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On Semi-parametric Modelling of Stock Prices with Lévy Processes

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Abstract

In this paper we investigate a Lévy process model for logarithmic asset returns, called Combined Gaussian and Multiple Poisson (Combined). This model consists of a Wiener process combined with several rescaled and independent Poisson processes. In order to see how well the model performed we compare it with two other Lévy processes models, namely the Normal Inverse Gamma process (NIG) and the Variance Gamma process (VG), as well as the Wiener process. In order to compare the models, we fitted them to devolatilized logarithmic returns of empirical data from S&P 500 index and the ABB stock, listed on the New York stock exchange. With the parameters obtained for the four different models, we simulated new datasets. To these simulated datasets we once again fitted the models. The performance of the models was investigated by calculating the Kolmogorov-Smirnov distance.

1 Introduction

1.1 Background

When it comes to trying to find a good model for the stock price behaviour, there has been a lot of focus on Lévy processes more general than the Wiener process. The reason for this is among others that these processes can capture the fat tails and different kurtosis, that empirical data of financial assets often show, in contrast to the Wiener process. Increments of the Wiener process has a skewness equal to zero and kurtosis equal to three. Empirical data often has a negative skewness and a higher kurtosis than three, see Schoutens [3].

1.2 Notations and Definitions

The Stock-Price Process The stock-price or other financial asset-price process will be denoted $S = \{S_t, t \geq 0\}$. We will work with the logarithmic asset returns, $\log S_t$. The Bachelier-Samuelson model of the stock-price is given by

$$S_t = S_0 \mathrm{e}^{\mu t + \lambda W_t},$$

where $\mu, \lambda \in \mathbb{R}$ are parameters and W_t is a Wiener process. The more general model we used is the following

$$S_t = S_0 \mathrm{e}^{X_t},$$

where X_t is a Lévy Process. In this model the log increments, $\log S_{t+s} - \log S_t$, have the same distribution as the increments $X_{t+s} - X_s$ of X.

Characteristic function In probability theory the characteristic function, $\phi(u)$ is the Fourier transform of the probability density function $f(x)$, that is

$$\phi(u) = \mathbf{E}[\mathrm{e}^{iuX}] = \int_{-\infty}^{\infty} \mathrm{e}^{iux} f(x) dx \ .$$

2 Lévy Processes

The Lévy processes we chose to compare our model with, were the Normal Inverse Gamma (NIG) and the Variance Gamma (VG). NIG was suggested by Barndorff-Nielsen [1] and VG was suggested by Madan, Carr and Chang [2], for use within mathematical finance.

2.1 Definition of a Lévy Process

If a stochastic process $L = \{L_t, t \geq 0\}$ satisfies the following conditions, then it is a Lévy process.

$L(0) = 0$ a.s.

L has independent increments, if $0 < t_1 < t_2 < t_3 < ... < t_n$, then $L_{t_1}, L_{t_2} - L_{t_1}, L_{t_3} - L_{t_2}, ..., L_{t_n} - L_{t_{n-1}}$ are independent.

L has stationary increments, $\{L_{t+s} - L_s, t \geq 0\} =_D \{L_t, t \geq 0\}$ for $s \geq 0$.

L is stochastically continuous, $\lim_{\mathrm{h}\to 0} \mathbf{P}[|X(t+h) - X(t)| > \varepsilon] = 0$ for $\varepsilon > 0$.

L is càdàg i.e., it is right-contionuous for $t \geq 0$ and has left limits for $t > 0$.

2.2 Normal Inverse Gaussian NIG Process

The Normal Inverse Gaussian NIG process starts at zero and has stationary and independent increments. The increments are NIG distributed. If $X^{(NIG)} = \{X_t^{(NIG)}, t \geq 0\}$ is a NIG process, then $X_t^{(NIG)}$ is NIG(α, β, $t\delta$, μ) distributed. The NIG(α, β, δ, μ) distribution has the following density function

$$f_{NIG}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha\delta}{\pi} \exp(\delta\sqrt{\alpha^2 - \beta^2} + \beta(x-\mu)) \frac{K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{\sqrt{\delta^2 + (x-\mu)^2}},$$

where $x \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\delta > 0$, $\alpha > 0$, $0 \leq |\beta| \leq \alpha$ are parameters, and K is a modified Bessel function of the third kind,

$$K_\lambda(x) = \frac{1}{2} \int_0^\infty y^{\lambda-1} \mathrm{e}^{-x(y-1/y)/2} dy \text{ for } \ x > 0.$$

The NIG(α, β, δ, μ) distribution has the following characteristic function

$$\phi_{NIG}(u; \alpha, \beta, \delta, \mu) = \exp(\delta(\sqrt{\alpha^2 + \beta^2} - \sqrt{\alpha^2 + (\beta+u)^2}) + \mu u).$$

2.3 Variance Gamma VG Process

The Variance Gamma (VG) process can be expressed as the difference between two independent Gamma processes. The Gamma process $X^{(Gamma)} = \{X_t^{(Gamma)}, t \geq 0\}$ starts in zero and has independent and stationary increments. The increments are Gamma distributed, that is $X_t^{(Gamma)}$ is $Gamma(at, b)$ distributed. So if $\{X(t), t \in T\}$ and $\{Y(t), t \in T\}$ are two Gamma processes, we can express a Variance Gamma density function in the following way,

$$f_{VG}(z) = f_{X+(-Y)}(z) = \int_{-\infty}^{\infty} f_X(z+s) f_Y(s) ds,$$

where f_X and f_Y are Gamma density functions. The Gamma density function is given by

$$f_G(x; a, b) = \frac{b^a}{\Gamma(a)} x^{a-1} \exp(-xb), \; x > 0.$$

2.4 Poisson Process

The Poisson process starts in zero and has independent and stationary increments. If $X^{(P)} = \{X_t^{(P)}, t \geq 0\}$ is a Poisson process, then $X_t^{(P)}$ is $Poisson(t\lambda)$ distributed. The Poisson density function is given by

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!} \text{ for } x = 0, 1, 2, ... \text{ and } \lambda > 0.$$

The Poisson distribution has the following characteristic function

$$\phi_P(u; \lambda) = \exp(\lambda(\exp(iu) - 1)).$$

2.5 Wiener process with drift

The Wiener process with drift starts in zero and has independent and stationary increments. If $W = \{W_t, t \geq 0\}$ is a Wiener process with drift then W_t is $\mathrm{N}(\mu t, \, \sigma^2 t)$ distributed. The Normal density function, with mean $\mu \in \mathbb{R}$ and variance $\sigma^2 > 0$ is given by

$$f_N(x; \sigma, \mu) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{1}{2} \frac{(x-\mu)}{\sigma}^2).$$

The $\mathrm{N}(\mu, \, \sigma^2)$ distribution has the following characteristic function

$$\phi_N(u; \sigma, \mu) = \exp(i\mu u - \frac{1}{2}\sigma^2 u^2).$$

3 The Combined Gaussian and Multiple Poisson Process (Combined)

We wanted to find a way to estimate the e Lévy measure. This was done by calculating an approximation for the cummulant characteristic function. The approximation we estimated is called Combined Gaussian and Multiple Poisson (Combined).

In probability theory the characteristic function $\phi(u)$, is the Fourier transform of the probability density function. The cumulant characteristic function $\Psi(u)$, is equal to the logarithm of the characteristic function, that is $\Psi(u) = \log\,\phi(u)$. The cumulant characteristic function $\Psi(u)$ satisfies the Lévy-Khintchine formula, shown below, for all infinitely divisible processes,

$$\Psi(u) = i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (\exp(iux) - 1 - iux1_{\{|x|<1\}})\nu(dx)$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure on $\mathbb{R} \setminus \{0\}$, such that

$$\int_{-\infty}^{+\infty} \min\{1, x^2\}\nu(\mathrm{dx}) < \infty\ .$$

So we have that for the Lévy process,

$$\begin{aligned}\mathbf{E}[e^{iuL(1)}] &= \phi(u) = \exp(\Psi(u)) \\ &= \exp(i\gamma u - \frac{1}{2}\sigma^2 u^2 + \int_{-\infty}^{+\infty} (\exp(iux) - 1 - iux1_{\{|x|<1\}})\nu(dx)) \\ &= \exp(i\gamma u - \frac{1}{2}\sigma^2 u^2) \exp(\int_{-\infty}^{+\infty} (\exp(iux) - 1 - iux1_{\{|x|<1\}})\nu(dx)),\end{aligned}$$

where we recognize $\exp(i\gamma u - \frac{1}{2}\sigma^2 u^2)$ as the characteristic function of the Normal distribution.

If we approximate the Lévy measure ν with point masses, we get the following expression

$$\text{point mass } \nu = \sum_{k=1}^{n} a_k \delta(x - b_k)$$

Then we can express

$$\int_{-\infty}^{+\infty} (\exp(iux) - 1 - iux1_{\{|x|<1\}})\nu(dx)$$

in the following way

$$\begin{aligned}&\sum_{k=1}^{n} (\exp(iub_k) - 1 - iub_k 1_{\{|b_k|<1\}})a_k \\ &= \sum_{k=1}^{n} (\exp(iub_k) - 1)a_k - iuc\end{aligned}$$

where $c \in \mathbb{R}$ is a constant.

If we study this expression we see that it is the characteristic functions of the sum of n rescaled Poisson distributed variables, since $\mathrm{e}^{\lambda(\mathrm{e}^{iu}-1)}$ is the characteristic function of a Poisson distributed variable, plus the constant c.

The approximation for the characteristic function $\phi(u)$, is then given by the product of one characteristic function for the Normal distribution and several characteristic functions for rescaled Poisson distributions, that is

$$\exp(i\mu u - \frac{1}{2}\sigma^2 u^2) \prod_k \exp(a_k(\mathrm{e}^{iub_k} - 1)),$$

as we can let the constant c go into the constant μ.

The characteristic function of the Combined Gaussian and Multiple Poisson distribution is then given by,

$$\phi_{Combined}(u; \sigma, \mu, a_1, ..., a_n, b_1, ..., b_n) = \exp(i\mu u - \frac{1}{2}\sigma^2 u^2) \prod_k \exp(a_k(\mathrm{e}^{iub_k} - 1)).$$

4 Method to estimate parameters and calculate goodness-of-fit

4.1 Maximum Likelihood ML

The Maximum Likelihood method estimates the value of the parameters θ, associated with a density function f that are the most likely given a random sample. Which means that if $x_1, x_2, ..., x_n$ is a sample from a distribution of a random variable X, with density function f and parameters θ, we want to find the θ that maximizes the likelihood function below

$$L(\theta) = \prod_{i=1}^{n} f(x_i; \theta).$$

4.2 Kolmogorov-Smirnov (KS)

The Kolmogorov-Smirnov distance is the maximum difference between the estimated distribution function and the empirical distribution function.

The Kolmogorov-Smirnov test statistic D_n, is given by

$$D_n = \sup_x |\hat{F}(x) - F_n(x)|,$$

where n is the sample size, $\hat{F}(x)$ is the fitted estimated cummulative ditribution function, CDF and $F_n(x)$ is the empirical cummulative distribution function.

$$F_n(x) = \frac{\text{number of observations} \leq x}{\text{total number of observations}}.$$

A useful formula to calculate the KS distance numerically is

$$D_n = \max_{i \leq n}(\max\{|(i-1)/n - \hat{F}(X_{(i)})|, |i/n - \hat{F}(X_{(i)})|\}),$$

where $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ denotes the ordered data set.

Since we do not know the analytical expression for the Combined distribution function or the Combined cumulative distribution function, we estimated the cumulative distribution function by simulating 100000 observations of it and caculating the empirical distribution function of the 100000 resulting data.

4.3 The Empirical Data

4.4 The Datasets

The empirical datasets that we fitted our four distributions parameters to were the S&P 500 Index and the ABB stock, listed on the New York stock exchange. The datasets were from the time period between 18 March 2000 to 17 March 2005. The closing values from this period were adjusted for dividends and splits. The logarithmic returns were then also devolatilized, see below.

4.5 Devolatilization

We devolatilized the datasets by choosing a window size, 20. We then split the logarithmic returns of our empirical datasets in bits of the window size. Then we calculated the standard deviation, $\sigma_1, \sigma_2, ..$, for each bit and the mean, σ_{mean} of all of these standard deviations. The standard deviations were then devided by this mean. The logarithmic returns in each bit of the dataset were then divided by the σ_n/σ_{mean}.

The log returns used in all estimations presented in this paper are devolatilizated.

5 Parameter estimation

We fitted the parameters of the Normal Inverse Gamma, the Variance Gamma, the Normal and the Combined Gaussian and Poisson distribution to the devolatilized logarithmic returns of the two chosen datasets.

5.1 The estimated Normal parameters

Displayed below are the estimated parameters that we obtained for the Normal distribution, when we fitted it to the two empirical datasets with the ML method. We also calculated the Kolmogorov-Smirnov distance for these parameters, the result is also shown below.

Asset	μ	σ	KS
ABB	-0.00115	0.0386	0.0341
S&P 500	-0.000259	0.0114	0.0237

Table 1: The estimated parameters for the Normal distribution.

5.2 The estimated NIG parameters

Displayed below are the estimated parameters that we obtained for the Normal Inverse Gamma distribution, when we fitted it to the two empirical datasets with the ML method. We also calculated the Kolmogorov-Smirnov distance for these parameters, the result is also shown below.

Asset	α	β	δ	μ	KS
ABB	38.1	-3.28	0.0563	0.00372	0.0218
S&P 500	21900	2630	2.79	-0.339	0.0238

Table 2: The estimated parameters for the NIG distribution.

5.3 The estimated VG parameters

Displayed below are the estimated parameters that we obtained for the Variance Gamma distribution, when we fitted it to the two empirical datasets with the ML method. We also calculated the Kolmogorov-Smirnov distance for these parameters, the result is also shown below.

Asset	a	b	d	KS
ABB	3.48	68.0	66.4	0.0246
S&P 500	2370	6030	6020	0.0237

Table 3: The estimated parameters for the VG distribution.

5.4 The estimated Combined Gaussian and Poisson parameters

To be able to estimate the parameters for the combined model, we inverse fourier transformed the expression for its characteristic function numerically.

$$f_{Combined}(u;\sigma,\mu,a,b) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_{Combined}(u;\sigma,\mu,a,b)\exp(\mathrm{iut})\mathrm{du}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(i\mu s - \frac{1}{2}\sigma^2 s^2)\prod_{l}\exp(a_i(\mathrm{e}^{i2\pi b_i u} - 1))\exp(\mathrm{iut})\mathrm{du}.$$

We then fitted it to the two empirical datasets, with the ML method. The estimation was done in steps. First we maximized for the parameters μ, σ, a_1 and b_1. When this was done we kept b_1 as fixed. In the next step we maximized the parameters μ, σ, a_1, a_2 and b_2. Then we kept both b_1, b_2 as fixed and so on. This was done up to ten times. We also calculated the Kolmogorov-Smirnov distance. The results are displayed below.

Asset	KS
ABB	0.0199
S&P 500	0.0210

Table 4: The calculated KS-distance for Combined.

6 Simulations with the estimated parameters

When the parameters for the four distributions were fitted to the two empirical datasets, ABB stock and S&P 500 index, we simulated eight datasets, one for each set of obtained parameters.

6.1 Simulating Normal Inverse Gamma and Variance Gamma distributed data

When we simulated the Normal Inverse Gamma and the Variance Gamma distributed data, we used the fact that,

If F is a distribution function and ξ is a random number uniformly distributet on [0,1], then $\eta = F^{-1}(\xi)$ is a variable with distribution function $F_\eta = P\{\eta \leq x\} = F(x)$.

This means that

$$\int_{-\infty}^{x} f_\eta(y)dy = \xi$$

$$x = F_\eta^{-1}(\xi)$$

7 Cross estimations of the distributions parameters

When we had simulated the eight data sets, we wanted to see how well the distributions would perform on data that was otherwise distributed. In order to do this we estimated the parameters for all four of the distributions to all eight of the simulated data sets.

7.1 Estimated parameters for the Normal distribution on simulated data

The Normal distribution's parameters were estimated for the eight simulated data sets. Shown below are the results and also the calculated Kolmogorov-Smirnov distance for these parameters.

Simulated distribution	μ	σ	KS
Normal	-0.00282	0.0399	0.0145
NIG	-0.000928	0.0378	0.0340
VG	-0.00214	0.0386	0.0353
Combined	-0.0494	0.0713	0.0288

Table 5: The estimated parameters for Normal distribution on data sets simulated with the parameters obtained from the ABB stock.

Simulated distribution	μ	σ	KS
Normal	-0.000362	0.0113	0.0218
NIG	-0.000484	0.0115	0.0210
VG	-0.000852	0.0115	0.0121
Combined	-0.0395	0.0488	0.0234

Table 6: The estimated parameters for Normal distribution on data sets simulated with the parameters obtained from the S&P 500 Index.

7.2 Estimated parameters for the NIG distribution on simulated data

We fitted the parameters of the Normal Inverse Gamma distribution to the eight simulated data sets. Shown below are the obtained parameters and also the calculated Kolmogorov-Smirnov distance for these parameters.

Simulated distribution	α	β	δ	μ	KS
Normal	391	-57.1	0.602	0.0861	0.0126
NIG	42.6	-3.87	0.0604	0.00459	0.0128
VG	41.5	-0.960	0.0622	0.000698	0.0192
Combined	499	-430	0.336	0.518	0.0218

Table 7: The estimated parameters for NIG distribution on data sets simulated with the parameters obtained from the ABB stock.

Simulated distribution	α	β	δ	μ	KS
Normal	4620	-96.5	0.589	0.0120	0.0218
NIG	5990	-82.5	0.794	0.0105	0.0210
VG	12200	-551	1.61	0.0730	0.0169
Combined	240	-55.5	0.525	0.0855	0.0230

Table 8: The estimated parameters for NIG distribution on data sets simulated with the parameters obtained from the S&P 500 Index.

7.3 Estimated parameters for the VG distribution on simulated data

We fitted the parameters of the Variance Gamma distribution to the eight simulated data sets. Shown below are the results and also the calculated Kolmogorov-Smirnov distances for these parameters.

Simulated distribution	a	b	d	KS
Normal	138	419	415	0.0142
NIG	2.78	62.9	61.6	0.0154
VG	2.91	63.9	63.9	0.0177
Combined	7.36	67.0	46.2	0.0178

Table 9: The estimated parameters for VG distribution on data sets simulated with the parameters obtained from the ABB stock.

Simulated distribution	a	b	d	KS
Normal	5700	9460	9450	0.0218
NIG	315	2170	2170	0.0206
VG	14800	14900	14900	0.0169
Combined	42.3	208	174	0.0221

Table 10: The estimated parameters for VG distribution on data sets simulated with the parameters obtained from the S&P 500 Index.

7.4 Results for the Combined distribution on simulated data

The Combined Gaussian and Poisson distribution's parameters were estimated for the eight simulated data sets in the same way as with the original datasets. Shown below are the results and also calculated the Kolmogorov-Smirnov distances for these parameters.

Simulated distribution	KS
Normal	0.0118
NIG	0.0120
VG	0.0194
Combined	0.0158

Table 11: The calculated KS-distance for the Combined distribution on data sets simulated with the parameters obtained from the ABB stock.

Simulated distribution	KS
Normal	0.0208
NIG	0.0206
VG	0.0105
Combined	0.0213

Table 12: The calculated KS-distance for the Combined distribution on data sets simulated with the parameters obtained from the ABB stock.

8 Results and Conclusions

We could see that the Combined model performed very well. In both the case with the two devolatilized empirical data sets and the case with the simulated data sets, it had either the smallest KS-distance or equally small as the smallest of the others, with one exeption. VG got a smaller KS-distance on the VG distributed simulated data, with the parameters obtained for the ABB stock.

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References

[1] Barndorff-Nielsen, O.E. (1998). Processes of normal inverse Gaussian type. *Finance Stoch.* **2** 41-68.

[2] Madan Dilip B., Carr Peter and Chang Eric C. *The Variance Gamma Process and Option Pricing.* European Finance Review 2: 79-105, 1998.

[3] W. Shoutens, *Lévy Processes in Finance, Pricing Financial Derivatives*, Wiley, 2003.