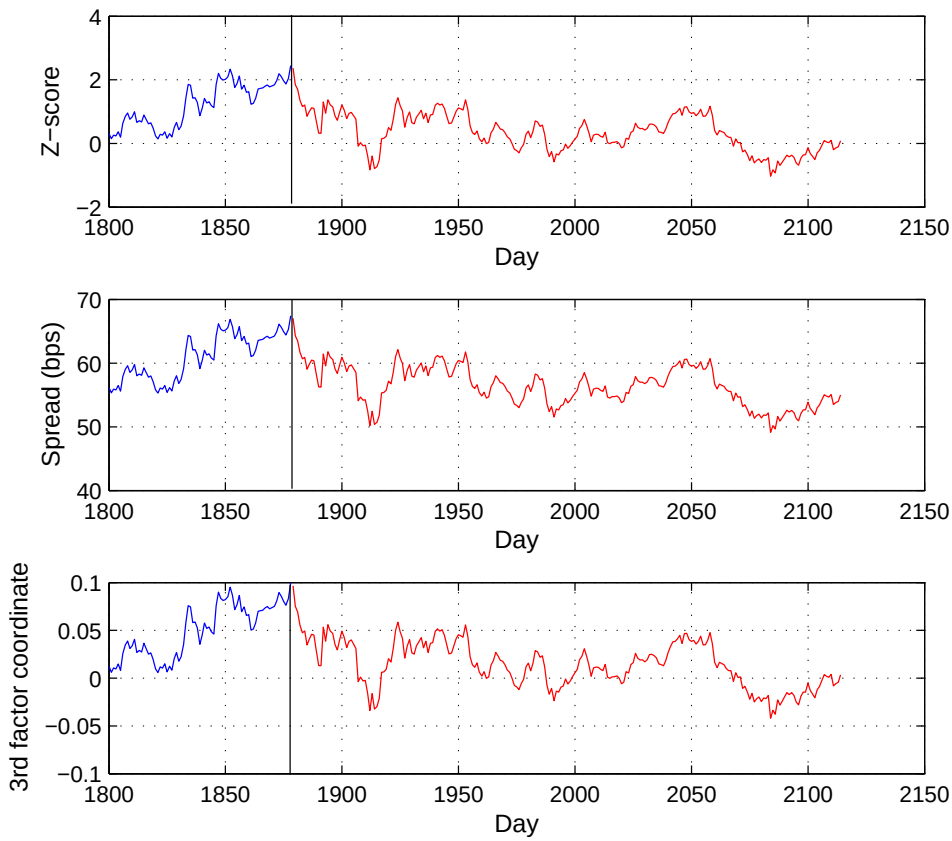

Figure 9: Z-score, spread and third factor coordinate.

In Figure 9 we see that all three tracking tools agree quite well.

Figure 10 below depicts the approximate profit/loss according to Equation 2.20, if we assume a $100 million notional for the middle position. Another assumption we have done is that the delta-vector stays constant after the trade date. This assumption is not unreasonable since swap deltas seem to be relatively stable over time.

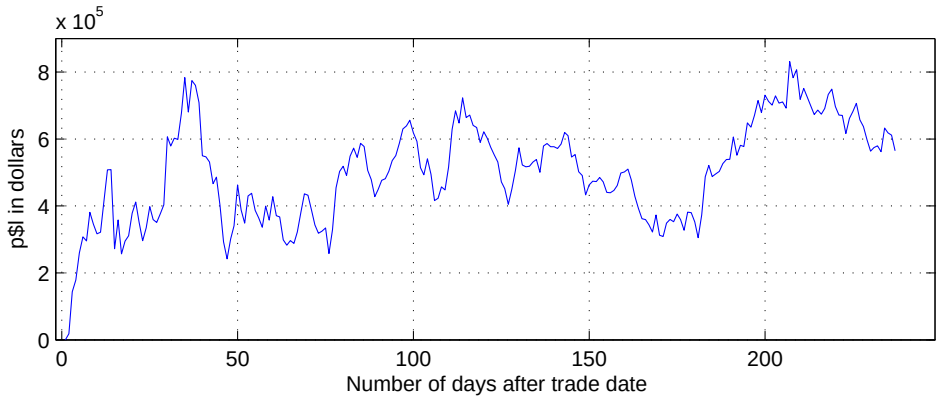

Figure 10: Total profit/loss after trade date.

If we compare Figures 9 and 10, we see that the profit/loss in Figure 10 is roughly the inverse of any of the curves in Figure 9 (after trade date). The profit increases after the trade date and reaches local maximum 35, 37, 114 and 207 days after trade date. Possible levels to lock in the profit are in the range of $\$(600 - 800)$ thousand.

Chapter 4

Conclusion

We have shown that principal component analysis can be used as a tool for finding relative value on the swap curve, but also as a tool for weighting butterfly strategies. The major benefit of PCA is the reduction in the number of factors that explain variation of the original time series data. The consideration of factors such as level, slope, and curvature enables the trader to hedge against chosen factors and stay exposed to others. The methodology of re-creating the swap curve using two factors enables the trader to explore for what tenors the third and higher factors have significance, and decide if the residual is something that can be traded or not.

PCA works on the premise that the relationship that held in the past will also be present in the future. There is no guarantee that the covariance matrix for a set of swap rates will stay the same. Empirical analysis also show that covariance matrices differ significantly depending on the chosen time window to calculate the underlying covariance matrix. Principal components, third factor coordinates, and "fair" curves, as well as butterfly weights, are therefore highly dependent on the time window that is used.

Some analysts argue that the longer time period that is used preceding the trade, the better. Then the analysis will capture many different interest rate environments but might not be applicable to the regime the trader faces at the moment. Others say that one should try to identify the current interest rate regime, and base the analysis from that time period only. I believe further empirical analysis may bring some clarity to this question.

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