Some aspects of Lévy processes in finance

Master's Thesis

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Abstract

This thesis deals with two aspects of Lévy processes in finance, namely change of time to model dependency in logreturns of financial data, and option pricing where change of measure is necessary. It is shown that a change of time with the so called integrated Cox Ingersoll Ross process, can be used to model dependency of squared logreturns of financial data. Option prices are compared in different Lévy models, using the Esscher change of measure.

Acknowledgements

First of all, I would like to thank my supervisor Patrik Albin for his wise supervisionship, and for the time he has taken to answer my questions. Also, I want to thank him for encouraging me to learn some serious probability theory. Then I would like to thank Mattias Bengtsson and Viktor Olsbo for fruitful discussions, and Erik Brodin for some ideas to this thesis. Last but not least, the author would like to thank his parents for their support.

Contents

1 Introduction

In this section we present some history and ideas. Formal definitions are given later. Throughout the thesis, the parameter t denotes time, where $t \in [0, T]$ for some $T > 0$. We assume that all random variables and stochastic processes are defined on a fixed and common probability space $(\Omega, \mathfrak{F}, \mathbf{P})$.

Let us denote by $S(t)$ the price process of a financial asset, for example a stock or an index. Over the years, several models have been proposed for $S(t)$. The most classical and widely used model is the so called Bachelier-Samuelson model, which is given by

$$S(t) = S(0)\mathrm{e}^{\mu t + \lambda W(t)}. \tag{1}$$

Here μ, $\lambda \in \mathbb{R}$ while $W(t)$ is a standard Wiener process, also known as standard Brownian motion. Actually, Bachelier's original model was the logarithm of the above model. However, this meant that there was a positive probability that the price of the modelled asset would become negative, which obviously is not a desirable property. Several years after Bacheliers model, Samuelson came up with the important idea to take the exponent. The Bachelier-Samuelson model is very popular, since it is easy to handle, and implies many nice formulas, for example in the theory of option pricing. However it is also well-known that the model has several unwanted properties, the most important of which are:

1. The increments of $\log(S(t))$ are normal distributed;

2. The increments of $\log(S(t))$ over disjoint intervals are independent.

The first of these properties implies that the increments are very light-tailed, that is, the probability for a large increment or decrement is small. Also it means the distribution of the increments of $\log(S(t))$ cannot be skewed.

It is possible to overcome this by replacing the Bachelier-Samuelson model with a more general model, which is given by

$$S(t) = S(0)\mathrm{e}^{X(t)}, \tag{2}$$

where $X(t)$ is a Lévy process. The Wiener process is the most important example of a Lévy process. Other examples of Lévy processes are the well-known normal inverse Gaussian process (NIG), and the less well-known Meixner process. We shall discuss these later.

Still, by the definiton of a Lévy process, the model (2) has the unwanted property that $\log(S(t))$ has independent increments. Several models have been proposed to overcome this disadvantage. One of these is the so called time-changed exponential Lévy model, given by

$$S(t) = S(0)\mathrm{e}^{X(\tau(t))}, \tag{3}$$

where $X(t)$ is a Lévy process and $\tau(t)$ is an increasing continuous process, that is independent of X. A slight modification of the above model, which is a bit easier to handle, is

$$S(t) = S(0)\mathrm{e}^{\mu t + X(\tau(t))}. \tag{4}$$

Here $\mathbf{E}\{X(t)\} = 0$ and $\mu \in \mathbb{R}$ while the rest is as before. Typically the "right" τ will result in dependent increments in the models 3 and 4.

The pricing of financial derivatives is an important subject. For this, it is necessary to find a so called equivalent martingale measure. In the thesis will we describe such a measure, and compare prices obtained with three different Lévy models.

The thesis is organized as follows: First we compare the fit of the normal, NIG and Meixner distributions to the logreturns of financial data.

Then we study how certain time changes can model the covariance structure of logreturns of financial data, and present som appealing properties of the time changed Lévy model.

Finally, we compare the prices of European call options in the Lévy models from the first section. Relevant facts and definitions are presented along thet way at the location where they are first needed.

2 Elements of Lévy processes

First of all, we define the concept of Lévy processes:

Definition 1 (Lévy Process) *A stochastic process $L = \{L(t)\}_{t\geq 0}$ is called a* Lévy process *if it has the following properties:*

1. *$L(0) = 0$ with probability one;*
2. *L has independent increments, that is, for $0 < t_1 < t_2 < ... < t_n$, the random variables $L(t_1)$, $L(t_2) - L(t_1)$,...,$L(t_n) - L(t_{n-1})$ are independent;*
3. *L has stationary increments, that is, $\{L(t+s) - L(s)\}_{t\geq 0} =_D \{L(t)\}_{t\geq 0}$ for $s \geq 0$;*
4. *L is stochastically continuous;*
5. *There is an event $\Omega_0 \in \mathfrak{F}$ with $\mathbf{P}(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$, $L(t,\omega)$ is right continous for $t \geq 0$ and has left limits for $t > 0$, that is, L is càdlàg.*

Here $=_D$ denotes equality of the finite dimensional distributions.

A standard Wienerprocess can be defined in the following way:

Definition 2 (Standard Wiener process) *A stochastic process $W = \{W(t)\}_{t\geq 0}$ is called a* standard Wiener process *if it is a Lévy process and the law of $W(1)$ is the standard normal distribution $N(0,1)$.*

$\phi_X(u)$ denotes the characteristic function $\mathbf{E}\{\mathrm{e}^{iuX}\}$ of a random variable X. For a Lévy process L, we have the following elementary properties for $s, t \geq 0$:

$$\phi_{L(s)} = \phi_{L(t)}^{s/t} \tag{5}$$

$$\mathbf{E}\{L(t)\} = t\,\mathbf{E}\{L(1)\} \tag{6}$$

$$\mathbf{Var}\{L(t)\} = t\,\mathbf{Var}\{L(1)\} \tag{7}$$

Here (6) and (7) should be understood as that the left and right hand sides are well-defined simultaneously and then agree. An immediate consequence of (5) is that if we know the law of $L(1)$ determines that of $L(t)$ for all $t \geq 0$.

Now we define the NIG and Meixner distributions.

Definition 3 (One-dimensional NIG distribution [2]) *A* one-dimensional NIG distribution *has the following density function*

$$f_{NIG}(x;\alpha,\beta,\delta,\mu) = \frac{\alpha\delta}{\pi}\exp\left(\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)\right)\frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}}$$

for $x \in \mathbb{R}$, where $\mu \in \mathbb{R}$, $\delta \geq 0$, $0 \leq |\beta| \leq \alpha$, and K is the modified Bessel function of the third kind.

Definition 4 (One-dimensional Meixner distribution [11]) *A* one-dimensional Meixner distribution *has the following density function*

$$f_{MXN}(x;a,b,d,m) = \frac{(2\cos(\frac{b}{2}))^{2d}}{2a\pi\Gamma(2d)}\mathrm{e}^{b(x-m)/a}\left|\Gamma(d+\frac{i(x-m)}{a})\right|^2$$

for $x \in \mathbb{R}$, where Γ is the gamma function, $a > 0$, $-\pi < b < \pi$, $d > 0$, and $m \in \mathbb{R}$.

Both the NIG and the Meixner distributions have semiheavy tails, that is, their tails have an exponential decay rate at infinity. A Lévy process $X(t)$ such that the law for $X(1)$ is NIG or Meixner distributed is called a NIG- or a Meixner process, respectively. The characteristic functions for the NIG- and Meixner distributions are given by

$$\phi_{\mathrm{NIG}}(u) = \exp\left[\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+u)^2})+\mu u\right] \tag{8}$$

$$\phi_{\mathrm{MXN}}(u) = \left(\frac{\cos(\frac{b}{2})}{\cosh(\frac{au-ib}{2})}\right)^{2d}\mathrm{e}^{imu} \tag{9}$$

From (5), (8) and (9) we see that if $X(t)$ is a NIG process and $Y(t)$ is a Meixner process, then we have

$$f_{X(t)}(x) = f_{\mathrm{NIG}}(x;\alpha,\beta,\delta t,\mu t) \tag{10}$$

and

$$f_{Y(t)}(x) = f_{\mathrm{MXN}}(x;a,b,dt,mt). \tag{11}$$

In other words, these two classes of distributions are closed under convolutions.

3 Fitting to distribution of empirical data

We now want to compare how good the normal, NIG and Meixner distributions fit to empirical financial data, respectively. First, stockprices for some different stocks are obtained. The data sets obtained consist of the closing prices for the stock denoted by $(S_i)_{1\leq i\leq n}$. The series of logreturns is obtained by $(X_i = \log(S_i/S_{i-1}))_{2\leq i\leq n}$. The parameters in the different distributions are then estimated using the maximum likelihood method (MLE), see for example [3]. We do the parameter estimation for two data sets, namely Olsen Dollar/DM data from 1 December 1985 and 2447 trading days ahead, which is a very well-known data set in mathematical finance, and then the ABB stock from 19 September 2000 to 17 September 2002 (500 trading days, data obtained from [14]). The parameters obtained are seen in the tables below.

Asset	α	β	δ	μ
ABB	18.66	-1.21788	0.0303724	-0.00163903
Olsen	1.81532	-0.012478	0.907547	-0.00157438

Table 1: Estimated NIG parameters

Asset	a	b	d	m
ABB	0.117576	-0.146326	0.234783	-0.00160889
Olsen	1.25544	-0.0175116	0.63259	-0.000847716

Table 2: Estimated Meixner parameters

Asset	μ	σ
ABB	-0.00363209	0.0409286
Olsen	-0.00781167	0.711162

Table 3: Estimated normal parameters

The following figures show how the fitted densities agree with the empirical data. Observe the seemingly superiority of the NIG and Meixner distributions compared to the normal distribution to fit the central part of the data.

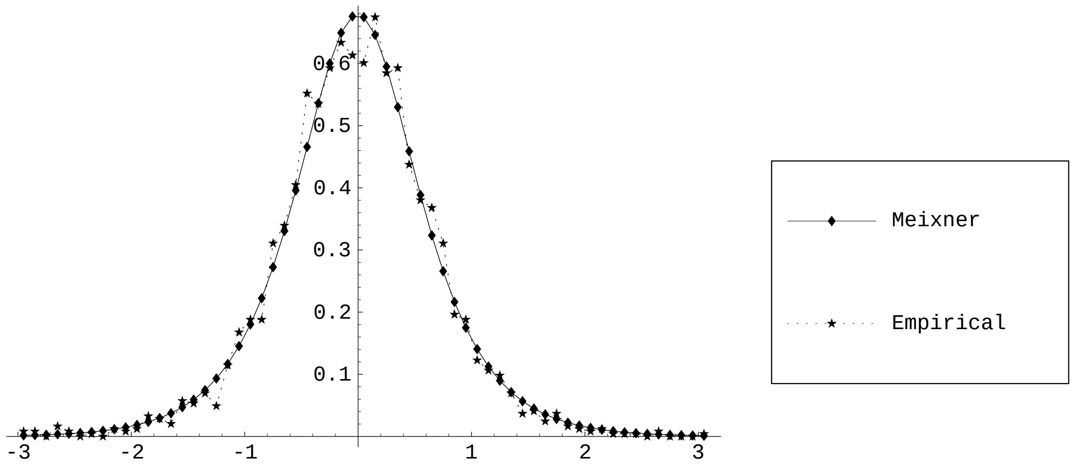

Figure 1: Fitted Meixner density and empirical histogram

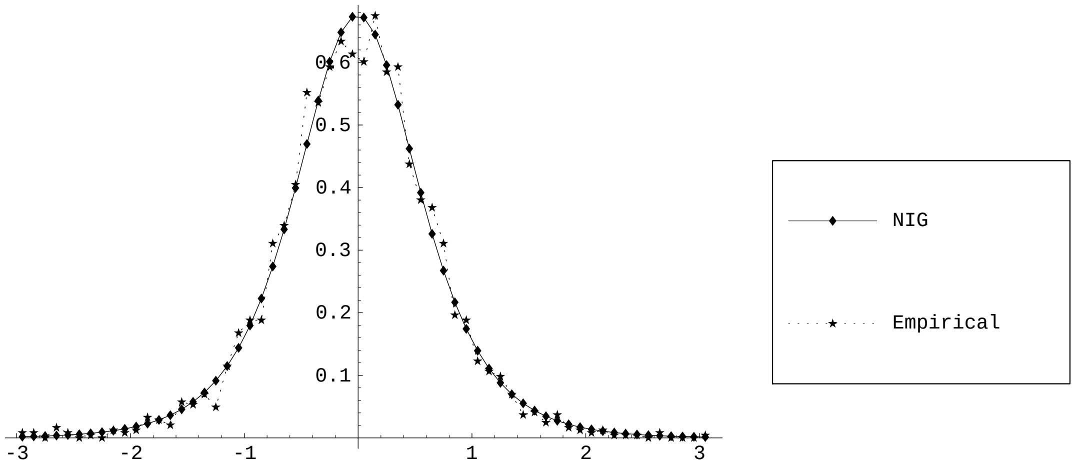

Figure 2: Fitted NIG density and empirical histogram

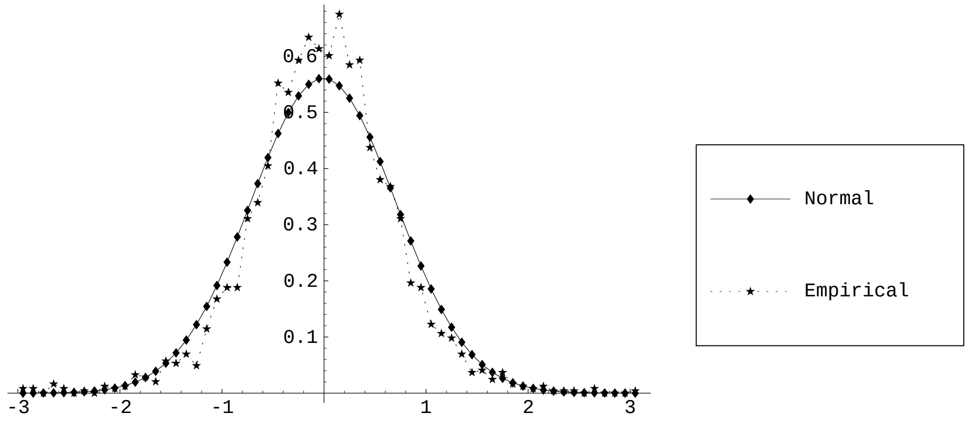

Figure 3: Fitted Normal density and empirical histogram

3.1 Goodness of fit

We investigate two measures of fit, namely the Kolmogorov distance and the Anderson & Darling test statistic.

The *Kolmogorov distance* is defined as

$$KD = \max_{x \in \mathbb{R}} |F_{\text{emp}}(x) - F_{\text{est}}(x)|, \tag{12}$$

where F_{est} is the estimated cummulative distribution function (CDF), and F_{emp} is the empirical CDF.

The *Anderson & Darling* statistic is defined as

$$AD = \max_{x \in \mathbb{R}} \frac{|F_{\text{emp}}(x) - F_{\text{est}}(x)|}{\sqrt{F_{\text{est}}(x)(1 - F_{\text{est}}(x))}} \tag{13}$$

The motivation for also using the Anderson & Darling statistic, is that it pays more attention to the tails of the distribution than, for example, does the Kolmogorov distance, the tails are very important in finance since it is here extreme events happen which may often be deciding factors in portfolio strategies. The tables below show the values for the two statistics. The simple rule is that the lower is the statistic, the better is the fit.

	NIG	Meixner	Normal
ABB	0.0251283	0.025671	0.0399513
Olsen	0.0119657	0.0123116	0.0348448

Table 4: Kolmogorov distance

	NIG	Meixner	Normal
ABB	0.0933217	0.12293	140.42
Olsen	0.0552062	0.0641442	2.11125

Table 5: Anderson & Darling statistic

3.2 Conclusions

For both data sets, the normal distribution was outperformed by the NIG and the Meixner distributions, when looking at the Kolmogorov distance and the Anderson & Darling statistic. For the Anderson & Darling statistic, the difference was very large. This must be due to the fact that the normal distribution has light tails, while the NIG and Meixner have semiheavy exponential tails, which apparently works much better. The figures also show that the central region is modelled better by the NIG and Meixner models. In addition, NIG and Meixner densities do look quite similar, and the difference between the two models in the goodness of fit is also quite small.

3.3 Further comments

The normal inverse Gaussian distribution was introduced in finance by Barndorff-Nielsen and his henchmen, back in the end of the nineties. The Meixner distribution was introduced in finance as late as in 2001 by Schoutens [11]. It might be of interest to know that the NIG distribution is a special case of the more general generalized hyperbolic distribution, introduced by Barndorff Nielsen [1] to model the logarithm of particle size, while the Meixner distribution is a special case of the generalized z-distribution, see [11] or [12]. These classes of generalized hyperbolic distributions and generalized z-distributions are quite different, and do not intesect. To the best knowledge of the author, there has not been any comparisons in the literature between the fit of the NIG and Meixner distributions.

4 Change of time

Cherny and Shiryaev [6] gave a list of properties a financial model should have, which we display with some slight modifications below:

1. The marginal distribution of the increments of $\log(S(t))$ should be skewed.

2. The marginal distribution of the increments of $\log(S(t))$ should have heavy tails.

3. The increments of $\log(S(t))$ should be stationary in time.

4. The increments of $\log(S(t))$ over disjoint intervals should be uncorrelated.

5. The realised variances of $\log(S(t))$ over disjoint intervals should be positively correlated.

Here the *realised variance* (also known as *realised volatility*) for one day with M intraday returns, is defined as

$$\sum_{j=1}^{M}\Big(X((t-1)\delta+\frac{\delta j}{M})-X((t-1)\delta+\frac{\delta(j-1)}{M})\Big)^2$$

where $\delta=\frac{1}{M}$.

In Section 1, we saw that the model $S(t)=S(0)\mathrm{e}^{X(t)}$, where $X(t)$ was a NIG or a Meixner process, satisfied the Properties 1 and 2 above. Moreover, these distributions provided an excellent fit to empirical data. By definition these models also satisfy Properties 3 and 4, but notably not Property 5.

We will now present a model which satisfies all five properties, namely

$$S(t)=S(0)\mathrm{e}^{\mu t+X(\tau(t))}, \tag{14}$$

where $\mu\in\mathbb{R}$, while X is a Lévy process with $\mathbf{E}\{X(1)\}=0$ and $\tau=\{\tau(t)\}_{t\geq 0}$ is an increasing càdlàg process, independent of X. In this model, the Lévy process undergoes a stochastic time change. This particular model has, to the best knowledge of the author, not been proposed before.

The next subsection lists some properties of the *time changed Lévy process* $X(\tau)=\{X(\tau(t))\}_{t\geq 0}$.

4.1 Properties of the time changed Lévy processes

Consider the time changed Lévy process $X(\tau)$, where $\tau(t)$ has stationary, positively correlated increments.

Assuming that $\mathbf{E}\{X(t)^2\} < \infty$ and $\mathbf{E}\{\tau(t)^2\} < \infty$, we now list several elementary properties of the process $X(\tau(t))$, that are left to the reader as an exercise to verify:

a. $\mathbf{E}\{X(\tau(t))\} = \mathbf{E}\{X(1)\}\mathbf{E}\{\tau(t)\}$

b. $\mathbf{E}\{X(\tau(t))^2\} = \mathbf{Var}\{X(1)\}\mathbf{E}\{\tau(t)\} + \mathbf{E}\{X(1)\}^2\mathbf{E}\{\tau(t)^2\}$

c. $X(\tau)$ has stationary increments.

d. If $\mathbf{E}\{X(t)\} = 0$, then it holds that

$$\begin{aligned}\mathbf{Cov}\{[X(\tau(t_4)) - X(\tau(t_3))]^2, [X(\tau(t_2)) - X(\tau(t_1))]^2\} \\ = \mathbf{E}\{X(1)^2\}^2\mathbf{Cov}\{\tau(t_4) - \tau(t_3), \tau(t_2) - \tau(t_1)\}\end{aligned} \tag{15}$$

e. $\mathbf{Cov}\{X(\tau(t_4)) - X(\tau(t_3)), X(\tau(t_2)) - X(\tau(t_1))\}$

$$= \mathbf{E}\{X(1)\}^2\mathbf{Cov}\{\tau(t_4) - \tau(t_3), \tau(t_2) - \tau(t_1)\}. \tag{16}$$

for $t_1 \leq t_2 \leq t_3 \leq t_4$.

Note that properties c-e also hold for the process $\mu t + X(\tau(t))$.

Returning to our model (14), we see that Properties c, d and e are what is needed for Properties 3, 5 and 4, respectively, for a financial model. For Properties 1 and 2, we can take X to be for example a NIG or Meixner process. By (16), we also see that, for the model (14) with $\mu = 0$ and $\mathbf{E}\{X(1)\} \neq 0$, the Property 4 for a financial model is not fulfilled. In [6], it is claimed that if the model (14) fulfills all five desirable properties for a financial model, for an appropiate choice of parameters, then this in effect means that the parameters must be chosen so that $\mathbf{E}\{X(1)\} = 0$. This seems a bit strange since there would then be no drift in the exponent. Our proposed model, on the other side, allows a non-zero drift in the exponent, and still takes care of all five properties, for a suitable choice of X that is.

4.2 Integrated positive stationary processes

One way to construct a process τ, with the above imposed restrictions is to let it be the integral of a positive, stationary, continuous and integrable process $y = \{y(t)\}_{t\geq 0}$. For such an integral the following elementary properties hold (see [2]):

Denoting $\mathbf{E}\{y(t)\} = \mu$, $\mathbf{Var}\{y(t)\} = \sigma^2$, $r(u) = \mathbf{Cor}\{y(t+u), y(t)\}$, $r^*(u) = \int_0^u r(s)ds$ and $r^{**}(u) = \int_0^u r^*(s)ds$, the process

$$\tau(t) = \int_0^t y(s)ds \tag{17}$$

satisfies

$$\mathbf{E}\{\tau(t)\} = \mu t, \tag{18}$$

and

$$\mathbf{Var}\{\tau(t)\} = 2\sigma^2 r^{**}(t). \tag{19}$$

Later on, we shall also need the following fact

$$\mathbf{Cov}\{\tau(t), \tau(t+s)\} = \sigma^2(r^{**}(t+s) + r^{**}(t) - r^{**}(s)). \tag{20}$$

Since y is stationary, it follows that τ has stationary increments.

Next, we discuss an example of a stationary positive process, which is very popular in finance, because of its convenient analytic properties.

4.3 The Cox Ingersoll Ross process (CIR)

The Cox Ingersoll Ross process, introduced by Cox, Ingersoll and Ross [5] to model the interest rate in finance, is also known as Feller's square root process. It is defined by the stochastic differential equation (SDE)

$$dy(t) = \kappa(\theta - y(t))dt + \sigma\sqrt{y(t)}dW(t). \tag{21}$$

Here W is a standard Wiener process and κ, σ, $\theta > 0$. If $y(0) > 0$, then $\mathbf{P}\{y(t) \geq 0\} = 1$. If the parameters are chosen so that $2\kappa\theta \geq \sigma^2$, then $\mathbf{P}\{y(t) > 0 \ \forall t \geq 0\} = 1$. In the case when $2\kappa\theta < \sigma^2$, we have $\mathbf{P}\{y(t) = 0 \text{ i.o.}\} = 1$.

A nice feature of the Cox-Ingersoll-Ross process is that the distribution of $y(t)|y(0)$ is known. If we let

$$\begin{cases} q = & 2\kappa\theta/\sigma^2 - 1 \\ c = & 2\kappa/[\sigma^2(1-\mathrm{e}^{-\kappa t})] \\ u = & cy(0)\mathrm{e}^{-\kappa t} \end{cases} \tag{22}$$

and pick a random variable Ξ which has a non-central chi-square distribution, with the parameter of non-centrality equal to $2u$ and $2(q+1)$ degrees of freedom, see [5], then

$$y(t)|y(0) =_D \frac{1}{2c}\Xi.$$

Since the density of the non-central chi-square distribution is known, so is that of $y(t)|y(0)$:

$$f_{y(t)|y(0)}(x) = c\,\mathrm{e}^{-u-cx}\left(\frac{cx}{u}\right)^{\frac{q}{2}} I_q(2\sqrt{ucx}), \tag{23}$$

where $I_q(\cdot)$ is the modified Bessel function of the first kind of order q.

Letting $t \to \infty$, noting that c and u depends on t and using well-known rules for the asymptotic behaviour of I_q at infinity, we see that the stationary distribution of the CIR process is the gamma-distribution $\Gamma(\nu, \lambda)$, with density function

$$f_{\Gamma(\nu,\lambda)}(x) = \frac{\lambda^\nu}{\Gamma(\nu)} x^{\nu-1}\mathrm{e}^{-\lambda x}, \tag{24}$$

where $\lambda = 2\kappa/\sigma^2$ and $\nu = 2\kappa\theta/\sigma^2$. Hence we have

$$\mathbf{E}\{y(t)\} = \theta \tag{25}$$

and

$$\mathbf{Var}\{y(t)\} = \frac{\theta\sigma^2}{2\kappa}. \tag{26}$$

Values of the CIR process are positively correlated. More precisely, we have

$$r(s) = \mathbf{Cor}\{y(t+s), y(t)\} = \mathrm{e}^{-\kappa|s|}. \tag{27}$$

Next, we consider the integral of the CIR process.

4.4 The integrated Cox Ingersoll Ross Process (intCIR)

Let

$$\tau(t) = \int_0^t y(u)du, \tag{28}$$

where y is a CIR process. Following [2], we call τ an intCIR process. The intCIR process was first proposed for time changes in financial models by Carr, Madan, Geman and Yor [4], but more about that later. The CIR process is stationary, so that the intCIR process has stationary increments. Since for the CIR process $r(u) = \mathrm{e}^{-\kappa u}$, $r^*(u) = \frac{1}{\kappa}(1 - \mathrm{e}^{-\kappa u})$, and $r^{**}(u) = \frac{1}{\kappa^2}(\mathrm{e}^{-\kappa u} - 1 + \kappa u)$ for $u > 0$ we have the following equations for the intCIR process for t, $s > 0$:

$$\mathbf{Var}\{\tau(t)\} = \frac{\theta\sigma^2}{\kappa^3}(\mathrm{e}^{-\kappa t} - 1 + \kappa t) \tag{29}$$

$$\mathbf{Cov}\{\tau(t), \tau(t+s)\} = \frac{\theta\sigma^2}{2\kappa^3}(\mathrm{e}^{-\kappa t} + \mathrm{e}^{-\kappa(s+t)} - \mathrm{e}^{-\kappa s} - 1 + 2\kappa t) \tag{30}$$

The density of the intCIR process is not known explicitly, but interestingly the characteristic function for $\tau(t)|y(0)$ is, and is given by

$$\phi_{\tau(t)|y(0)}(u) =$$

$$\exp\left(\frac{\theta\kappa^2}{\sigma^2} + \frac{2iuy(0)}{\kappa + \sqrt{\kappa^2 - 2iu\sigma^2}\coth\frac{1}{2}\sqrt{\kappa^2 - 2iu\sigma^2}}\right)\times$$

$$\times\left(\cosh\left(\tfrac{1}{2}\sqrt{\kappa^2 - 2iu\sigma^2}\right) + \frac{\kappa\sinh\left(\frac{1}{2}\sqrt{\kappa^2 - 2iu\sigma^2}\right)}{\sqrt{\kappa^2 - 2iu\sigma^2}}\right)^{-2\theta\kappa/\sigma^2}$$

For later use, we calculate the second and third moments for $\tau(1)$:

$$\mathbf{E}\{\tau(1)^2\} = \frac{\theta\sigma^2}{\kappa^3}(\mathrm{e}^{-\kappa} - 1 + \kappa) + \theta^2 \tag{31}$$

$$\mathbf{E}\{\tau(1)^3\} = \int_0^\infty i\left\{\frac{d^3}{d\,u^3}\phi_{\tau(1)|y(0)}(u)\right\}_{u=0\,,y(0)=x} f_{\Gamma(\nu,\omega)}(x)\,dx \tag{32}$$

4.5 Change of time with the intCIR process

In this section we consider the process $X(\tau)$, where X is a Lévy process with $\mathbf{E}\{X(t)\} = 0$, and τ is the intCIR process.

For $0 < t_1 < t_2 < t_3 < t_4$ with $t_4 - t_3 = t_2 - t_1 = \delta$ we get by straightforward calculations, using (15), and (30) that

$$\mathbf{Cov}\{[X(\tau(t_4)) - X(\tau(t_3))]^2, [X(\tau(t_2)) - X(\tau(t_1))]^2\} = C \cdot \mathrm{e}^{-\kappa(t_4 - t_2)}, \tag{33}$$

where $C = \mathbf{E}\{X_1^2\}^2 \frac{\theta\sigma^2}{\kappa^3}(\cosh\kappa\delta - 1)$.

Barndorff-Nielsen [2] suggests the superposition of a finite number of independent intCIR processes with parameters κ_i, θ_i and σ_i. However, he does not derive any properties for time changes of this kind. Now, if we let $\tau = \sum_{i=1}^n \tau_i$, where the τ_i's are independent intCIR processes, and at the same time let M be a positive integer, straightforward calculations give

$$\mathbf{Cov}\Big\{\sum_{l=1}^{M}\Big[X\Big(\tau\Big(t_4 - \frac{l-1}{M}\delta\Big)\Big) - X\Big(\tau\Big(t_4 - \frac{l}{M}\delta\Big)\Big)\Big]^2,$$

$$,\sum_{k=1}^{M}\Big[X\Big(\tau\Big(t_2 - \frac{k-1}{M}\delta\Big)\Big) - X\Big(\tau\Big(t_2 - \frac{k}{M}\delta\Big)\Big)\Big]^2\Big\}$$

$$= \sum_{i=1}^{n} C_i \cdot \mathrm{e}^{-\kappa_i(t_4-t_2)}$$

where

$$C_i = \mathbf{E}\{X(1)^2\}^2 \frac{\theta_i\sigma_i^2}{\kappa_i^2}\Big(\cosh\Big(\frac{\kappa_i\delta}{M}\Big) - 1\Big)\Big(\sum_{l=1}^{M}\sum_{k=1}^{M}\mathrm{e}^{-\kappa_i\delta\frac{k-l}{M}}\Big)$$

In other words we have an autocovariance function for the realised variance with M intraday returns which look like

$$\mathrm{acf}(t) = \sum_{i=1}^{n} C_i \mathrm{e}^{-\kappa_i t}, \tag{34}$$

where the constants C_i are given above.

Now we check how good this fits with the covariance structure of the realised variance of real financial data, the Olsen data set. The fitting is done using the routine NonLinearFit in *Mathematica*. In the figure below we see that while one intCIR process is not enough, two is much better.

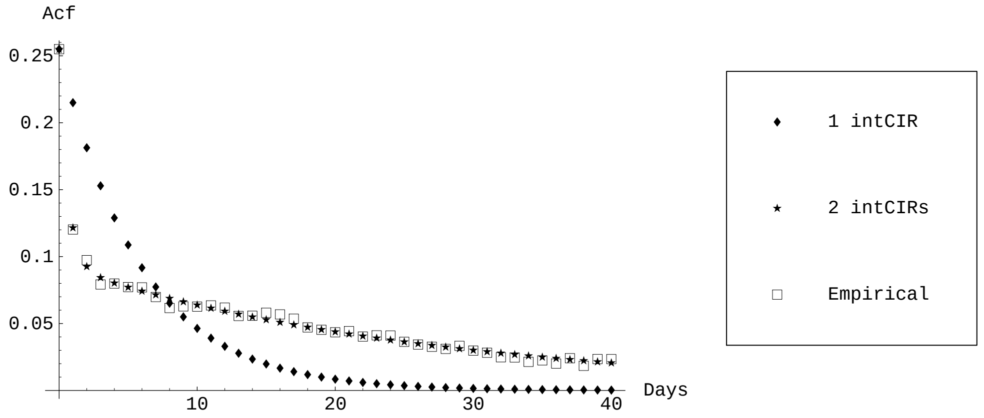

Figure 4: Fitting to the empirical acf with one and two intCIR processes as time in a Lévy process

For the Olsen data set we get in the case of one intCIR $\kappa = 0.17$, $C_1 = 0.25$ and in the case of two, we obtain $\kappa_1 = 1.62$, $\kappa_2 = 0.038$, $C_1 = 0.16$, $C_2 = 0.09$.

4.6 Parameter estimation

While the density of the CIR process is known, the density of the intCIR is not. So we cannot use the maximum likelihood method for estimating the parameters in the time changed model, and other methods are needed. We now consider the simplest case of the model (14), namely that when τ is a single intCIR process and $X = W$ where $W(t)$ is standard Wiener process. This is a four parameter model with parameters μ, κ, σ, and θ. Here κ can be got from estimation of the exponential decay rate of the autocovariance function. An estimate for μ is the sample mean of the series of daily logreturns. Since

$$\mathbf{E}\{W(\tau(1))^2\} = \theta, \tag{35}$$

the sample centralized second moment for the data set is an estimate for θ. For the last parameter σ we can use the estimated constant C_1 from the autocovariance function. For the Olsen dataset, the sample mean is -0.00781167, and the sample centralized second moment is equal to 0.505751. To summarize, we get the following parameter values:

$$\mu = -0.00781167,\ \theta = 0.505751, \kappa = 0.1705,\ \sigma = 0.414.$$

We note that for this data set, $2\kappa\theta = 0.172 \geq 0.171 = \sigma^2$, which means that the CIR process in this case is strictly positive.

We now move on to estimate parameters in the case where $\tau(t) = \tau_1(t) + \tau_2(t)$, where τ_1 and τ_2 are independent intCIR processes. In this model, there are seven parameters. Obviuosly, we have

$$\mathbf{E}\{W(\tau(1))^2\} = \theta_1 + \theta_2. \tag{36}$$

Both κ_1 and κ_2 we get directly from the estimate of the autocovariance function. Also, we have the estimated constants C_1 and C_2. One more estimator is needed. It turns out that the fourth moment of $W(\tau(1))$ is decided by the autocovariance function, the second moment and κ, so we calculate instead the sixth moment:

$$\mathbf{E}\{W(\tau(1))^6\} = \mathbf{E}\{\mathbf{E}\{W(\tau(1))^6|\tau(1)\}\} = 15\mathbf{E}\{\tau(1)^3\}$$

$$= 15(\mathbf{E}\{\tau_1(1)^3\} + \mathbf{E}\{\tau_2(1)^3\} + 3\mathbf{E}\{\tau_1(1)^2\}\theta_2 + 3\mathbf{E}\{\tau_2(1)^2\}\theta_1),$$

where we plug in the expressions for the earlier calculated second and third moments for the intCIR process. The sample centralized sixth moment for the data set is 9.36. To summarize we get the following parameters:

$$\mu = -0.00781167,\ \theta_1 = 0.4297,\ \kappa_1 = 1.62,\ \sigma_1 = 1.007$$

$$\theta_2 = 0.0878,\ \kappa_2 = 0.038,\ \sigma_2 = 0.2824$$

We see that $2\kappa_1\theta_1 = 1.36 > 1.01 = \sigma_1^2$ while, $2\kappa_2\theta_2 = 0.007 < 0.08 = \sigma_2^2$ which means that τ_1 is strictly increasing but that the CIR process in τ_2 will hit the zero infinitely many times.

4.7 Simulated results

In this section, we present simulation results for the CIR, the intCIR, and finally the exponential Wiener process with an intCIR process time change. We present results using the parameters from the estimation in the case of one intCIR process. We do not present simulations from the case of two. This is mainly because the numerical difficulties which arise when trying to simulate a CIR process for which $2\kappa\theta < \sigma^2$. Such a CIR process will often be extremely close to zero.

For the CIR process y, the distribution for $y(t)|y(0)$ is the non-centralized chi-squared. Hence this process can be simulated by simulating non-centralized chi-squared random numbers, which can be done by means of using the *Mathematica* software. A simulated sample path of the CIR process for 100 days using the parameters from the previuos section and the time step $dt = 0.01$, is seen in Figure (5) below. The starting value $y(0)$ is simulated from the stationary distribution of the CIR process.

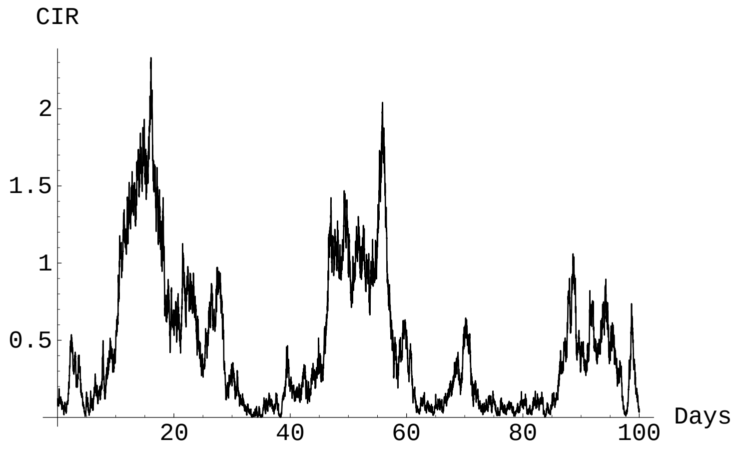

Figure 5: Simulation of the CIR process with parameters $\kappa = 0.1705$, $\theta = 0.505751$, and $\sigma = 0.414$.

The intCIR process $\tau(t) = \int_0^t y(s)\, ds$ can be simulated by approximating the integral using step functions, that is $\tau(t_n) = \sum_{i=1}^{n} \frac{1}{2}(t_i - t_{i-1})(y(t_i) + y(t_{i-1}))$ for an increasing sequence of times $\{t_i\}_{i=0}^{\infty}$. Using the simulated path of the CIR process in (5), the resulting simulated path of the intCIR process is seen in Figure (6) below.

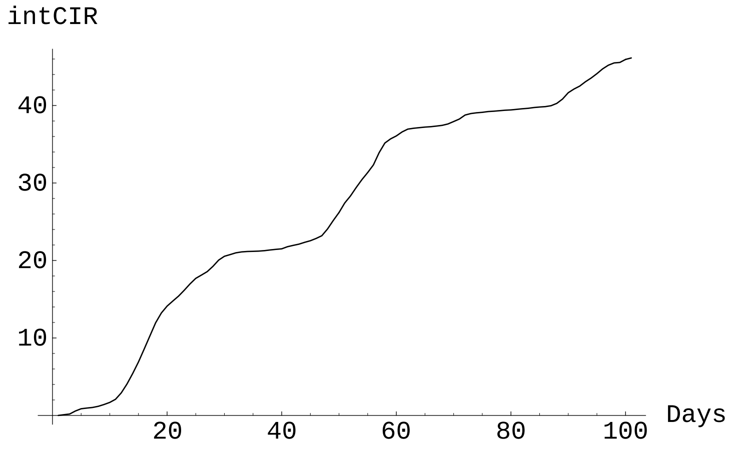

Figure 6: Simulation of the intCIR process using the above simulation of the CIR process.

With the simulations of the CIR and intCIR processes in hand, the time changed Lévy process is easily simulated in the following way. With sample values $(y_k)_{k=0}^n$ with $y_0 = 0$ from the intCIR simulation, we get that

$$(X(y_k))_{k=1}^n = (\sum_{l=1}^{k}(X(y_l) - X(y_{l-1})))_{k=1}^n =_D (\sum_{l=1}^{k}\Xi_l)_{k=1}^n$$

where the Ξ_l are independent stochastic variables with $\Xi_l =_D X(y_l - y_{l-1})$. Figure 7 shows a simulated path of the expontial time changed Brownian motion, using the intCIR simulation from above.

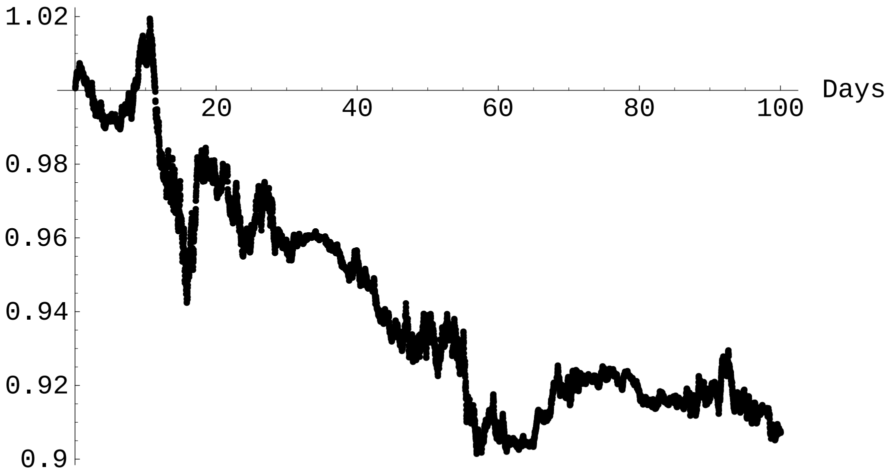

Figure 7: Simulation of the time changed exponential Brownian motion

The corresponding intraday logreturns for the simulated time changed exponential Brownian motion is shown in Figure 8.

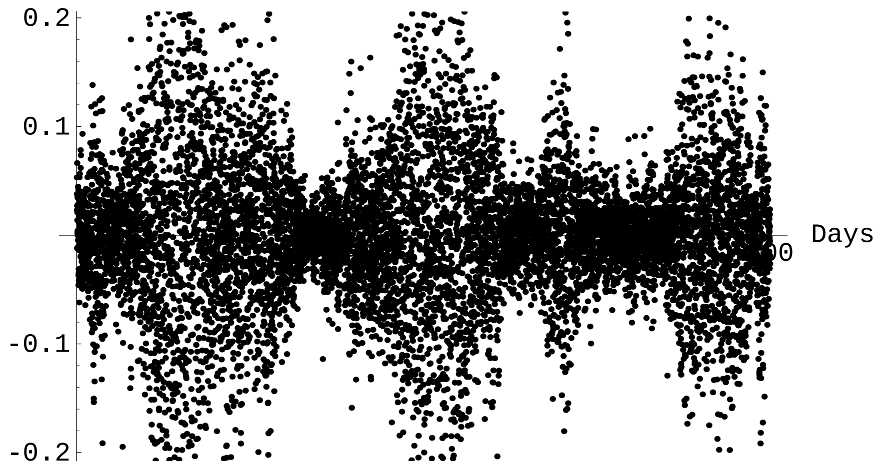

Figure 8: Intraday logreturns from the simulation of the time changed exponential Brownian motion, 100 days and 100 observations per day.

We also check that the acf for the realised variance for the simulation fits to the theoretical acf. In the next figure, we see the acf for $M = 100$ using a simulation of 2500 days (a quarter million values). One must simulate so many days to get a fairly accurate estimate of the acf of the simulated process.

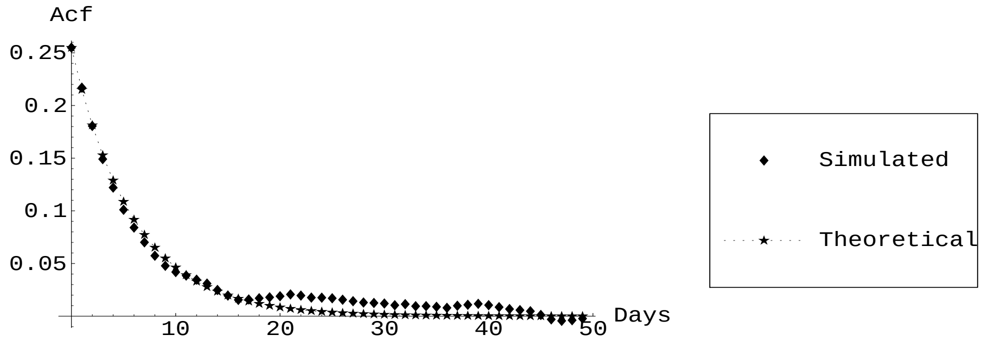

Figure 9: Sample acf from the simulation, compared with the theoretical acf.

Finally, it is of interest to study the empirical histogram from the simulation, compared with the empirical histogram from the dataset from which all the parameters where obtained. Also, we compare these histograms with the empirical histogram from a simulation using the maximum likelihood estimated parameters for the normal distribution for this data set from the first chapter.

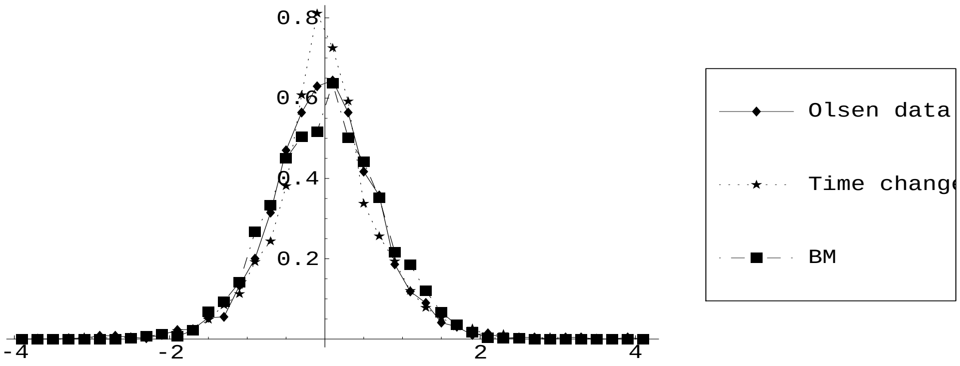

Figure 10: Comparison between empirical histograms.

In Figure (10) we see that the time changed exponential Brownian motion model seems to have a sharper peak at zero than has real observed data.

4.8 Conclusions

We showed theoretically that the model (14,) where τ is the integral of the superposition of two CIR processes, can model the covariance structure of squared logreturns from financial data. Parameters in the model when using Brownian motion and one intCIR process where estimated from the acf and moment estimators. We noted a sharper peak in the density from this model than for real data.

4.9 Further comments

The model (14), where τ is the intCIR, process was first proposed by Carr, Geman, Madan and Yor [4]. Their main concerns however, were not to fit the model to the logreturns of empirical financial data, but to consider pricing of european call options (for definition of european call options, see next section) in this model. This is difficult, a much more complicated matter than in the usual exponential Lévy model case. They calculate a pricing formula, and then estimates the parameters in the model by minimizing the squares of the distance between observed market prices and the theoretical prices. They get a remarable good fit. To the knowledge of the author of this thesis, no one has tried to fit the time changed model to logreturns before. Also, it seems that no one has bothered to check theoretically that a fit to the correlation structure of squared logreturns can be done, as in this thesis. Simulation studies as in this thesis also seem to be abscent in the literature. It is obvious that there are many more things to be done in this field. Here we just studied time changes in Brownian motion. A more advanced approach would be to change time in for example a NIG och Meixner process. In [2] there are examples of other stationary positive processes which can be integrated and used as time changes Lévy processes.

5 Option pricing with Lévy models

In this section, we will assume that we have a financial market consisting of two assets. One is a risk free asset B, the dynamics of which is given as

$$dB(t) = r\,B(t)\,dt, \tag{37}$$

where r is a constant called the *interest rate*. The second one is a stock, with price process S given by

$$S(t) = S(0)\mathrm{e}^{X(t)}, \tag{38}$$

where X is a Lévy process.

The aim of this section is to compare European call-option prices, for three different cases for X, namely NIG, Meixner, and the Brownian motion with drift (i.e. the Bachelier-Samuelson model).

A European call option in the stock, is a paper that gives its holder the right, but not the obligation, to buy one share at time T, to a fixed price K. The time T is called the time of maturity, and K is called the strike price. Thus, at time T, the option pays to its owner

$$\Phi(S(T)) = (S(T) - K)^+, \tag{39}$$

where $x^+ = \max(0, x) = x\,\mathbf{1}_{\{x>0\}}$ for $x \in \mathbb{R}$. The price of such an option at time 0 is defined as

$$v(T, S(T)) = \mathrm{e}^{-rT}\mathbf{E}^{\mathbf{Q}}\{\Phi(S(T))\}, \tag{40}$$

where the measure $\mathbf{Q}$ is a probability measure equivalent to $\mathbf{P}$ such that $\mathrm{e}^{-rt}S(t)$, the discounted price process of the stock, is a $\mathbf{Q}$-martingale with respect to $\sigma(X)$. Here we say that $\mathbf{P}$ and $\mathbf{Q}$ are equivalent when they have the same null-sets.

In the classical Bachelier-Samuelson model, there is only one such measure $\mathbf{Q}$. In this case, the measure is defined by the relation (see for example [9])

$$d\mathbf{Q} = \exp\Big[\frac{r - \mu - \sigma^2/2}{\sigma}W(T) - \frac{1}{2}\Big(\frac{r - \mu - \sigma^2/2}{\sigma}\Big)^2 T\Big]\,d\mathbf{P}, \tag{41}$$

However, in virtually all other models where X is a Lévy proces, the measure $\mathbf{Q}$ is not unique (see for example [10]). Many measures have been proposed in the literature. Unfortunately, most of these are not given by explicit formulas as in (41), but rather implicitely. Luckily, there is one change of measure which we will now consider that is given explicitely and has convenient analytic properties. It is called the Esscher measure, and was first proposed for option pricing by Gerber and Shiu [7].

5.1 The Esscher change of measure

The Esscher measure is defined by the relation

$$d\mathbf{Q} = \mathrm{e}^{\nu X(T) - T\log M(\nu)} d\mathbf{P}, \tag{42}$$

where $M(u) = \mathbf{E}^{\mathbf{P}}\{\mathrm{e}^{uX(1)}\}$ is the moment generating function of $X(1)$, and the parameter ν is defined as the solution to the equation (see [10])

$$r = \log \frac{M(\nu+1)}{M(\nu)}. \tag{43}$$

In other words, $\Lambda(T) = \mathrm{e}^{\nu X(T) - T\log M(\nu)}$ is the Radon-Nikodym derivative of $\mathbf{Q}$ with respect to $\mathbf{P}$. It is then an elementary exercise to verify that the process $\mathrm{e}^{-rt}S(0)\mathrm{e}^{X(t)}$, where X is a Lévy process with moment generating function $M(u)^t$, is indeed a $\mathbf{Q}$-martingale with respect to $\sigma(X)$.
In the Bachelier-Samuelson case, we have $M(\nu) = \mathrm{e}^{\mu\nu + \frac{1}{2}\sigma^2\nu^2}$, giving $\nu = (r-\mu)/\sigma^2 - 1/2$, which is inserted in (42), giving (41).

Now we want to calculate European call option prices for the three different models NIG, Meixner and Bachelier-Samuelson. In the Bachelier Samuelson case, the price is given by the famous Black-Scholes formula

$$C_{\mathrm{BS}} = S(0)\Phi(h) - K\mathrm{e}^{-rT}\Phi(h - \sigma\sqrt{T}), \tag{44}$$

where

$$h = \frac{S(0)/K + (r + \sigma^2/2)T}{\sigma\sqrt{T}}. \tag{45}$$

To calculate the price in the NIG and Meixner models, we first need to know their densities under $\mathbf{Q}$:

$$\phi_{\mathrm{MXN},\mathbf{Q}}(u) = \mathbf{E}^{\mathbf{Q}}\{\mathrm{e}^{iuX(1)}\} = \left(\frac{\cos\left(\frac{a\nu+b}{2}\right)}{\cosh\left(\frac{au - i(a\nu+b)}{2}\right)}\right)^{2d} \mathrm{e}^{imu}. \tag{46}$$

Here the right hand side is the characteristic function of a random variable with density $f_{\mathrm{MXN}}(x; a, a\nu+b, d, m)$. In effect, this means that this change of measure only affects the skewness of the distribution.

In this case the parameter ν is given explicitly by (see [11])

$$\nu = -\frac{1}{a}\left[b + 2\arctan\left(\frac{-\cos\left(\frac{a}{2}\right) + \mathrm{e}^{(m-r)/(2d)}}{\sin\left(\frac{a}{2}\right)}\right)\right]. \tag{47}$$

In the NIG case, the density for $X(1)$ changes to

$$f_{\mathbf{Q},\mathrm{NIG}}(x) = f_{\mathrm{NIG}}(x;\alpha,\beta+\nu,\delta,\mu). \tag{48}$$

Also here, the Esscher measure change changes the skewness. In contrast to the Meixner case, where ν is given by (47), we have to find the parameter ν by numerically solving the equation

$$r = \mu + \delta(\sqrt{\alpha^2-(\beta+\nu)^2} - \sqrt{\alpha^2-(\beta+\nu+1)^2}). \tag{49}$$

So, by (40), (46), (48), (10) and (11), we get the formulas for the European call prices, in the NIG and Meixner cases,

$$C_{\mathrm{MXN}} = \mathrm{e}^{-rT}\int_{\log K/S(0)}^{\infty} (S(0)-K)\, f_{\mathrm{MXN}}(x;a,a\nu+b,dT,mT)\,dx \tag{50}$$

$$C_{\mathrm{NIG}} = \mathrm{e}^{-rT}\int_{\log K/S(0)}^{\infty} (S(0)-K)\, f_{\mathrm{NIG}}(x;\alpha,\beta+\nu,\delta T,\mu T)\,dx \tag{51}$$

From these formulas we see that it was a crucial part of the derivation that the Meixner and NIG distributions are closed under convolution. If this would not have been the case, one would have to use numerical Fourier methods to get the densities for $X(T)$ for $T \neq 1$.

To compare the prices in the different models, we used the MLE-estimated ABB-parameters, and calculated the expressions (44), (50) and (51), for different maturities T, and different strike price-stock prices ratios $K/S(0)$. For the ABB dataset, $\nu_{\mathrm{NIG}} = 1.723$, changing $\beta = -1.218$ to $\beta_{\mathbf{Q}} = 0.505$, while $\nu_{\mathrm{MXN}} = 1.735$, changing $b = -0.146$ to $b_{\mathbf{Q}} = 0.058$.

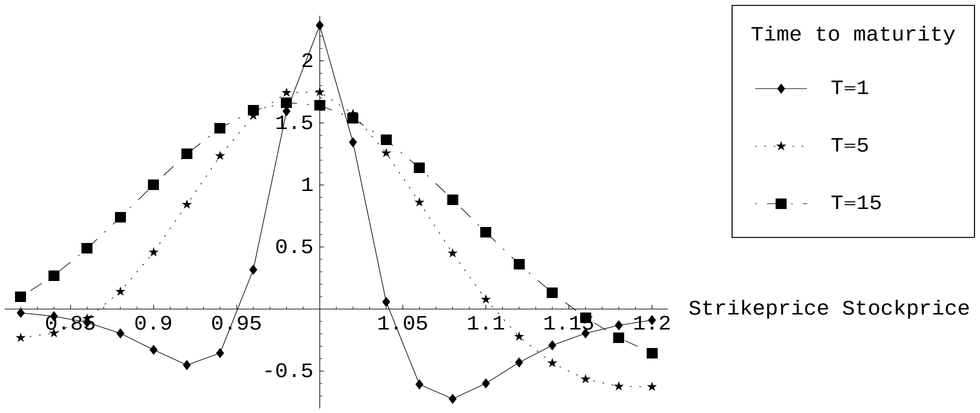

Figure 11: BS-prices - NIG-prices for S=1000, using ABB parameters.

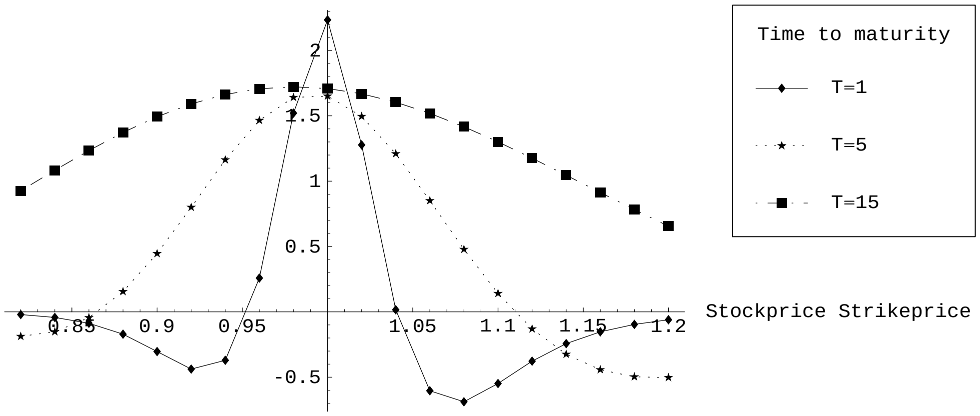

Figure 12: BS-prices - Meixner-prices for S=1000, using ABB parameters.

Ratio	BS	NIG	Meixner	NIG/BS	Meixner/BS
0.82	180	180.031	180.021	1.00017	1.00011
0.86	140.001	140.109	140.088	1.00077	1.00062
0.90	100.062	100.39	100.365	1.00328	1.00304
0.94	61.132	61.487	61.503	1.0058	1.00606
0.98	28.094	26.500	26.573	0.943264	0.945878
1.02	8.383	7.039	7.106	0.839688	0.847636
1.06	1.466	2.073	2.070	1.41417	1.41186
1.10	0.144	0.743	0.693	5.14535	4.7969
1.14	0.00805224	0.300	0.251	37.2919	31.1822
1.18	0.00026113	0.132	0.096	503.646	368.603

Table 6: European call-prices in the different models, for different strike-price/stockprice ratios using S=1000 and estimated ABB parameters, T=1.

5.2 Comparison on the market

In this section, we study how well the different pricing formulas fit to real life option prices. The data set we use was obtained from Ph.D. Wim Schoutens, and consists of 77 ($= n$) prices of european call options for different strikes and maturities on the SP500-index (Standard and Poor's), from the 18th of April 2002. On this day, the SP500 closed at 1124.47 USD. Also, the annual interest rate was $r = 0.0475$. The data set is available at the following URL: http://www.dd.chalmers.se/˜f98joty/SP50020020418.txt. We try to find model parameters, such that the mean absolute error, as percentage of the mean option price for the data set (= 61.85 USD), is minimized. The reason for using this statistic, which we call the *ape*-statistic, is that this is also used in for example [4]. More formally, we minimize the following;

$$\frac{\sum_{i=1}^{n} |C_{\text{market}}(i) - C_{\text{model}}(i)|}{\sum_{i=1}^{n} C_{\text{market}}(i)}, \tag{52}$$

where C_{market} is the observed market price, and C_{model} is the model price in the different models. If the model is perfect, then (52) is zero. The minimization is carried out using the FindMinimum function in the software package Mathematica. In the Black-Scholes case, we just have one variable, σ, hence in this case, the minimization is very fast. In the other two cases however, the minimization is in several variables, and the formulas are also more complicated. For the Meixner case, the minimization took over 1.5 hours on a quite powerful personal computer (1.05 GHz and 240 Mb RAMmemory). Unfortunately the minimization in the NIG case failed due to unresolved numerical problems (possibly because of the extreme flatness of the score function). However, with the parameters below we obtain a quite low *ape*-statistic also in this case. In the table below, we see the lowest *ape*-statistics obtained in the different models. The paremeters for which these were obtained were $\sigma = 0.13$, $\alpha = 128.569$, $\beta = 0$, $\delta = 2.093$, $\mu = 0.380$, $a = 0.0186$, $d = 82.362$, and $m = -0.609$ (note that the parameter b disapears in the Meixner model with the Esscher measure change).

Model	ape
BS	2.75
Meixner	2.69
(NIG)	(2.84)

Table 7: Ape values for different models in percent for the SP500 option set from the 18th of April 2002.

As we can see, for this data set, the models does not differ much in the sense of getting a low *ape*-value. In the figure (13) below, we compare the Meixner model with the observed market prices.

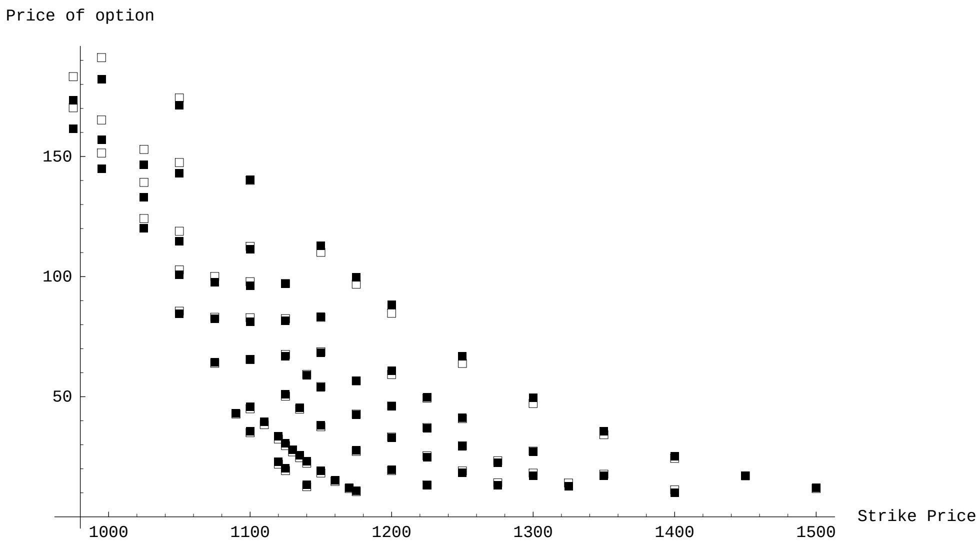

Figure 13: The observed market prices (filled), and the theoretical Meixner prices (empty), using the parameters giving the lowest ape-value.

5.3 Conclusions

We see that in absolute values, the differences in the prices of the European call prices in the three different models are not very large. However, when the strike price is much bigger than the stockprice, we see that the prices in the NIG and Meixner cases are some thousand times larger than in the Bachelier-Samuelson case. This reflects the fact that the NIG and Meixner distributions have fatter tails than the normal, that is, there is a much larger probability for the stockprice to rise steep. For our market study, we found that all three models fitted to the observed European call options prices equally well, which may be a bit surprising, since there are more parameters in the NIG and the Meixner models. However, we just used one data set, so one shall not make any too hasty conclusions. It is also possible, that if we had used a data set with a larger spread on the strike price stock price quota, the NIG and/or Meixner would have performed better than the Black-Scholes model. If the market uses any of the pricing formulas described here, then we understand that it uses different parameters for different strike prices, and different times to maturity.

5.4 Further comments

The pricing of options with the Esscher transform for the exponential NIG-model, and other models stemming from the generalized hyperbolic distribution driven stock price, was considered by Prause [10]. For the exponential Meixner model, this was done by Schoutens [11]. However, there does not seem to be any articles in the literature where the prices in these quite different models are considered simultaneously and compared.

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