IMPORTANT DEFINITIONS, THEOREMS AND EXAMPLES, AND DIFFICULT PROOFS, IN WEAK CONVERGENCE

After 'Billingsley (1999): Convergence of Probability Measures'

GRADUATE COURSE IN WEAK CONVERGENCE, SPRING 2002

- $\underline{(S,d),S',S''}$ metric spaces with Borel sets $\underline{\mathfrak{S},\mathfrak{S}',\mathfrak{S}''}$; $\underline{S_r(x)} \equiv \{y\in S : d(x,y)<r\}$, $\underline{S_r[x]} \equiv \{y\in S : d(x,y)\le r\}$ and $\underline{A_r} \equiv \{y\in S : d(y,A)\le r\}$ for $r>0$, $x\in S$, $A\subseteq S$;
- $\underline{\mathbb{C}_B(S)} \equiv \{$bounded and continuous $f:S\to\mathbb{R}\}$; $\underline{\|f\|_{\mathbb{C}_B(S)}} \equiv \sup_{x\in S}|f(x)|$;
- $\underline{\mathrm{BL}(S)} \equiv \{$bounded and Lipschitz uniformly continuous $f:S\to\mathbb{R}\}$; $\underline{\|f\|_{\mathrm{BL}(S)}} \equiv \|f\|_{\mathbb{C}_B(S)} + \sup\{|f(x)-f(y)|/d(x,y) : S\ni x\neq y\in S\}$ (cf. **MT 20**);
- $\underline{Pf} \equiv \int_S f\,dP$ for $P\in\underline{\mathcal{P}(S)}\equiv\{P$ probability measure on $S\}$ and $f\in\mathbb{C}_B(S)$;
- $\underline{(\mathcal{T},\varrho)}$ compact metric; $\underline{\mathcal{C}(\mathcal{T})}\equiv\{$continuous $f:\mathcal{T}\to\mathbb{R}\}$; $\underline{\|f\|_{\mathcal{C}(\mathcal{T})}}\equiv \sup_{x\in\mathcal{T}}|f(x)|$;
- $\mathcal{C}(\mathcal{T})\ni x\to\underline{\pi_{t_1\ldots t_k}}x\equiv(x(t_1),\ldots,x(t_k))\in\mathbb{R}^k$; $\mathbb{R}^\infty\ni x\to\underline{\pi_{i_1\ldots i_k}}x\equiv(x_{i_1},\ldots,x_{i_k})\in\mathbb{R}^k$;
- $\underline{D}\equiv\{$càdlàg $f:[0,1]\to E\}$ (continu á droite limites á gauche);
- $\underline{B}$ separable real Banach space, $\underline{\text{dual } B^\star}$ and $\underline{\text{bidual } B^{\star\star}}$ (cf. **MT 19, 34**).

1. Measures on Metric Spaces.

Theorem 1.1. *$P\in\mathcal{P}(S)$ is* <u>*regular*</u>*, i.e., $P(A) = \inf_{\text{open } G\supseteq A} P(G) = \sup_{\text{closed } F\subseteq A} P(F)$ for $A\in\mathfrak{S}$.* (Cf. **MT 14**.)

Theorem 1.2. *$P,Q\in\mathcal{P}(S)$ agree if $Pf=Qf$ for $f\in\mathrm{BL}(S)$.*

Theorem 1.3. *For S separable and complete $P\in\mathcal{P}(S)$ is* <u>*tight*</u>*, i.e., $\sup_{\text{compact } K\subseteq S} P(K)=1$.*

Definition. <u>π-system</u> $\Pi\subseteq\mathfrak{S}$ closed under finite intersection. $\mathcal{D}\subseteq\mathfrak{S}$ <u>Dynkin system</u> if $S\in\mathcal{D}$, $B\backslash A\in\mathcal{D}$ if $\mathcal{D}\ni A\subseteq B\in\mathcal{D}$, and $\bigcup_{i=1}^\infty A_i\in\mathcal{D}$ if $\{A_i\}_{i=1}^\infty\subseteq\mathcal{D}$ increasing.

Lemma. (DYNKIN) *If π-system $\Pi\subseteq\mathcal{D}$ Dynkin system, then $\sigma(\Pi)\subseteq\mathcal{D}$.*

Definition. $\mathcal{A}_P\subseteq\mathfrak{S}$ <u>separating for P</u> $\in\mathcal{P}(S)$ if $Q\in\mathcal{P}(S)$ agrees with P if $Q(\cdot)=P(\cdot)$ on $\mathcal{A}_P$. $\mathcal{A}\subseteq\mathfrak{S}$ <u>separating</u> if $P,Q\in\mathcal{P}(S)$ agree if $P(\cdot)=Q(\cdot)$ on $\mathcal{A}$.

Proposition. *π-system Π with $\sigma(\Pi)=\mathfrak{S}$ is separating.*

Example 1.1. $P\in\mathcal{P}(\mathbb{R}^k)$ is tight, with $\{(-\infty,y] : y\in\mathbb{R}^k\}$, $\{(-\infty,y) : y\in\mathbb{R}^k\}$, $\{(y,\infty) : y\in\mathbb{R}^k\}$ and $\{[y,\infty) : y\in\mathbb{R}^k\}$ separating.

Example 1.2. $(\underline{\mathbb{R}^\infty}, \sum_{i=1}^\infty \frac{1\wedge|x_i-y_i|}{2^i})$ is separable and complete, $P\in\mathcal{P}(\mathbb{R}^\infty)$ tight, and <u>finite dimensional sets $\mathbb{R}^\infty_f$</u> $\equiv\{\bigcap_{i=1}^k\pi_{n_i}^{-1}(H_i),\ H_i$ open, $n_i,k\in\mathbb{N}\}$ separating.

Example 1.3. $\underline{\mathcal{C}} \equiv \mathcal{C}[0,1]$ is separable and complete, $P \in \mathcal{P}(\mathcal{C})$ tight, and $\underline{\text{finite dimensional sets } \mathcal{C}_f} \equiv \{\pi_{t_i}^{-1}(H_i) : H_i \text{ open}, t_i \in [0,1], k \in \mathbb{N}\}$ separating, cf. **MT 1**.

2. Properties of Weak Convergence.

Definition. $\mathcal{P}(S) \ni \underline{P_n \Rightarrow P} \in \mathcal{P}(S)$ if $P_n f \to Pf$ for $f \in \mathbb{C}_B(S)$.

Definition. Give $\mathcal{P}(S)$ weak $\underline{\text{topology}}$ generated by $\mathcal{P}(S) \ni P \to Pf \in \mathbb{R}$, $f \in \mathbb{C}_B(S)$.

Proposition. *$P_n \to P$ in $\mathcal{P}(S)$ iff. $P_n \Rightarrow P$ in $\mathcal{P}(S)$!* (Cf. **MT 8**.)

Example 2.1. $\delta_{x_n} \Rightarrow \delta_x$ in $\mathcal{P}(S)$ iff. $x_n \to_S x$.

Definition. $A \in \mathfrak{S}$ $\underline{P\text{-continuity set}}$ of $P \in \mathcal{P}(S)$ if $P(\partial A) = 0$.

Theorem 2.1. (PORTMANTEAU) (Cf. **MT 10**.) *Equivalent are*

(i) *$P_n \Rightarrow P$ in $\mathcal{P}(S)$;*

(ii) *$P_n f \to Pf$ for $f \in \mathrm{BL}(S)$;*

(iii) *$\limsup_n P_n F \le PF$ for closed $F \subseteq S$;*

(iv) *$\liminf_n P_n G \ge PG$ for open $G \subseteq S$;*

(v) *$\lim_n P_n A = PA$ for P-continuity $A \in \mathfrak{S}$.*

Theorem 2.2. *For $P \in \mathcal{P}(S)$, if $\mathcal{A}_P$ π-system with opens in S countable unions of sets in $\mathcal{A}_P$, then $P_n \Rightarrow P$ in $\mathcal{P}(S)$ if $P_n(A) \to P(A)$ for $A \in \mathcal{A}_P$.*

Theorem 2.3. *For S separable, $P_n \Rightarrow P$ in $\mathcal{P}(S)$ if $P_n(A) \to P(A)$ for A in π-system $\mathcal{A}_P$ such that $\{A \in \mathcal{A}_P : x \in \mathrm{Int}(A), A \subseteq S_\varepsilon(x)\} \ne \emptyset$ for $x \in S$ and $\varepsilon > 0$.*

Definition. $\mathcal{A} \subseteq \mathfrak{S}$ $\underline{\text{convergence determining}}$ if, for any $P \in \mathcal{P}(S)$, $P_n(A) \to P(A)$ for P-continuity $A \in \mathcal{A}$ implies $P_n \Rightarrow P$ in $\mathcal{P}(S)$. (Cf. **MT 13**.)

Theorem 2.4. *If S separable, π-system $\mathcal{A}$ is convergence determining if* $\{P$-continuity $A \in \mathcal{A} : x \in \mathrm{Int}(A), A \subseteq S_\varepsilon(x)\} \ne \emptyset$ *for $P \in \mathcal{P}(S)$, $x \in S$, $\varepsilon > 0$.* (Cf. **MT 15**.)

Example. Finite intersections of open balls convergence determining if S separable.

Example 2.4. $\mathbb{R}_f^\infty$ is convergence determining for $\mathcal{P}(\mathbb{R}^\infty)$.

Example 2.5. $\mathcal{C}_f$ is $\underline{\text{not}}$ convergence determining for $\mathcal{P}(\mathcal{C})$.

Theorem 2.6. *$P_n \Rightarrow P$ in $\mathcal{P}(S)$ iff. subsequences have subsequences $\Rightarrow P$ in $\mathcal{P}(S)$.*

Proposition. $\underline{D_h} \equiv \{x \in S : h$ *not continuous at* $x\} \in \mathfrak{S}$ *for* $h : S \to S'$.

Theorem 2.7. (MAPPING THEOREM) *For $h : S \to S'$ measurable with $P(D_h) = 0$, $P_n h^{-1} \Rightarrow P h^{-1}$ in $\mathcal{P}(S')$ if $P_n \Rightarrow P$ in $\mathcal{P}(S)$.*

Example 2.6. $P_n \Rightarrow P$ in $\mathcal{P}(\mathbb{R}^\infty)$ iff. $P_n \pi_k^{-1} \Rightarrow P \pi_k^{-1}$ in $\mathcal{P}(\mathbb{R}^k)$ for $k \in \mathbb{N}$.

Example 2.7. $P_n \Rightarrow P$ in $\mathcal{P}(\mathcal{C}) \underset{\not\Leftarrow}{\Rightarrow} P_n\pi^{-1}_{t_1,\ldots,t_k} \Rightarrow P\pi^{-1}_{t_1,\ldots,t_k}$ in $\mathcal{P}(\mathbb{R}^k)$ for $k \in \mathbb{N}$.

Theorem 2.8 (MT 12). *For S', S'' separable, $P_n \Rightarrow P$ in $\mathcal{P}(S' \times S'')$ iff. $P_n(A' \times A'') \to P(A' \times A'')$ for $P(\cdot \times S'')$-continuity $A' \in \mathfrak{S}'$ and $P(S' \times \cdot)$-continuity $A'' \in \mathfrak{S}''$.*

Corollary. *For S', S'' separable, $P'_n \times P''_n \Rightarrow P' \times P''$ in $\mathcal{P}(S' \times S'')$ iff. $P'_n \Rightarrow P'$ in $\mathcal{P}(S')$ and $P''_n \Rightarrow P''$ in $\mathcal{P}(S'')$.*

3. Convergence in Distribution.

Definition. $X_n \Rightarrow X$ for S-valued r.v.'s if $\mathbf{P}X_n^{-1} \Rightarrow \mathbf{P}X^{-1}$ in $\mathcal{P}(S)$.

Definition. $A \in \mathfrak{S}$ X-continuity set of S-valued r.v. X if $\mathbf{P}\{X \in \partial A\} = 0$.

Proposition. (PORTMANTEAU) *For S-valued r.v. X_n, X, equivalent are*

(i) $X_n \Rightarrow X$ *in* S;

(ii) $\mathbf{E}\{f(X_n)\} \to \mathbf{E}\{f(X)\}$ *for* $f \in \mathrm{BL}(S)$;

(iii) $\limsup_n \mathbf{P}\{X_n \in F\} \le \mathbf{P}\{X \in F\}$ *for closed* $F \subseteq S$;

(iv) $\liminf_n \mathbf{P}\{X_n \in G\} \ge \mathbf{P}\{X \in G\}$ *for open* $G \subseteq S$;

(v) $\lim_n \mathbf{P}\{X_n \in A\} = \mathbf{P}\{X \in A\}$ *for X-continuity sets A.*

Definition. $X_n \to_{\mathbf{P}} X$ in S if $\mathbf{P}\{d(X, X_n) > \varepsilon\} \to 0$ for $\varepsilon > 0$.

Proposition. *$X_n \to_{\mathbf{P}} X$ in S iff. $d(X, X_n) \Rightarrow 0$ in $\mathcal{P}(\mathbb{R})$. $X_n \to_{\mathbf{P}} x$ in S iff. $X_n \Rightarrow x$ in $\mathcal{P}(S)$ for (non-random) $x \in S$.* (Cf. **MT 9**.)

Theorem 3.1. *$Y_n \Rightarrow X$ in $\mathcal{P}(S)$ if $X_n \Rightarrow X$ in $\mathcal{P}(S)$ and $d(X_n, Y_n) \Rightarrow 0$ in $\mathcal{P}(\mathbb{R})$.*

Definition. $P \in \mathcal{P}(S)$ has density $f : S \to \mathbb{R}$ wrt. measure μ on S if $P(A) = \int_A f \, d\mu$ for $A \in \mathfrak{S}$.

Theorem. (SCHEFFÉ) *If $f, f_n : S \to \mathbb{R}$ are densities wrt. measure μ on S with $f_n \to f$ a.e. (μ), then $\int_S |f_n - f| \, d\mu \to 0$.* (Cf. **MT 3-5**.)

Corollary. *If $P_n, P \in \mathcal{P}(S)$ have densities f, f_n wrt. measure μ on $(S, \mathfrak{S})$, and $f_n \to f$ a.e. (μ), then $P_n \Rightarrow P$ in $\mathcal{P}(S)$.*

Theorem 3.4. *If $X_n \Rightarrow X$ in $\mathcal{P}(\mathbb{R})$, then $\mathbf{E}\{|X|\} \le \liminf_n \mathbf{E}\{|X_n|\}$.*

Definition. R.v.'s X_n uniformly integrable if $\lim_{x \uparrow \infty} \sup_n \int_{|X_n| > x} |X_n| \, d\mathbf{P} = 0$.

Theorems 3.5-3.6 (MT 2). *For $\{X_n\}_{n=1}^\infty \subseteq \mathbb{L}^1(\Omega)$ with $X_n \Rightarrow X$ in $\mathcal{P}(\mathbb{R})$, $\mathbf{E}\{|X_n|\} \to \mathbf{E}\{|X|\} < \infty$ iff. $\{X_n\}_{n=1}^\infty$ are uniformly integrable. Then $\mathbf{E}\{X_n\} \to \mathbf{E}\{X\}$.*

5. Prohorov's Theorem.

Definition. $\Pi \subseteq \mathcal{P}(S)$ relatively compact if each sequence in Π has subsequence that converges in $\mathcal{P}(S)$. (Cf. **MT 27**.)

Proposition (MT 28). *$P_n \Rightarrow P$ in $\mathcal{P}(S)$ if $\{P_n\}_{n=1}^{\infty}$ is relatively compact and $P_n(A) \to P(A)$ for $A \in \mathcal{A}$, where $\mathcal{A} \subseteq \mathfrak{S}$ contains a separating class of continuity sets for each $Q \in \mathcal{P}(S)$.*

Example 5.1. Let $\{P_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(\mathcal{C})$ be relatively compact. Then $P_n \Rightarrow P$ in $\mathcal{P}(\mathcal{C})$ if $P_n \pi_{t_1,\ldots,t_k}^{-1}(B) \to P\pi_{t_1,\ldots,t_k}^{-1}(B)$ for B a finite intersection of $P\pi_{t_1,\ldots,t_k}^{-1}$-continuity balls in $\mathbb{R}^k$. Hence $P_n \Rightarrow P$ if $P_n \pi_{t_1,\ldots,t_k}^{-1} \Rightarrow P\pi_{t_1,\ldots,t_k}^{-1}$ in $\mathcal{P}(\mathbb{R}^k)$ for $k \in \mathbb{N}$.

Definition. $\Pi \subseteq \mathcal{P}(S)$ tight if $\sup\{\inf\{P(K) : P \in \Pi\} : S \supseteq K \text{ compact}\} = 1$.

Theorem 5.1-5.2. (PROHOROV) *$\Pi \subseteq \mathcal{P}(S)$ tight $\Rightarrow$ Π relatively compact. For S separable and complete, $\Pi \subseteq \mathcal{P}(S)$ relatively compact $\Rightarrow$ Π tight.*

Definition. $\mathcal{M}(\mathcal{T}) \equiv \{$signed finite Borel measures on $\mathcal{T}\}$ with $w^\star$-topology [i.e., $\{\nu \in \mathcal{M}(\mathcal{T}) : |\nu f - \mu f| < \varepsilon\}$, $f \in \mathcal{C}(\mathcal{T})$, $\varepsilon > 0$, subbasic open], $\mathcal{M}^+(\mathcal{T}) \equiv \{\mu \in \mathcal{M}(\mathcal{T}) :$ μ non-negative$\}$ and $\mathcal{M}_1^+(\mathcal{T}) = \{\mu \in \mathcal{M}^+(\mathcal{T}) : \mu(\mathcal{T}) \le 1\}$.

Lemma. *$\mathcal{M}_1^+(\mathcal{T})$ is sequentially compact.*

Math Proof. By Riesz Representation Theorem, $\mathcal{C}(\mathcal{T})^\star = \mathcal{M}(\mathcal{T})$ (**MT 18**) with $\|\mu\|_{\mathcal{M}(\mathcal{T})} = (\mu^+ + \mu^-)(\mathcal{T})$. By Alaoglus's Theorem, unit ball $(0)_1^\star$ in $\mathcal{M}(\mathcal{T})$ is $w^\star$-compact, and so is $w^\star$-closed subset $\mathcal{M}_1^+(\mathcal{T})$. $w^\star$-topology on $(0)_1^\star$ is metrizable iff. $B = \mathcal{C}(\mathcal{T})$ is separable (**MT 22**), which it is (**MT 24**)! Since $\mathcal{M}_1^+(\mathcal{T})$ metrizable and compact, it is sequentially compact. $\square$

Direct Proof. Take $\{\mu_i\} \subseteq \mathcal{M}_1^+(\mathcal{T})$ and dense $\mathcal{F} = \{f_i\}_{i=1}^{\infty}$ in $\mathcal{C}(\mathcal{T})$ (**MT 24**). Since $\{\mu_n f_1\}_{n=1}^{\infty}$ is bounded, there is convergent subsequence $\{\mu_n^{(1)} f_1\}_{n=1}^{\infty}$. Since $\{\mu_n^{(1)} f_2\}_{n=1}^{\infty}$ is bounded, there is convergent subsequence $\{\mu_n^{(2)} f_2\}_{n=1}^{\infty}$. Continue like that ... . Diagonalize so $Lf_i \equiv \lim_n \mu_n^{(n)} f_i = \lim_n \mu_n^{(i)} f_i$ exists for $i \in \mathbb{N}$.

Extend L from $\mathcal{F}$ to $\mathcal{C}(\mathcal{T})$: For $g \in \mathcal{C}(\mathcal{T})$, take $\mathcal{F} \ni g_i \to_{\mathcal{C}(\mathcal{T})} g$. Then $\{Lg_i\}_{i=1}^{\infty}$ is Cauchy, since $|Lg_i - Lg_j| \le \lim_n \mu_n^{(i \vee j)} |g_i - g_j| \le \|g_i - g_j\|_{\mathcal{C}(\mathcal{T})}$, so $Lg \equiv \lim_i Lg_i$ exists, and is independent of particular sequence $\mathcal{F} \ni g_i \to_{\mathcal{C}(\mathcal{T})} g$, since

$$|L\hat{g}_i - Lg| = \lim_j |L(\hat{g}_i - g_j)| \le \|\hat{g}_i - g\|_{\mathcal{C}(\mathcal{T})} + \limsup_j \|g - \hat{g}_i\|_{\mathcal{C}(\mathcal{T})} = \|\hat{g}_i - g\|_{\mathcal{C}(\mathcal{T})}$$

for any sequence $\mathcal{F} \ni \hat{g}_i \to_{\mathcal{C}(\mathcal{T})} g$. In same fashion, L is linear, continuous, and (trivially) positive [i.e., $Lg \ge 0$ for non-negative $g \in \mathcal{C}(\mathcal{T})$], so Riesz Representation Theorem gives $Lg = \mu g$, some $\mu \in \mathcal{M}^+(\mathcal{T})$, which is in $\mathcal{M}_1^+(\mathcal{T})$ (take $g = 1$).

To show $\mu_n^{(n)} \to_{w^\star} \mu$, pick $g \in \mathcal{C}(\mathcal{T})$ and $\mathcal{F} \ni g_i \to_{\mathcal{C}(\mathcal{T})} g$. Since $(\mu_n^{(n)} - \mu) g_i \to 0$, $\lim_n |\mu_n^{(n)} g - \mu g| \le \limsup_n \big(|(\mu_n^{(n)} - \mu)(g - g_i)| + |(\mu_n^{(n)} - \mu) g_i|\big) \le 2\|g - g_i\|_{\mathcal{C}(\mathcal{T})}$. $\square$

Proof of $\boxed{\Rightarrow}$ *in Theorem.* Take $\{P_i\}_{i=1}^{\infty} \subseteq \Pi$, and compacts $K_1 \subseteq K_2 \subseteq \ldots \subseteq S$ with $\sup\{P_i(K_n^c) : i \in \mathbb{N}\} \leq \frac{1}{n}$. By Lemma, restriction $\{P_i|K_1\}_{i=1}^{\infty}$ has $w^\star$-convegent subsequence $\{P_i^{(1)}|K_1\}_{i=1}^{\infty}$. By Lemma, $\{P_i^{(1)}|K_2\}_{i=1}^{\infty}$ has $w^\star$-convegent subsequence $\{P_i^{(2)}|K_2\}_{i=1}^{\infty}$. Continue like that ... .

Since $P^{(k)}(\cdot\cap K_k) \equiv \lim_i P_i^{(k)}(\cdot\cap K_k)$ increases with k, $P \equiv \lim_k P^{(k)}(\cdot\cap K_k)$ exists. Here P is finitely additive, by additivity of $P^{(k)}(\cdot\cap K_k)$. Further, for $\varepsilon > 0$,

$$\sup_{\text{compact } K\subseteq A} P(K) \geq \sup_{\text{compact } K\subseteq A} P_i^{(k)}(K\cap K_k) - \varepsilon$$
$$= P_i^{(k)}(A\cap K_k) - \varepsilon \to P(A) - \varepsilon \quad \text{as } i\to\infty \text{ and } k\to\infty.$$

So $P(A) = \sup_{\text{compact } K\subseteq A} P(K)$, i.e., tightness. Tight finitely additive probability setfunctions are additive (**MT 17**). And $P_i^{(i)} \Rightarrow P$ in $\mathcal{P}(S)$, since for closed $F \subseteq S$

$$P(F) \geq \lim_i P_i^{(n)}(F\cap K_n) = \lim_i P_i^{(i)}(F\cap K_n) \geq \limsup_i P_i^{(i)}(F) - \tfrac{1}{n}. \quad \square$$

Corollary. *If $\{P_n\}_{n=1}^{\infty} \subseteq \mathcal{P}(S)$ is tight, and each convergent subsequence converges to P in $\mathcal{P}(S)$, then $P_n \Rightarrow P$ in $\mathcal{P}(S)$.*

B-valued Random Variables.

Definition. Mean $E \in B^{\star\star}$ of B-valued r.v. X is $B^\star \ni \eta \to E(\eta) \equiv \mathbf{E}\{\eta(X)\} \in \mathbb{R}$. If mean $E \in B^{\star\star}$ lies in B (cf. **MT 22**), it is Pettis-mean of X.

Definition. B-valued r.v. X has Bochner mean $E\{X\} \equiv \lim_n \int_\Omega X_n \, d\mathbf{P} \in B$ if $\big\{\sum_{i=1}^k x_i \mathbf{1}_{A_i} : x_i \in B,\ A_i \text{ events in } \Omega,\ k \in \mathbb{N}\big\} \ni X_n \to_{\text{a.e.}} X$ with $\mathbf{E}\{\|X - X_n\|\} \to 0$.

Proposition (MT 34). *B-valued r.v. X has Bochner mean iff. $\mathbf{E}\{\|X\|\} < \infty$. If so X has same Pettis mean, and $\|\mathbf{E}\{X\}\| \leq \mathbf{E}\{\|X\|\}$.*

Definition. Covariance $Q : B^\star \to B^{\star\star}$ of B-valued r.v. X is $B^\star \ni \eta \to Q(\eta) \equiv \mathbf{Cov}\{\eta(X), \cdot(X)\} \in \mathbb{R}$.

Theorem. *B-valued r.v. X has mean $E \in B^{\star\star}$ if $\mathbf{E}\{|\eta(X)|\} < \infty$ for $\eta \in B^\star$, and covariance $Q : B^\star \to B^{\star\star}$ if $\mathbf{E}\{(\eta(X))^2\} < \infty$ for $\eta \in B^\star$.*

Proof. $B^\star \ni \eta \to L(\eta) \equiv \eta(X) \in \mathbb{L}^1(\Omega)$ is linear with $\{(\eta, L(\eta)) : \eta \in B^\star\}$ closed, since $\eta_\alpha \to_{B^\star} \eta$ and $L(\eta_\alpha) = \eta_\alpha(X) \to_{\mathbb{L}^1(\Omega)} Y$ give $\eta_\alpha(X) \to \eta(X)$, $\omega \in \Omega$, so $\eta_\alpha(X) \to_{\mathbf{P}} \eta(X)$ and $\mathbb{L}^1(\Omega)$-limit Y must be $\eta(X)$. By Closed Graph Theorem L is bounded, so linear map $E(\eta) = \mathbf{E}\{\eta(X)\}$ has $|E(\eta)| \leq \mathbf{E}\{|\eta(X)|\} = \|L(\eta)\|_{\mathbb{L}^1(\Omega)} \leq \|L\| \|\eta\|_{B^\star}$, i.e., is in $B^{\star\star}$. Q is similar (**MT 35**). $\square$

Definition. B-valued r.v. X $[P \in \mathcal{P}(B)]$ Gaussian if $\eta(X)$ $[P\eta^{-1}]$ Gaussian for $\eta \in B^\star$.

Corollary. *Gaussian B-valued r.v. has well-define mean and covariance.*

Direct Proof. $(Q\eta)(\nu)=\mathbf{Cov}\{\eta(X),\nu(X)\}$ is linear in ν. If $\nu_\alpha\to_{B^\star}0$, then $\nu_\alpha(X)\to 0$, $\omega\in\Omega$, so that $\mathbf{Var}\{\nu_\alpha(X)\}\to 0$ by nature of Gaussian r.v.'s. Hence $|(Q\eta)(\nu_\alpha)|\leq\sqrt{\mathbf{Var}\{\eta(X)\}}\sqrt{\mathbf{Var}\{\nu_\alpha(X)\}}\to 0$. E is similar. □

Definition. Process $\{X(t)\}_{t\in T}$ Gaussian if $\sum_{i=1}^k a_i X(t_i)$ N-distributed, $a_i\in\mathbb{R}$, $t_i\in T$, $k\in\mathbb{N}$, and standard Wiener if zero-mean Gaussian with $\mathbf{E}\{X(s)X(t)\}=s\wedge t$.

Example. Pick continuous Gaussian process $\{X(t)\}_{t\in[0,1]}$ with (continuous) mean $m(t)=\mathbf{E}\{X(t)\}$ and covariance $r(s,t)=\mathbf{Cov}\{X(s),X(t)\}$. Since $X_n(t)\equiv X(\lfloor n t\rfloor/n)+(nt-\lfloor nt\rfloor)X(\lceil nt\rceil/n)\to_{\mathcal{C}} X$ surely, for $F\in\mathrm{BV}[0,1]=\mathcal{C}^\star$, $FX_n\to FX$ surely. Since FX_n is Gaussian, FX is, so $X\sim\mathrm{N}_{\mathcal{C}}(E,Q)$. (Conversely, a $\mathcal{C}$-valued Gaussian r.v. trivially corresponds to continuous Gaussian process.)

For $F,G\in\mathrm{BV}[0,1]$, by Fubini, $E(F)=\mathbf{E}\{FX\}=\mathbf{E}\{\int_0^1 X\,dF\}=\int_0^1 m\,dF$, and

$$(QF)(G)=\mathbf{Cov}\{FX,QX\}=\mathbf{Cov}\{\textstyle\int_0^1 X\,dF,\int_0^1 X\,dG\}=\int_0^1\int_0^1 r(s,t)\,dF(s)dG(t).$$

Since $E(F)=F(m)=m(F)$, mean is Pettis. For standard Wiener process W

$$\begin{aligned}\textstyle\int_0^1\int_0^1 r_W\,dFdG &= \textstyle\int_0^1 t(\int_t^1 dF(s))dG(t)+\int_0^1 s(\int_s^1 dG(t))dF(s)\\ &= \textstyle\int_0^1 t(F(1)-F(t))dG(t)+\int_0^1 s(G(1)-G(s))dF(s)\\ &= \textstyle\big[-t(F(1)-F(t))(G(1)-G(t))\big]_0^1+\int_0^1(F(1)-F(t))(G(1)-G(t))\,dt.\end{aligned}$$

Characteristic Functions (chf.).

Definition. Chf. of B-valued r.v. X is $B^\star\ni\phi_X(\eta)\equiv\mathbf{E}\{\mathrm{e}^{i\eta(X)}\}\in\mathbb{C}$.

Proposition. *Chf. of B-valued r.v. X is $w^\star$-continuous and non-negative definite.*

Proof. For $\{\eta_\alpha\}_{\alpha\in A}\subseteq B^\star$ with $\eta_\alpha\to_{w^\star}\eta$, $\eta_\alpha(X)\to\eta(X)$ surely, so $\eta_\alpha(X)\Rightarrow\eta(X)$ in $\mathcal{P}(\mathbb{R})$ and $\phi_X(\eta_\alpha)=\phi_{\eta_\alpha(X)}(1)\to\phi_{\eta(X)}(1)=\phi_X(\eta)$. Further, $\sum_{k,\ell=1}^n a_k\overline{a_\ell}\phi_X(\eta_k-\eta_\ell)=\mathbf{E}\{\sum_{k,\ell=1}^n a_k\overline{a_\ell}\mathrm{e}^{i[\eta_k(X)-\eta_\ell(X)]}\}=\mathbf{E}\{|\sum_k^n a_k\mathrm{e}^{i\eta_k(X)}|^2\}$. □

Proposition. *Distributions of B-valued r.v.'s agree iff. their chf. do.*

Proof. $\boxed{\Rightarrow}$ $\mathbf{P}X^{-1}=\mathbf{P}Y^{-1}$ on Dynkin System containing π-system $\Pi\equiv\{\bigcap_{i=1}^k\{x\in B:|\eta_i(x-y_i)|<\varepsilon_i\},\eta_i\in B^\star,y_i\in B,\varepsilon_i>0,k\in\mathbb{N}\}$, since $\mathbf{E}\{\mathrm{e}^{i\sum_{\ell=1}^k a_\ell\eta_\ell(X)}\}=\mathbf{E}\{\mathrm{e}^{i\sum_{\ell=1}^k a_\ell\eta_\ell(Y)}\}$. For $z\in B$, there is $\eta_z\in(0)_1^\star$ with $\eta_z(z)=\|z\|$ (**MT 33**), so for countable $w^\star$-dense $D^\star\subseteq(0)_1^\star$ (that is $w^\star$-compact metric, by **MT 22**)

$$(\star)\qquad \{y\in B:\|x-y\|<r\}=\textstyle\bigcap_{\eta\in D^\star}\{y\in B:|\eta(x-y)|<r\},$$

so $\sigma(\Pi)$ hosts B-balls. Here definition of $\|\cdot\|^\star$ gives $\subseteq$, while $\supseteq$ is from $D^\star$ includes items arbitrarily $w^\star$-close to η_{x-y}. [Notice $(\star)$ means $(0)_1^\star$ separates points in B, since $\eta(x)=\eta(\hat{x})$ for $\eta\in(0)_1^\star\Rightarrow\{y:\|x-y\|<r\}=\{y:\|\hat{x}-y\|<r\}$, $r>0$]. □

Corollary. *$X_n \Rightarrow X$ in $\mathcal{P}(B)$, some X, iff. $\{\mathbf{P}X_n^{-1}\}$ is relatively compact and $\phi_{X_n} \to \phi$ on $B^\star$, some ϕ. Then $\phi_X = \phi$.*

Example. Distribution of B-valued Gaussian X given by E, Q, since $\phi_X(\eta) = e^{iE\eta - \frac{1}{2}(Q\eta)(\eta)}$, and given by marginal distributions of Gaussian process $\{\eta(X)\}_{\eta \in B^\star}$. To $E \in B^{\star\star}$ and symmetric non-negative $Q : B^\star \to B^{\star\star}$, there is Gaussian process $\{x(\eta)\}_{\eta \in B^\star}$ with $\mathbf{E}\{x(\eta)\} = E(\eta)$ and $\mathbf{Cov}\{x(\eta), x(\nu)\} = (Q\eta)(\nu)$. If Q is covariance of B-valued Gaussian r.v. X, $\eta(X) =_d x(\eta)$ for $\eta \in B^\star$ (**MT 36**).

Example. For $\xi_1, \xi_2, \ldots$ iid. standardized r.v.'s, the $\mathcal{C}$-valued r.v.'s $X_n(t) \equiv n^{-1/2}\big(\sum_{j=1}^{\lfloor nt \rfloor} \xi_j + (nt - \lfloor nt \rfloor)\xi_{\lceil nt \rceil}\big)$ satisfy $\phi_{X_n}(F) \to \phi_W(\eta)$ for continuous $F \in$ BV$[0,1]$, since elementary chf. $\phi_{\xi_1}(t)$ is $1 - \frac{1}{2}t^2(1 + o(1))$, so that

$$\begin{aligned}
\phi_{X_n}(F) = \mathbf{E}\{e^{i\int_0^1 X_n\, dF}\} &= \mathbf{E}\{\exp[[iX_n(t)\,(F(t) - F(1))]_0^1 + i\textstyle\int_0^1 X_n'(t)\,(F(1) - F(t))\,dt]\} \\
&= \mathbf{E}\{\exp[0 + i\textstyle\sum_{k=1}^n \int_{(k-1)/n}^{k/n} \sqrt{n}\,\xi_k\,(F(1) - F(t))\,dt]\} \\
&\sim \textstyle\prod_{k=1}^n \big[1 - \frac{1}{2}\big(\sqrt{n}\int_{(k-1)/n}^{k/n}(F(1) - F(t))\,dt\big)^2\big] \\
&= \exp\{\textstyle\sum_{k=1}^n \ln\big[1 - \frac{1}{2}n\big(\int_{(k-1)/n}^{k/n}(F(1) - F(t))\,dt\big)^2\big]\} \\
&\sim \exp\{-\tfrac{1}{2}\textstyle\sum_{k=1}^n \int_{(k-1)/n}^{k/n}(F(1) - F(t))^2\,dt\} = \phi_W(F).
\end{aligned}$$

Chf. for infinite dimensional r.v.'s are chiefly used to show unicity, since tightness issue is difficult. We give flavour of theory by listing few results from the literature.

Hilbert-Schmidt topology on real separable Hilbert space H is weak topology generated by the Hilbert-Schmidt operators [i.e., all linear functions $L : H \to H$ with $\sum_{k=1}^\infty \|Le_k\|^2 < \infty$ for some ortonormal basis $\{e_k\}_{k=1}^\infty \subseteq H$].

Theorem. (Sazonov) Let H be a real separable Hilbert space. A function $\phi : H \to \mathbb{C}$ is chf. for some H-valued r.v. iff. it is continuous in the Hilbert-Schmidt topology, with $\phi(0) = 1$, and non-negative definite.

For converse, notice linear $L : H \to H$ is continuous iff. bilinear form $H \times H \ni (x, y) \to L(x, y) \equiv \langle Lx, Ly \rangle \in \mathbb{R}$ is.

Theorem. (Mouchtari) If there exists weak topology on $B^\star$ generated by class of bilinear forms $B^\star \times B^\star \ni (x, y) \to L(x, y) \in \mathbb{R}$, such that $\phi : B^\star \to \mathbb{C}$ is chf. iff. it is non-negative definite, with $\phi(0) = 1$, and continuous, then B is Hilbert space.

Complete linear metric space Φ over $\mathbb{R}$ countably Hilbert if metric is $\sum_{k=1}^\infty 2^{-k}(1 \wedge \|\cdot\|_k)$ for Hilbertian norms $\{\|\cdot\|_k\}_{k=1}^\infty$ with $\|\cdot\|_k$-Cauchy $\|\cdot\|_\ell$-convergent sequences $\|\cdot\|_k$-convergent for $k, \ell \in \mathbb{N}$. Countably Hilbert space Φ nuclear if $\sum_{n=1}^\infty \|\eta_n\| < \infty$ iff. $\sum_{n=1}^\infty |\eta_n(x)| < \infty$ for $x \in \Phi$, whenever $\{\eta_n\}_{n=1}^\infty \subseteq \Phi^\star$.

Countably Hilbert spaces are *much* more general than Hilbertian ones. Nuclear spaces are *very* special, but include e.g., the Schwarz spaces $\mathcal{D}$ and $\mathcal{S}$.

A r.v. X with values in a topological vector space (TVS) E, with dual $E^\star \equiv \{\text{continuous linear } \eta : E \to \mathbb{R}\}$, has chf. $E^\star \ni \eta \to \phi_X(\eta) \equiv \mathbf{E}\{e^{i\eta(X)}\}$.

Theorem. (BOULICAUT) Let E be a separable metrizable HLCTVS (Hausdorff locally convex TVS). We have, for any individually tight $P_n, P \in \mathcal{P}(E)$ with chf. ϕ_n, ϕ, that $\phi_n(\eta) \to \phi(\eta)$ for $\eta \in E^\star$ implies $P_n \Rightarrow P$ in $\mathcal{P}(E)$, iff. E is nuclear.

6. Miscellany [METRICS FOR WEAK CONVERGENCE (CF. MT 30)].

Theorem. *For S separable, $\mathcal{P}(S)$ is metrized by dual bounded Lipschitz norm $\|P\|^\star_{\mathrm{BL}} \equiv \sup\{Pf : f \in BL(S), \|f\|_{\mathrm{BL}} \le 1\}$.* (Cf. **MT 21**.)

Proof. We show $\|P_\alpha - P\|_{\mathrm{BL}^\star} \to 0$ iff. $P_\alpha f \to Pf$ for $f \in \mathrm{BL}(S)$. $\boxed{\Rightarrow}$ is trivial, since $|P_\alpha f - Pf| \le \|f\|_{\mathrm{BL}} \|P_\alpha - P\|^\star_{\mathrm{BL}}$. For $\boxed{\Leftarrow}$, pick $\varepsilon > 0$, dense $\{x_i\}_{i=1}^\infty \subseteq S$, and P-continuity $S_{r_i}(x_i)$, $r_i \in [\frac{\varepsilon}{4}, \frac{\varepsilon}{2}]$, so $B_i \equiv S_{r_i}(x_i) \backslash (\cup_{j=1}^{i-1} S_{r_j}(x_j))$ partition S. Since $|f(x) - f(x_i)| \le \frac{\varepsilon}{2} \|f\|_{\mathrm{BL}}$, $x \in B_i$, and $|f| \le \|f\|_{\mathrm{BL}}$, for $f \in \mathrm{BL}(S)$, Scheffé gives

$$|P_\alpha f - Pf| = \Big|\sum_i \Big(\int_{B_i} (f - f(x_i))\, dP_\alpha + f(x_i)(P_\alpha(B_i) - P(B_i)) + \int_{B_i} (f(x_i) - f)\, dP\Big)\Big|$$
$$\le \Big(\varepsilon + \sum_{i=1}^\infty |P_\alpha(B_i) - P(B_i)|\Big) \|f\|_{\mathrm{BL}} \to \varepsilon \|f\|_{\mathrm{BL}}. \qquad \square$$

Theorem 6.8. *For S separable, $\mathcal{P}(S)$ is separable and metrized by Prohorov metric $\rho(P, Q) \equiv \inf\{\varepsilon \ge 0 : P(F) \le Q(F_\varepsilon) + \varepsilon$ for closed $F \subseteq S\}$.*

Proof. $\rho(P, P) = 0$. If $\rho(P, Q) = 0$, $P(F) \le Q(F)$, closed F, since $Q(F_\varepsilon) \downarrow Q(F)$, so $P(G) = 1 - P(G^c) \ge 1 - Q(G^c) = Q(G)$, open G, i.e., $P = Q$ by regularity. If $P(F) \le Q(F_\varepsilon) + \varepsilon$, closed F, $P(F_\varepsilon) = 1 - \lim_{\hat\varepsilon \downarrow \varepsilon} P((F_{\hat\varepsilon})^c) \ge 1 - \lim_{\hat\varepsilon \downarrow \varepsilon} Q(((F_{\hat\varepsilon})^c)_\varepsilon) - \varepsilon \ge Q(F) - \varepsilon$ since $F \subseteq (((F_{\hat\varepsilon})^c)_\varepsilon)^c$ (see **MT 11**) so symmetry. $P(F) \le R(F_{\hat\varepsilon}) + \hat\varepsilon \le Q((F_{\hat\varepsilon})_{\tilde\varepsilon}) + \hat\varepsilon + \tilde\varepsilon \le Q(F_{\hat\varepsilon + \tilde\varepsilon}) + \hat\varepsilon + \tilde\varepsilon$ (**MT 11**) if $\rho(P, R) < \hat\varepsilon$, $\rho(R, Q) < \tilde\varepsilon$, so Δ-inequailty.

If $P_n \to_\rho P$, then, given $\varepsilon > 0$, $P_n(F) \le P(F_\varepsilon) + \varepsilon$ for closed $F \in S$, for n large enough, so that $\limsup_n P_n(F) \le P(F)$, i.e., $P_n \Rightarrow P$ in $\mathcal{P}(S)$.

Let $P_n \Rightarrow P$ in $\mathcal{P}(S)$, so that $P_n \to_{\|\cdot\|^\star_{\mathrm{BL}}} P$. Given $\varepsilon > 0$, for closed $F \subseteq S$, $f_F \equiv (1 - d(\cdot, F)/\varepsilon))^+ \in \mathrm{BL}(S)$ with $\|f_F\|_{\mathrm{BL}} \le 1 + 1/\varepsilon$ and $\mathbf{1}_F \le f_F \le \mathbf{1}_{F_\varepsilon}$. Hence $P(F) \le Pf_F \le P_n f_F + \|P - P_n\|^\star_{\mathrm{BL}} \|f_F\|_{\mathrm{BL}} \le P_n(F_\varepsilon) + \varepsilon$ for $\|P - P_n\|^\star_{\mathrm{BL}} \le \frac{\varepsilon}{1 + \frac{1}{\varepsilon}}$.

Let $\{A_k\}_{k=1}^\infty \subseteq \mathfrak{S}$ partition S with $d(A_k) \le \frac{1}{n}$, and $a_k \in A_k$. Put $N_n \equiv \{\sum_{k=1}^\infty r_k \delta_{a_k} \in \mathcal{P}(S) : r_k \in \mathbb{Q}\}$. For $P \in \mathcal{P}(S)$, pick $Q = \sum_{k \le \ell} r_k \delta_{a_k} \in N_n$ with $\sum_{k=1}^\ell |P(A_k) - r_k| \le \frac{1}{2n}$, where $\mathbf{P}\{\cup_{k > \ell} A_k\} < \frac{1}{2n}$. Then $\rho(P, Q) \le \frac{1}{n}$, since for $F \subseteq S$ closed,

$$P(F) \le \sum_{k \le \ell : F \cap A_k \ne \emptyset} P(A_k) + \frac{1}{2n} \le \sum_{k \le \ell : F \cap A_k \ne \emptyset} r_k + \frac{1}{n} \le Q(F_{1/n}) + \frac{1}{n}. \quad \square$$

Theorem 6.8. *$(\mathcal{P}(S),\rho)$ and $(\mathcal{P}(S),\|\cdot\|_{\mathrm{BL}}^{\star})$ are separable complete for S such.*

Proof. By inspection of above proof, $\|\cdot\|_{\mathrm{BL}}^{\star}$-Cauchy is ρ-Cauchy, so is enough do latter. For such, is enough show tight, by Prohorov (cf. **MT 42**). By familiar arguing, this follows if, given $\varepsilon,\delta>0$, $P_n(\cup_{i=1}^{j}S_\delta[x_i])\geq 1-\varepsilon$, $n\in\mathbb{N}$, some $\{x_i\}_{i=1}^{j}\subseteq S$. But $\rho(P_k,P_n)\leq(\varepsilon\wedge\delta)/2$, $k,n\geq n_0$, and $P_n(\cup_{i=1}^{j}S_{\delta/2}[x_i])\geq 1-\varepsilon/2$, $n\leq n_0$, give $P_n(\cup_{i=1}^{j}S_\delta[x_i])\geq P_n\big((\cup_{i=1}^{j}S_{\delta/2}[x_i])_{\delta/2}\big)\geq P_{n_0}(\cup_{i=1}^{j}S_{\delta/2}[x_i])-\varepsilon/2\geq 1-\varepsilon$, $n\geq n_0$. $\square$

7. Weak Convergence in $\mathcal{P}(\mathcal{C}(\mathcal{T}))$.

Theorem 7.3. *$\Pi\subseteq\mathcal{P}(\mathcal{C}(\mathcal{T}))$ is relatively compact iff. the following two holds*

(i) *$\{P\pi_t^{-1}:P\in\Pi\}$ is tight for $t\in\mathcal{T}$; and*

(ii) *To each $\varepsilon>0$ there exist $\delta>0$ and finite $\Gamma\subseteq\Pi$ such that*

$$P\big(\{x\in\mathcal{C}(\mathcal{T}):\sup\{|x(s)-x(t)|:s,t\in\mathcal{T},\ \varrho(s,t)\leq\delta\}>\varepsilon\}\big)\leq\varepsilon \qquad \textit{for}\quad P\in\Pi\backslash\Gamma.$$

Proof. $\boxed{\Rightarrow}$ Since $S=\mathcal{C}(\mathcal{T})$ separable (**MT 24**) and complete, Prohorov shows Π tight, so for $\varepsilon>0$, $\inf\{P(K):P\in\Pi\}\geq 1-\varepsilon$, some compact $K\subseteq S$. By Ascoli, $K\subseteq\{x\in S:\sup_{s,t\in\mathcal{T},\,\varrho(s,t)\leq\delta}|x(t)-x(s)|\leq\varepsilon\}$, some $\delta>0$, which gives **(ii)**, with $\Gamma=\emptyset$. By Ascoli, for $t\in\mathcal{T}$, $K\subseteq\{x\in S:|x(t)|\leq M\}$, some $M>0$, so **(i)** follows from

$$\inf\{P\pi_t^{-1}([-M,M]):P\in\Pi\}=\inf\{P\big(\{x\in S:|x(t)|\leq M\}\big):P\in\Pi\}\geq 1-\varepsilon.$$

$\boxed{\Leftarrow}$ By Prohorov's Theorem, it suffices show Π tight. Since Γ tight, proof of $\Rightarrow$ gives **(ii)** for Γ, so may put $\Gamma=\emptyset$ in **(ii)**. Take $\delta_k>0$ with $P\big(\{x\in S:\sup\{|x(s)-x(t)|:s,t\in\mathcal{T},\ \varrho(s,t)\leq\delta_k\}>\frac{1}{k}\}\big)\leq 2^{-k}\varepsilon$ for $P\in\Pi$. Let $\mathcal{T}\subseteq(\{t_1,\ldots,t_m\})_{\delta_1}$ ($\mathcal{T}$ totally bounded), where by **(i)**, $P\big(\{x\in S:|x(t_i)|>M\}\big)\leq\frac{\varepsilon}{2m}$, some $M>0$. By Ascoli, the following set $K=\mathrm{clos}(K)$ is compact, with $P(K^c)\leq\varepsilon$ for $P\in\Pi$,

$$K\equiv\bigcap_{i=1}^{m}\{x\in S:|x(t_i)|\leq M\}\cap\bigcap_{k=1}^{\infty}\{x\in S:\sup_{s,t\in\mathcal{T},\,\varrho(s,t)\leq\delta_k}|x(s)-x(t)|\leq\tfrac{1}{k}\}.\ \square$$

Corollary (MT 33). *For $\mathcal{T}$ connected,* **(i)** *in* Theorem 7.3 *may be replaced with*

(i′) *$\{P\pi_t^{-1}:P\in\Pi\}$ is tight for some $t\in\mathcal{T}$.*

Corollary (MT 33). **(ii)** *in* Theorem 7.3 *may be replaced with*

(ii′) *To each $\varepsilon>0$ there exist $\delta>0$ and finite $\Gamma\subseteq\Pi$ such that*

$$P\big(\{x\in\mathcal{C}(\mathcal{T}):\sup\{|x(s)-x(t)|:s\in\mathcal{T},\ \varrho(s,t)\leq\delta\}>\varepsilon\}\big)\leq\varepsilon \quad \textit{for}\quad P\in\Pi\backslash\Gamma,\ t\in\mathcal{T}.$$

Corollary (MT 33). *$P_n\Rightarrow P$ in $\mathcal{P}(\mathcal{C}(\mathcal{T}))$, some P, iff. the following two hold*

(i) *$P_n\pi_{t_1,\ldots,t_k}^{-1}\Rightarrow$ in $\mathcal{P}(\mathbb{R}^k)$ for $t_i\in\mathcal{T}$, $k\in\mathbb{N}$* (where the limit must be $P\pi_{t_1,\ldots,t_k}^{-1}$)*;*

(ii) *To each $\varepsilon>0$ there exist $\delta>0$ and $n_0\in\mathbb{N}$ such that*

$$P_n\big(\{x\in\mathcal{C}(\mathcal{T}):\sup\{|x(s)-x(t)|:s\in\mathcal{T},\ \varrho(s,t)\leq\delta\}>\varepsilon\}\big)\leq\varepsilon \quad \textit{for}\quad n\geq n_0,\ t\in\mathcal{T}.$$

Criterion. (Cf. **MT 38**) *$\mathcal{C}$-valued r.v.'s $\{X_n\}_{n=1}^\infty$ relatively compact if* $\lim_{M\to\infty}$ $\sup_n \mathbf{P}\{|X_n(\hat{t})|>M\}=0$, *some* $\hat{t}\in[0,1]$, *and for some constants* $C,\beta,\gamma,x_0>0$

$$\sup_n \mathbf{P}\{|X_n(s)-X_n(t)|>x\}\leq C\,|t-s|^{1+\gamma}/x^\beta \qquad \textit{for} \quad s,t\in[0,1],\ x\in(0,x_0].$$

Proof. **(i′)** from first condition, and **(ii)** from second, since picking $a\in(0,\gamma/\beta)$,

$$\begin{aligned}
&\mathbf{P}\big\{\sup_{s\in[t-\delta,t+\delta]}|X_n(s)-X_n(t)|>\varepsilon\big\}\\
&\leq \mathbf{P}\big\{\textstyle\bigcup_{k=1}^\infty\bigcup_{\frac{\ell}{2^k}\in[-\delta,\delta]}\{|X_n(t+\frac{\ell}{2^k})-X_n(t)|>(1-\frac{1}{2^{ak}})\varepsilon\}\big\}\\
&\leq \mathbf{P}\big\{\textstyle\bigcup_{\frac{\ell}{2}\in[-\delta,\delta]}\{|X_n(t+\frac{\ell}{2})-X_n(t)|>(1-\frac{1}{2^a})\varepsilon\}\big\}\\
&\qquad+\sum_{k=1}^\infty \mathbf{P}\Big\{\textstyle\bigcup_{\frac{i}{2^{k+1}}\in[-\delta,\delta]}\{|X_n(t+\frac{i}{2^{k+1}})-X_n(t)|>(1-\frac{1}{2^{a(k+1)}})\varepsilon\},\\
&\qquad\qquad\qquad\textstyle\bigcap_{\frac{j}{2^k}\in[-\delta,\delta]}\{|X_n(t+\frac{j}{2^k})-X_n(t)|\leq(1-\frac{1}{2^{ak}})\varepsilon\}\Big\}\\
&\leq 0+\sum_{k=1}^\infty\sum_{\frac{i}{2^{k+1}}\in[-\delta,\delta]}\mathbf{P}\big\{|X_n(t+\tfrac{2i+1}{2^{k+1}})-X_n(t)|-|X_n(t+\tfrac{2i(+2)}{2^{k+1}})-X_n(t)|>(1-\tfrac{1}{2^a})\tfrac{\varepsilon}{2^{ak}}\big\}\\
&\leq \textstyle\sum_{k=1}^\infty 2^{k+3}\delta\, c\, 2^{-(k+1)(1+\gamma)}\big/\big((1-\frac{1}{2^a})\frac{\varepsilon}{2^{ak}}\big)^\beta\leq\varepsilon \qquad \text{for}\quad \delta>0\quad\text{small enough.}\quad\Box
\end{aligned}$$

8. Wiener Measure and Donsker's Theorem.

Theorems 8.1-8.2 (MT 37). (Donsker) *For iid. standardized r.v.'s $\xi_1,\xi_2,\ldots$,* $\{X_n(t)\}_{t\in[0,1]}\equiv\{n^{-1/2}(\sum_{j=1}^{\lfloor nt\rfloor}\xi_j+(nt-\lfloor nt\rfloor)\xi_{\lceil nt\rceil})\}_{t\in[0,1]}\Rightarrow W$ *in* $\mathcal{P}(\mathcal{C})$.

Proof. For $[\frac{k}{n},\frac{k+1}{n})\ni(t-\delta)\vee 0<(t+\delta)\wedge 1\in[\frac{\ell-1}{n},\frac{\ell}{n})$, by Doob-Kolmogorov Inequality

$$\begin{aligned}
\mathbf{P}\big\{\sup_{s\in[t-\delta,t+\delta]}|X_n(t)-X_n(s)|>\varepsilon\big\}&\leq 2\mathbf{P}\big\{\sup_{j\in\{k,\ldots,\ell\}}|X_n(\tfrac{j}{n})-X_n(\tfrac{k}{n})|>\tfrac12\varepsilon\big\}\\
&=2\mathbf{P}\big\{\sup_{i\in\{1,\ldots,\ell-k\}}|\textstyle\sum_{j=k+1}^{k+i}\xi_j|>\frac12\varepsilon\sqrt{n}\big\}\\
&\leq 2\mathbf{E}\big\{(\textstyle\sum_{j=1}^{\ell-k}\xi_j)^2\big\}/(\frac12\varepsilon\sqrt{n})^2=8(\ell-k)/(\varepsilon^2 n)\\
&\leq 8\big((n(t+\delta)+1)-(n(t-\delta)-1)\big)/(\varepsilon^2 n)\leq\varepsilon
\end{aligned}$$

for $n\geq n_0$, $\delta\leq\delta_0$, giving **(ii′)**. **(i′)** is trivial $(t=0)$. So relatively compact. By example, convergent subsequences go to Gaussian with covariance $\min\{s,t\}$. (We found limit chf. for continuous $F\in\mathrm{BV}[0,1]$. Such F are enough by **MT 26**.) $\Box$

12. Geometry of D.

Definition. • $\underline{\Lambda}\equiv\{\text{increasing bijective }\lambda:[0,1]\to[0,1]\}$;

• $\Lambda\ni\lambda\to\underline{\|\lambda\|_\Lambda}\equiv\sup_{s,t\in[0,1],\,s\neq t}|\ln\frac{\lambda(t)-\lambda(s)}{t-s}|\in[0,\infty)$;

• $D\times D\ni(x,y)\to\underline{d_1(x,y)}\equiv\inf\{\|\lambda\|_\Lambda\vee\|x-y\circ\lambda\|_{\mathcal{C}}:\lambda\in\Lambda\}\in[0,\infty)$.

Theorem 12.2. *Skorokhod J_1-space* (D,d_1) *complete separable metric space.*

Proof. $\lambda=I$ gives $d_1(x,y)<\infty$ and $d_1(x,x)=0$. $d_1(x,y)\neq0$ for $x(1)\neq y(1)$, since $\lambda(1)=1$. $|x(\hat{t})-y(\hat{t})|>0$, $\hat{t}<1$, gives $|x(t)-y(t)|\geq\varepsilon$, $t\in[\hat{t},\hat{t}+\delta]$, some $\varepsilon,\delta>0$, so $\|x-y\circ\lambda\|_{\mathcal{C}}\geq\varepsilon$, or $\|\lambda-I\|_{\mathcal{C}}\geq\delta$ so $\sup|\frac{\lambda(t)-\lambda(s)}{t-s}-1|\geq\sup|\frac{\lambda(t)-t}{t}|\geq\delta$, i.e., $\|\lambda\|_{\Lambda}\geq|\ln(1\pm\delta)|$, so $d_1(x,y)>0$. If $\|\hat{\lambda}\|_{\Lambda}\leq\hat{\varepsilon}$, $\|x-z\circ\hat{\lambda}\|_{\mathcal{C}}\leq\hat{\varepsilon}$ and $\|\tilde{\lambda}\|_{\Lambda}\leq\tilde{\varepsilon}$, $\|y-z\circ\tilde{\lambda}\|_{\mathcal{C}}\leq\tilde{\varepsilon}$, then $\|x-y\circ\hat{\lambda}\circ\tilde{\lambda}^{-1}\|_{\mathcal{C}}\leq\|x-z\circ\hat{\lambda}\|_{\mathcal{C}}+\|z\circ\hat{\lambda}-y\circ\hat{\lambda}\circ\tilde{\lambda}^{-1}\|_{\mathcal{C}}\leq\hat{\varepsilon}+\tilde{\varepsilon}$, $\|\hat{\lambda}\circ\tilde{\lambda}^{-1}\|_{\Lambda}\leq\sup|\ln\frac{\hat{\lambda}\circ\tilde{\lambda}^{-1}(t)-\hat{\lambda}\circ\tilde{\lambda}^{-1}(s)}{\tilde{\lambda}^{-1}(t)-\tilde{\lambda}^{-1}(s)}|+\sup|\ln\frac{\tilde{\lambda}^{-1}(t)-\tilde{\lambda}^{-1}(s)}{t-s}|\leq\hat{\varepsilon}+\tilde{\varepsilon}$. So d_1 metric.

$\{\sum_{i=1}^{n-1}e_i\mathbf{1}_{[\frac{i-1}{n-1},\frac{i}{n-1})}+e_n\mathbf{1}_{\{1\}}:e_1,\ldots,e_n\in\mathbb{Q},\ n\in\mathbb{N}\}$ dense in D, so separable.

Enough show Cauchy $\{y_n\}_{n=1}^{\infty}\subseteq D$ with $d_1(y_n,y_m)<2^{-n}$, $m\geq n$, converges (**MT 42**). Taking $\lambda_n\in\Lambda$ with $\|\lambda_n\|_{\Lambda}<2^{-n}$ and $\|y_n-y_{n+1}\circ\lambda_n\|_{\mathcal{C}}<2^{-n}$, we have $\|\lambda_n-I\|_{\mathcal{C}}\leq C2^{-n}$, somce $C>0$. Putting $\mu_k^n\equiv\lambda_{k+n}\circ\ldots\circ\lambda_n$, this give, for $k\leq\ell$,

$$\|\mu_\ell^n-\mu_k^n\|\leq\sum_{i=k}^{\ell-1}\|\lambda_{i+1+n}\circ\ldots\circ\lambda_n-\lambda_{i+n}\circ\ldots\circ\lambda_n\|=\sum_{i=k}^{\ell-1}\|\lambda_{i+1+n}-I\|\leq C2^{-(k+n)},$$

so $\mu_k^{(n)}\to_{\mathcal{C}}\hat{\lambda}_n$. Here $\hat{\lambda}_n\in\Lambda$, since $\hat{\lambda}_n(0)=0$, $\hat{\lambda}_n(1)=1$, and strictly increasing

$$|\ln\frac{\hat{\lambda}_n(t)-\hat{\lambda}_n(s)}{t-s}|=\lim_k|\ln\frac{\mu_k^n(t)-\mu_k^n(s)}{t-s}|=\lim_k\Big|\ln\frac{\lambda_n(t)-\lambda_n(s)}{t-s}+\sum_{j=0}^{k-1}\ln\frac{\mu_{j+1}^n(t)-\mu_{j+1}^n(s)}{\mu_j^n(t)-\mu_j^n(s)}\Big|$$
$$\leq\|\lambda_n\|_{\Lambda}+\sum_{j=0}^{k-1}\|\lambda_{n+j+1}\|_{\Lambda}\leq 2^{-n}.$$

Since $\hat{\lambda}_n=\hat{\lambda}_{n+1}\lambda_n$, so $\hat{\lambda}_{n+1}^{-1}\hat{\lambda}_n=\lambda_n$, we get $\|y_n\circ\hat{\lambda}_n^{-1}-y\|_{\mathcal{C}}\to 0$, some y, since

$$\|y_k\circ\hat{\lambda}_k^{-1}-y_n\circ\hat{\lambda}_n^{-1}\|\leq\sum_{j=n}^{k-1}\|y_{j+1}\circ\hat{\lambda}_{j+1}^{-1}-y_j\circ\hat{\lambda}_j^{-1}\|=\sum_{j=n}^{k-1}\|y_{j+1}\circ\lambda_j-y_j\|\leq 2^{1-n},$$

$k>n$. Here $y\in D$, so $y_n\to_D y$, since $\|\hat{\lambda}_n^{-1}\|_{\Lambda}=\|\hat{\lambda}_n\|_{\Lambda}\leq 2^{-n}\to 0$. □

Definition. For $x:[0,1]\to\mathbb{R}$,

- $\underline{w_x(\delta)}\equiv\sup_{s,t\in[0,1],\,|s-t|\leq\delta}|x(s)-x(t)|$;
- $\underline{w'_x(\delta)}\equiv\inf_{0=t_0<\ldots<t_k=1,\,t_i-t_{i-1}>\delta}\sup_{s,t\in[t_{i-1},t_i)}|x(s)-x(t)|$;
- $\underline{w''_x(\delta)}\equiv\sup_{0\leq r\leq s\leq t\leq 1,\,t-r\leq\delta}|x(s)-x(r)|\wedge|x(t)-x(s)|$;
- $\underline{j_x}\equiv\sup_{t\in(0,1]}|x(t^-)-x(t)|$.

Lemma 12.1 (MT 40). *For* $x:[0,1]\to\mathbb{R}$,

- $w'_x(\delta/2)\leq w_x(\delta)\leq 2w'_x(\delta)+j_x$;
- $w''_x(\delta)\vee|x(\delta)-x(0)|\vee|x(1-0)-x(1-\delta)|\leq w'_x(2\delta)$;
- $w'_x(\frac{\delta}{2})\leq 24\big(w''_x(\delta)\vee|x(\delta)-x(0)|\vee|x(1-0)-x(1-\delta)|\big)$;
- $x\in D$ *iff.* $\lim_{\delta\downarrow 0}w'_x(\delta)=0$;
- $x\in D$ *is measurable;*
- $\sup_{t\in[0,1]}|x(t)|<\infty$ *for* $x\in D$;
- $\#\{t\in(0,1]:|x(t^-)-x(t)|>\delta\}<\infty$ *for* $x\in D$.

Proof. Everything easy, except third inequality, which has long boring proof. □

Theorem 12.3. $A\subseteq D$ *is relatively compact iff. following two hold*

(i) $\sup_{x\in A}\|x\|_{\mathcal{C}}<\infty$;

(ii) $\lim_{\delta\downarrow 0}\sup_{x\in A}w'_x(\delta)=0$.

Proof. $\boxed{\Leftarrow}$ Since D separable complete, enough show A totally bounded. Take ε-net $e_1,\dots,e_n$ for $\{x(t):x\in A,\,t\in[0,1]\}$. Let $\sup_{x\in A} w'_x(3\delta)\le\varepsilon$, some $\delta>0$, and $|\ln\frac{\delta\pm m^{-1}}{\delta}|\le\varepsilon$, some $\mathbb{N}\in m\ge\delta^{-1}$. For $x\in A$, $\|x-y\circ\lambda\|_{\mathcal{C}}\le\varepsilon$, some $y\in\{\sum_{i=1}^m e_{j_i}\mathbf{1}_{[\frac{i-1}{m},\frac{i}{m})}+e_{j_{m+1}}\mathbf{1}_{\{1\}}:1\le j_i\le n\}$, some pice-wise linear $\lambda\in(0)^{\Lambda}_{\varepsilon}$. $\square$

Theorem 12.4. $A\subseteq \underline{D}\equiv D[0,1]$ *is relatively compact iff. following four hold*

(i) $\sup_{x\in A}\|x\|_{\mathcal{C}}<\infty$*;*

(ii) $\lim_{\delta\downarrow 0}\sup_{x\in A} w''_x(\delta)=0$*;*

(iii) $\liminf_{\delta\downarrow 0}\sup_{x\in A}|x(\delta)-x(0)|=0 \quad \big(\lim_{\delta\downarrow 0}\sup_{x\in A}|x(\delta)-x(0)|=0\big)$*;*

(iv) $\liminf_{\delta\downarrow 0}\sup_{x\in A}|x(1-0)-x(1-\delta)|=0 \quad \big(\lim_{\delta\downarrow 0}\sup_{x\in A}|x(1-0)-x(1-\delta)|=0\big)$.

Proof. Theorem 12.3, previous lemma, and monotonicity of $w'_x(\cdot)$ and $w''_x(\cdot)$. $\square$

Theorem 12.5. $\pi_{t_1,\dots,t_k}$ *measurable.* π_0,π_1 *continuous. For* $t\in(0,1)$, π_t *continuous at* x *iff.* x *continuous at* t. *For* $T\ni 1$ *dense in* $[0,1]$, $\sigma(\{\pi_t^{-1}H:t\in T,\,H\subseteq\mathbb{R}$ Borel$\})=\sigma(J_1)$ *and* $\{\pi^{-1}_{t_1,\dots,t_k}H_1\times\ldots\times H_k:t_i\in T,\,H_i\subseteq\mathbb{R}$ Borel, $k\in\mathbb{N}\}$ *separating.*

Proof. Continuity easy. $D\ni x\to h_\varepsilon(x)\equiv\frac{1}{\varepsilon}\int_t^{t+\varepsilon}x\in\mathbb{R}$ continuous, since $d_1(x_n,x)\to 0\Rightarrow x_n\to x$ a.e., so $\pi_t=\lim_\varepsilon h_\varepsilon$ and $\pi_{t_1,\dots,t_k}$ measurable. $V_n\circ\pi_{t_0,\dots,t_n}x\equiv\sum_{i=1}^n x(t_i)\mathbf{1}_{[t_{i-1},t_i)}+x(t_n)\mathbf{1}_{\{t_n\}}\to_{d_1}x$ as $t_i-t_{i-1}\to 0$ (as for separability), so I $\sigma(\{\pi_t^{-1}H:t\in T,\,H\subseteq\mathbb{R}$ Borel$\})$-measurable, since $\pi_{t_0,\dots,t_n}$ is, and V_n continuous. $\square$

13. Weak Convergence in $\mathcal{P}(D)$.

Proposition. $\underline{T_P}\equiv\{t\in[0,1]:P(\pi_t$ continuous at $t)=1\}\ni 0,1$ *and* $\#T_P^c\le\aleph_0$ *for* $P\in\mathcal{P}(D)$. $t\in T_P\cap(0,1)$ *iff.* $\{x\in D:x(t-0)\ne x(t)\}$ *P-null.*

Proof. Last statement by π_t continuous at x iff. x is at t. $0,1\in T$ by π_0,π_1 continuous. $\#\{t:P(x\in D:|x(t^-)-x(t)|>\frac{1}{n})>\frac{1}{m}\}<\infty$, so $\#\{t:P(x\in D:|x(t^-)-x(t)|>\frac{1}{n})>0\}\le\aleph_0$, and same for $\#\{t:P(x\in D:|x(t^-)-x(t)|>0)>0\}$, by $P(\bigcap_{k=1}^\infty\bigcup_{\ell=k}^\infty\{x\in D:|x(t_\ell^-)-x(t_\ell)|>\frac{1}{n}\})=0$ for infinite $\{t_k\}_{k=1}^\infty\subseteq[0,1]$. $\square$

Theorem 13.1. $P_n\Rightarrow P$ *in* $\mathcal{P}(D)$ *some* P, *iff.* $\{P_n\}_{n=1}^\infty$ *tight and* $P_n\pi^{-1}_{t_1,\dots,t_k}\Rightarrow$ *in* $\mathcal{P}(\mathbb{R}^k)$, t_i *in some dense* $T\ni 1$ *in* $[0,1]$, $k\in\mathbb{N}$. *Then* $P\pi^{-1}_{t_1,\dots,t_k}\Leftarrow P_n\pi^{-1}_{t_1,\dots,t_k}$, $t_i\in T$.

Proof. Tightness necessary, by D separable complete. Given tightness, sufficient show convergence on separating class, which $\{\pi^{-1}_{t_1,\dots,t_k}H_1\times\ldots\times H_k:t_i\in T,\,H_i\subseteq\mathbb{R}$ Borel, $k\in\mathbb{N}\}$ is. Convergence on $T=T_P$ necessary, by Mapping Theorem. $\square$

Theorem 13.2. $\{P_n\}_{n=1}^\infty\subseteq\mathcal{P}(D)$ *tight iff. following two hold*

(i) $\lim_{M\to\infty}\limsup_n P_n\big(x\in D:\|x\|_{\mathcal{C}}>M\big)=0$*;*

(ii) $\lim_{\delta\downarrow 0}\limsup_n P_n\big(x\in D:w'_x(\delta)>\varepsilon\big)=0$ *for* $\varepsilon>0$.

Proof. $\boxed{\Leftarrow}$ Let $P_n(x\in D: w'_x(\delta_k)>\frac{1}{k})\leq 2^{-k}\frac{\varepsilon}{2}$ and $P_n(x\in D: \|x\|_{\mathcal{C}}>M)\leq\frac{\varepsilon}{2}$, $n\in\mathbb{N}$, so $K\equiv\bigcap_{k=1}^{\infty}\{x\in D: w'_x(\delta_k)\leq\frac{1}{k}\}\cap\{x\in D: \|x\|_{\mathcal{C}}\leq M\}$ compact, by Theorem 12.3, with $P_n(K)\geq 1-\varepsilon$. $\boxed{\Rightarrow}$ Immediate from Theorem 12.3. $\square$

Corollary. **(i)** *in* Theorem 13.2 *may be replaced with either of*
(i′) $\lim_{M\to\infty}\limsup_n P_n\big(x\in D: |x(t)|>M\big)=0$ *for dense* $T\ni 1$ *in* $[0,1]$*;*
(i″) $\lim_{M\to\infty}\limsup_n P_n\big(x\in D: j_x>M\big)=0$ *and* **(i′)** *holds for* $t=0$.

Proof. By **(ii)**, pick $\delta>0$ with $P_n(x\in D: w'_x(\delta)\leq\frac{1}{2})\geq 1-\frac{\varepsilon}{2}$, $n\in\mathbb{N}$. Let $0=s_0<\ldots<s_m=1$, $s_i\in T$, $s_i-s_{i-1}<\delta$, so for x with $w'_x(\delta)\leq\frac{1}{2}$, and $0=t_0<\ldots<t_k=1$, $t_j-t_{j-1}>\delta$, with $\sup_{s,t\in[t_{j-1},t_j)}|x(t)-x(s)|\leq 1$, since $s_i\in[t_{j-1},t_j)$ some i, $\|x\|_{\mathcal{C}}\leq\sup_i|x(s_i)|+1$, so by **(i′)**, for $M>0$ with $P_n(x\in D: |x(s_i)|>M-1)\leq\frac{\varepsilon}{2m}$, $n\in\mathbb{N}$, we get **(i)** of Theorem 13.2. Sufficiency of **(i″)** from $w_x(\delta)\leq 2w'_x(\delta)+j_x$. $\square$

Corollary (MT 45). $P_n\Rightarrow P$ *in* $\mathcal{P}(D)$*, some* P*, iff.* **(ii)** *of* Theorem 13.2 *holds and* $P_n\pi^{-1}_{t_1,\ldots,t_k}\Rightarrow$ *in* $\mathcal{P}(\mathbb{R}^k)$ *for* $t_i\in T_P$, $k\in\mathbb{N}$ (where limit must be $P\pi^{-1}_{t_1,\ldots,t_k}$).

Theorem 13.3. $P_n\Rightarrow P$ *in* $\mathcal{P}(D)$*, some* P*, iff.* $P_n\pi^{-1}_{t_1,\ldots,t_k}\Rightarrow$ *in* $\mathcal{P}(\mathbb{R}^k)$, t_i *in some dense* $T\ni 1$ *in* $[0,1]$, $k\in\mathbb{N}$, *and following two hold*
(i) $\liminf_{\delta\downarrow 0}\limsup_n P_n\big(x\in D: |x(1-0)-x(1-\delta)|>\varepsilon\big)=0$ *for* $\varepsilon>0$*;*
(ii) $\lim_{\delta\downarrow 0}\limsup_n P_n\big(x\in D: w''_x(\delta)>\varepsilon\big)=0$ *for* $\varepsilon>0$.

Proof. **(ii)** of Theorem 13.2 can be replaced with **(ii)**- **(iv)** of Theorem 12.4. So **(i)**-**(ii)** necessary. Sufficient, since $\limsup_n P_n(x\in D: |x(\delta_k)-x(0)|>\varepsilon)\geq\eta$, some $\eta>0$ and $T_P\ni\delta_k\downarrow 0$, gives $P(x\in D: |x(0+0)-x(0)|\geq\varepsilon)\geq\eta$, contradicting cádlág. $\square$

Corollary (MT 45). *If content with if,* **(i)** *in* Theorem 13.3 *may be replaced with*
(i′) $\lim_{\delta\downarrow 0}P\big(x\in D: |x(1)-x(1-\delta)|>\varepsilon\big)=0$ *for* $\varepsilon>0$.

Proof. **(ii)** gives $\limsup_n\mathbf{P}\{|X_n(1)-X_n(1-0)|\wedge|X_n(1-0)-X_n(1-\delta_k)|>\varepsilon\}\to 0$ and **(i′)** $\limsup_n\mathbf{P}\{|X_n(1)-X_n(1-\delta_k)|>\varepsilon\}\to 0$, some $\delta_k\downarrow 0$, so that $\limsup_n\mathbf{P}\{|X_n(1-0)-X_n(1-\delta_k)|>\varepsilon\}\to 0$, i.e., **(i)**. (`Billingsley's` proof seems incomplete!) $\square$

Theorem 13.4. *For* $X_n\Rightarrow X$ *in* $\mathcal{P}(D)$, $\mathbf{P}\{X\in\mathcal{C}\}=1$ *iff.* $j_{X_n}\Rightarrow 0$ *in* $\mathcal{P}(\mathbb{R})$.

Proof. By Mapping Theorem, $j_{X_n}\Rightarrow j_X$, since $x_n\to_D x$ easily gives $j_{x_n}\to j_x$. For D-valued X, by Lemma 12.1, $j_X=0$ equivalent with X $\mathcal{C}$-valued. $\square$

Corollary. $P_n\Rightarrow P$ *in* $\mathcal{P}(D)$ *with* $P(\mathcal{C})=1$ *if* $P_n\pi^{-1}_{t_1,\ldots,t_k}\Rightarrow P\pi^{-1}_{t_1,\ldots,t_k}$ *in* $\mathcal{P}(\mathbb{R}^k)$ *for* $t_i\in[0,1]$, $k\in\mathbb{N}$, *and* **(ii)** [or **(ii′)**] *of* Theorem 7.3 *hold.*

Proof. Hypothesis of Theorems 13.1-2 hold, since $\|x\|_{\mathcal{C}}\leq|x(0)|+\frac{2}{\delta}w_x(\delta)$ and $w'_x(\frac{\delta}{2})\leq w_x(\delta)$, so $\Rightarrow$ in $\mathcal{P}(D)$. And Theorem 13.4 applies, since $j_x\leq w_x(\delta)$, $\delta>0$. $\square$

Theorem 13.5. *$X_n \Rightarrow X$ in $\mathcal{P}(D)$, some X, if $(X_n(t_1),\dots,X_n(t_k)) \Rightarrow$ in $\mathcal{P}(\mathbb{R}^k)$, t_i in some dense $T \ni 1$ in $[0,1]$, $k \in \mathbb{N}$, and following two hold*

(i) $\liminf_{\delta\downarrow 0} \limsup_n \mathbf{P}\{|X_n(1-0)-X_n(1-\delta)|>\varepsilon\} = 0$ *for* $\varepsilon>0$;

(ii) *there are constants* $K,\beta,\gamma>0$ *such that*

$$\mathbf{P}\big\{|X_n(s)-X_n(r)|\wedge|X_n(t)-X_n(s)|>\lambda\big\} \le K|t-r|^{1+\gamma}/\lambda^\beta \quad \textit{for } 0\le r<s<t\le 1,\ \lambda>0.$$

Proof. Enough show **(ii)** of Theorem 13.3. Let $D_k \equiv \{2^{-k}i : i=0,\dots,2^k\}$,

$$A_k \equiv \max\{|X_n(s)-X_n(r)|\wedge|X_n(t)-X_n(s)| : r,s,t\in D_k,\ s-r=t-s=2^{-k},\ t-s\le\delta\}$$
$$B_k \equiv \max\{|X_n(s)-X_n(r)|\wedge|X_n(t)-X_n(s)| : r,s,t\in D_k,\ r\le s\le t,\ t-s\le\delta\}$$
$$C \equiv \sup\{|X_n(s)-X_n(r)|\wedge|X_n(t)-X_n(s)| : r,s,t\in[0,1],\ r\le s\le t,\ t-s\le\delta/2\}$$

and, for $t\in D_k$, define $t'_k(t)\in D_{k-1}$ by $t'_k(t)\equiv t$ if $t\in D_{k-1}$ and

$$t'_k(t) \equiv \begin{cases} t-2^{-k} & \text{if } t\notin D_{k-1} \text{ and } |X_n(t)-X_n(t-2^{-k})|\le|X_n(t)-X_n(t+2^{-k})| \\ t+2^{-k} & \text{if } t\notin D_{k-1} \text{ and } |X_n(t)-X_n(t-2^{-k})|>|X_n(t)-X_n(t+2^{-k})| \end{cases}.$$

Notice $|X_n(t)-X_n(t'_k(t))|\le A_k$, $t\in D_k$, so for $\hat{t},\tilde{t}\in D_k$,

$$\begin{aligned} |X_n(\hat t)-X_n(\tilde t)| &\le |X_n(\hat t)-X_n(t'_k(\hat t))|+|X_n(t'_k(\hat t))-X_n(t'_k(\tilde t))|+|X_n(t'_k(\tilde t))-X_n(\tilde t)| \\ &\le |X_n(t'_k(\hat t))-X_n(t'_k(\tilde t))|+2A_k. \end{aligned}$$

For $r,s,t\in D_k$ with $r<s<t$, $t'_k(r)\le t'_k(s)<t'_k(t)$, so $B_k\le B_{k-1}+2A_k$, giving $B_k \le 2\sum_{j=1}^k A_j$, since $A_0=B_0=0$, and $C\le 2\sum_{j=1}^\infty A_j$, by right-continuity. Thus

$$\begin{aligned} \mathbf{P}\{C\ge\varepsilon\} &\le \textstyle\sum_{k=1}^\infty \mathbf{P}\{2A_k\ge(1-\frac{1}{\theta})\theta^k\varepsilon\} \\ &\le \textstyle\sum_{k=1}^\infty\sum_{i=1}^{2^k-1}\mathbf{P}\big\{|X_n(\frac{i}{2^k})-X_n(\frac{i-1}{2^k})|\wedge|X_n(2^{-k}(\frac{i+1}{2^k})-X_n(\frac{i}{2^k})|\ge\frac{1-\frac{1}{\theta}}{2}\theta^k\varepsilon\big\} \\ &\le \textstyle\sum_{k=1}^\infty 2^k K(2^{-k})^{1+\gamma/2}\delta^{\gamma/2}/(\frac{1-\frac{1}{\theta}}{2}\theta^k\varepsilon)^\beta \to 0 \quad \text{as } \delta\downarrow 0, \quad \text{some } \theta>0. \quad \square \end{aligned}$$

Corollary. **(i)** *in* Theorem 13.5 *may be replaced with*

(i′) $X(1)-X(1-\delta)\Rightarrow 0$ *in* $\mathcal{P}(\mathbb{R})$ *as* $\delta\downarrow 0$.

Proof. This is same exchange of **(i′)** to **(i)** we have seen before. □

Corollary (MT 45). **(ii)** *in* Theorem 13.5 *may be replaced with*

(ii′) $\mathbf{E}\big\{|X_n(s)-X_n(r)|^\beta\wedge|X_n(t)-X_n(s)|^\beta\big\}\le K|t-r|^{1+\gamma}$.

14. Applications [Functional Central Limit Theorems].

Theorem 14.1. (Donsker) *For iid. standardized r.v.'s* $\{\xi_j\}_{j=1}^\infty$, $\{X_n(t)\}_{t\in[0,1]} \equiv \{n^{-1/2}\sum_{j=1}^{\lfloor nt\rfloor}\xi_j\}_{t\in[0,1]} \Rightarrow W$ *in* $\mathcal{P}(D)$.

Theorem 14.3. *For uniform iid. $[0,1]$-valued r.v.'s* $\{\eta_j\}_{j=1}^\infty$, $\{Y_n(t)\}_{t\in[0,1]}\equiv\{\sqrt{n}\,(\frac{1}{n}\sum_{j=1}^n \mathbf{1}_{[0,t]}(\eta_j)-t)\}_{t\in[0,1]}\Rightarrow W^\circ(s)$ *in* $\mathcal{P}(D)$, *where* $\{W^\circ(s)(t)\}_{t\in[0,1]}$ *is Brownian bridge, i.e., zero-mean Gaussian with* $\mathbf{E}\{W^\circ(s)W^\circ(t)\}=s\wedge t-st$.

Proof. $\sum_{i=1}^k a_i(\mathbf{1}_{[0,t_i]}(\eta)-t_i)$ is zero-mean with variance $\sum_{i,j=1}^k a_ia_j(t_i\wedge t_j-t_it_j)$,

$$\textstyle\sum_{i=1}^k a_iY_n(t_i)=\frac{1}{\sqrt{n}}\sum_{j=1}^n\sum_{i=1}^k a_i(\mathbf{1}_{[0,t_i]}(\eta_j)-t_i)\Rightarrow \mathrm{N}\big(0,\mathbf{Var}\{\sum_{i=1}^k a_iW^\circ(t_i)\}\big)$$

in $\mathcal{P}(\mathbb{R})$, by elementary CLT. So fidi's $\Rightarrow$ in $\mathcal{P}(\mathbb{R}^k)$, by Cramér-Wold Device. **(i′)** of Theorem 13.5 holds since $\mathbf{E}\{[W^\circ(1)-W^\circ(1-\delta)]^2\}=\ldots=\delta\to 0$ as $\delta\downarrow 0$. For **(ii′)** of Theorem 13.5, notice that for $0\le r<s<t\le 1$, some constant $C>0$,

$$\begin{aligned}
&\mathbf{E}\big\{(Y_n(s)-Y_n(r))^2(Y_n(t)-Y_n(s))^2\big\}\\
&=\mathbf{E}\Big\{\big(\tfrac{1}{\sqrt{n}}\textstyle\sum_{j=1}^n[\mathbf{1}_{(r,s]}(\eta_j)-(s-r)]\big)^2\big(\tfrac{1}{\sqrt{n}}\sum_{j=1}^n[\mathbf{1}_{(s,t]}(\eta_j)-(t-s)]\big)^2\Big\}\\
&=\tfrac{1}{n}\mathbf{E}\big\{[\mathbf{1}_{(r,s]}(\eta)-(s-r)]^2[\mathbf{1}_{(s,t]}(\eta)-(t-s)]^2\big\}\\
&\quad+\tfrac{n-1}{n}\mathbf{E}\big\{[\mathbf{1}_{(r,s]}(\eta)-(s-r)]^2\big\}\,\mathbf{E}\big\{[\mathbf{1}_{(s,t]}(\eta)-(t-s)]^2\big\}\\
&\quad+2\tfrac{n-1}{n}\big[\mathbf{E}\big\{[\mathbf{1}_{(r,s]}(\eta)-(s-r)][\mathbf{1}_{(s,t]}(\eta)-(t-s)]\big\}\big]^2\le C(s-r)(t-s)\le C(t-r)^2.
\end{aligned}$$

Theorem 14.3. *For iid.* $[0,1]$*-valued r.v.'s* $\{\xi_j\}_{j=1}^\infty$, $\{Y_n(t)\}_{t\in[0,1]}\equiv\{\sqrt{n}(\frac{1}{n}\sum_{j=1}^n \mathbf{1}_{[0,t]}(\xi_j)-F_\xi(t))\}_{t\in[0,1]}\Rightarrow Y$ *in* $\mathcal{P}(D)$, *where* $\{Y(t)\}_{t\in[0,1]}$ *is zero-mean Gaussian with* $\mathbf{E}\{Y(s)Y(t)\}=F_\xi(s\wedge t)-F_\xi(s)F_\xi(t)$ *for* $s,t\in[0,1]$.

Proof. $F^{\leftarrow}(s)\equiv\inf\{t\in[0,1]:s\le F_\xi(t)\}$ has $F^{\leftarrow}(s)\le t$ iff. $s\le F_\xi(t)$, so $F^{\leftarrow}(\eta)=_d\xi$, and $\{Y_n^\eta(F_\xi(t))\}_{t\in[0,1]}=_d\{Y_n(t)\}_{t\in[0,1]}$. Since $D\ni x\to h(x)\equiv x\circ F_\xi\in D$ continuous on $\mathcal{C}$, general result by Mapping Theorem ($\mathbf{P}\{Y^\eta\in\mathcal{C}\}=1$). □

23. Euclidian Chf. [Mainly after `Feller Vol. II`].

$\mathbb{R}^k$-valued r.v. X has chf. $\phi_X(t)=\mathbf{E}\{e^{i\langle t,X\rangle}\}=\mathbf{E}\{e^{i\sum_{j=1}^k t_jX_j}\}$, $t\in\mathbb{R}^k$.

Theorem. (Unicity) *For* F_X*-continuity* $(-\infty,x]\subseteq\mathbb{R}^k$,

$$F_X(x)=\lim_{\varepsilon\downarrow 0}\int_{z\le x}\int_{y\in\mathbb{R}^k}\prod_{j=1}^k e^{-iy_jz_j}\,\frac{1}{\sqrt{2\pi}\,\varepsilon}f_{\mathrm{N}(0,\varepsilon^{-2})}(y_j)\phi_X(y)\,dy\,dz.$$

Proof. $\boxed{\Leftarrow}$ By weak convergence in $\mathcal{P}(\mathbb{R}^k)$, as $\varepsilon\downarrow 0$,

$$\begin{aligned}
\int_{z\le x}dF_X(z)&\leftarrow\int_{z\le x}\int_{t\in\mathbb{R}^k}\prod_{j=1}^k f_{\mathrm{N}(0,\varepsilon^2)}(z_j-t_j)\,dF_X(t)\,dz\\
&=\int_{z\le x}\int_{t\in\mathbb{R}^k}\prod_{j=1}^k\frac{1}{\sqrt{2\pi}\,\varepsilon}\phi_{\mathrm{N}(0,\varepsilon^{-2})}(t_j-z_j)\,dF_X(t)\,dz\\
&=\int_{z\le x}\int_{y\in\mathbb{R}^k}\prod_{j=1}^k\frac{1}{\sqrt{2\pi}\,\varepsilon}f_{\mathrm{N}(0,\varepsilon^{-2})}(y_j)\Big(\int_{t\in\mathbb{R}^k}e^{iy_j(t_j-z_j)}\,dF_X(t)\Big)\,dydz.\quad\square
\end{aligned}$$

Corollary. (Independence) $\mathbb{R}^k$*-valued r.v.'s* X,Y *independent iff.* $\phi_{X,Y}=\phi_X\phi_Y$.

Theorem. (Inversion) $\mathbb{R}^k$*-valued r.v.* X *with integrable chf.* ϕ_X *is absolutely continuous with bounded continuous density* $f_X(x)=\frac{1}{(2\pi)^k}\int_{t\in\mathbb{R}^k}e^{-i\langle t,x\rangle}\phi_X(t)\,dt$.

Proof. By integrability and inspection of previous proof, as $\varepsilon\downarrow 0$, the density

$$\int_{t\in\mathbb{R}^k}\prod_{j=1}^k f_{\mathrm{N}(0,\varepsilon^2)}(z_j-t_j)\,dF_X(t)=\int_{y\in\mathbb{R}^k}\prod_{j=1}^k e^{-iy_jz_j}\,\frac{1}{2\pi}\phi_{\mathrm{N}(0,\varepsilon^2)}(y_j)\phi_X(y)\,dy$$

$$\to \int_{y\in\mathbb{R}^k} e^{-i\langle y,z\rangle} \tfrac{1}{(2\pi)^k} \phi_X(y)\, dy \equiv f(z),$$

with f bounded continuous. By bounded convergence, for bounded Borel $I \subseteq \mathbb{R}^k$,

$$\int_I dF_X \leftarrow \int_{z\in I}\int_{t\in\mathbb{R}^k} \prod_{j=1}^k f_{\mathrm{N}(0,\varepsilon^2)}(z_j - t_j)\, dF_X(t)\, dz \to \int_{z\in I} f(z)\, dz \quad \text{as} \quad \varepsilon\downarrow 0. \ \square$$

Theorem. (Lévy-Cramér, Convergence) *$X_n \Rightarrow X$ in $\mathcal{P}(\mathbb{R}^k)$, some X, iff. $\phi_{X_n} \to \phi$, some ϕ that is continuous at zero. If so $\phi_X = \phi$.*

Proof. $\boxed{\Leftarrow}$ By Unicity Theorem, enough show tight. By continuity, as $K \to \infty$,

$$\begin{aligned}\limsup_n \mathbf{P}\{|(X_n)_j| \geq 2K\} &\leq \limsup_n 2\int_{|x|\geq 2K} \left(1 - \tfrac{\sin(x/K)}{x/K}\right) dF_{(X_n)_j}(x) \\ &\leq \limsup_n 2\int_{\mathbb{R}} \left(1 - \tfrac{\sin(x/K)}{x/K}\right) dF_{(X_n)_j}(x) \\ &= \limsup_n K\int_{x\in\mathbb{R}}\int_{|t_j|\leq 1/K} (1 - e^{it_j x_j})\, dt_j\, dF_{(X_n)_j}(x) \\ &= K\int_{|t_j|\leq 1/K} (1 - \phi(\ldots, 0, t_j, 0, \ldots))\, dt_j \to 0. \ \square\end{aligned}$$

Corollary. (Cramér-Wold Device) *$X_n \Rightarrow X$ in $\mathcal{P}(\mathbb{R}^k)$, some X, iff. $\langle t, X_n\rangle \Rightarrow$ in $\mathcal{P}(\mathbb{R})$ for $t \in \mathbb{R}^n$. If so $\phi_X(t) = \lim_n \phi_{\langle t, X_n\rangle}(1)$.*

Theorem. (Bochner) *$\phi : \mathbb{R} \to \mathbb{C}$ is chf. for some real r.v. iff. continuous, non-negative definite* i.e., $\sum_{k,\ell=1}^n a_k \overline{a_\ell} \phi(t_k - t_\ell) \geq 0$, $t_k \in \mathbb{R}$, $a_k \in \mathbb{C}$, $n \in \mathbb{N}$], *with* $\phi(0) = 1$.

Proof. Continuity and $\phi(0) = 1$ necessary, so enough show continuous ϕ with $\phi(0) = 1$ chf. iff. bounded and $\int_{\mathbb{R}^2} e^{ix(s-t)} \phi(s-t)\, dF(s) dF(t) \geq 0$, $x \in \mathbb{R}$, all F, since $\hat{a}_k = a_k / (\sum_{k=1}^n |a_k|) = p_k e^{ixt_k}$, $p_k \geq 0$, $dF = \sum_{k=1}^n p_k \delta_{t_k}$ in this gives non-negative definite, and non-negative definite gives $a\phi(t) + \overline{a}\phi(-t) + 1 + |a|^2 \geq 0$, so $\overline{\phi(t)} = \phi(-t)$ ($a = 1$, i) and $|\phi(t)| \leq 1$ [$a = -\overline{\phi(t)}$], so integral exists, non-negative for $dF = \sum_{k=1}^n p_k \delta_{t_k}$ as above, and in general by approximate with such (cf. proof of Theorem 6.8).

For r.v. $\xi \sim \phi$ and iid. r.v.'s η, $\hat{\eta} \sim F$, $|\phi_\eta(y)|^2 = \phi_{\eta - \hat{\eta}}(y) = \int_{\mathbb{R}^2} e^{iy(s-t)} dF(s) dF(t) \geq 0$, so $0 \leq \int_{\mathbb{R}^3} e^{iy(s-t)}\, dF(s) dF(t) dF_{\xi + x}(y) = \int_{\mathbb{R}^2} e^{ix(s-t)} \phi(s-t)\, dF(s) dF(t)$.

By Continuity Theorem, bounded continuous ϕ with $\phi(0) = 1$ chf. if $\phi\, \phi_{\mathrm{N}(0,\varepsilon^{-2})}$ chf., $\varepsilon > 0$. Notice $f(x) \equiv \frac{1}{2\pi} \int_{\mathbb{R}} e^{-ixy} \phi(y) \phi_{\mathrm{N}(0,\varepsilon^{-2})}(y)\, dy \geq 0$, $x \in \mathbb{R}$, since equal

$$\tfrac{\varepsilon}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ixy} \phi(y)\, dF_{\mathrm{N}(0,\varepsilon^2)}(y) = \tfrac{\varepsilon}{\sqrt{2\pi}} \int_{\mathbb{R}^2} e^{-ix(s-t)} \phi(s-t)\, dF_{\mathrm{N}(0,\varepsilon^2/2)}(s) dF_{\mathrm{N}(0,\varepsilon^2/2)}(t),$$

and bounded continuous. f is integrable, since by Inversion Theorem,

$$\int_{\mathbb{R}} \phi_{\mathrm{N}(t,\delta^2)}(x) f(x)\, dx = \int_{\mathbb{R}} f_{\mathrm{N}(t,\delta^2)}(y) \phi(y) \phi_{\mathrm{N}(0,\varepsilon^{-2})}(y)\, dy \leq \|\phi\| < \infty,$$

so $\int_{\mathbb{R}} f \leq \liminf_{\delta\downarrow 0} \int_{\mathbb{R}} \phi_{\mathrm{N}(0,\delta^2)} f < \|\phi\|$ by Fatou's Lemma. Now Dominated Convergence and $\mathrm{N}(t, \delta^2) \Rightarrow t$ in $\mathcal{P}(\mathbb{R})$ as $\delta \downarrow 0$ show $\phi\, \phi_{\mathrm{N}(0,\varepsilon^{-2})}$ chf., since

$$\int_{\mathbb{R}} e^{ixt} f(x)\, dx = \lim_{\delta\downarrow 0} \mathbf{E}\{(\phi\phi_{\mathrm{N}(0,\varepsilon^{-2})})(\mathrm{N}(t, \delta^2))\} = (\phi\phi_{\mathrm{N}(0,\varepsilon^{-2})})(t). \ \square$$

24. Basic Theory for Processes.

Definition. For set $T\neq\emptyset$, stochastic process $\{X(t)\}_{t\in T}$ with parameter in T is family of r.v.'s defined on common probability space, that we assume completed.

Definition. The distributions $\{\{F_{X(t_1),\ldots,X(t_k)}\}_{t\in T^k}\}_{k\in\mathbb{N}}$ fidi's of $\{X(t)\}_{t\in T}$.

Theorem. (KOLMOGOROV) *For family of distributions* $\{\{F_t:\mathbb{R}^k\to[0,1]\}_{t\in T^k}\}_{k\in\mathbb{N}}$ *there exists process* $\{X(t)\}_{t\in T}$ *with these fidi's iff. following two hold*

(i) $F_{\ldots,t_{i-1},t_j,t_{i+1},\ldots,t_{j-1},t_i,t_{j+1},\ldots}(\ldots,x_{i-1},x_j,x_{i+1},\ldots,x_{j-1},x_i,x_{j+1},\ldots)=F_t(x)$;

(ii) $\lim_{x_{k+1}\to\infty}F_{t,t_{k+1}}(x,x_{k+1})=F_t(x)$.

Proof. $\boxed{\Leftarrow}$ Enough find probability P on smallest (so called cylinder) σ-algebra of $\mathbb{R}^T$ that has $P\pi_t^{-1}$ well-defined with valued F_t, $t\in T^k$, since for $X(\omega;t)=\omega(t)$, $\omega\in\mathbb{R}^T$, $t\in T$, $P\big(\bigcap_{i=1}^k\{X(t_i)\in H_i\}\big)=P\big(\bigcap_{i=1}^k\{\omega(t_i)\in H_i\}\big)=P\pi_{t_1\ldots t_k}^{-1}(H_1\times\ldots\times H_k)$. Define finitely additive P on algebra $\mathfrak{A}$ of finite unions of $\pi_{t_1\ldots t_k}^{-1}(H_1\times\ldots\times H_k)$ by $P\pi_t^{-1}=F_t$. Enough show P σ-additive, since it then extends to $\sigma(\mathfrak{A})$. Enough show P tight (**MT 17**). Holds by tightness of fidi's. □

Definition. Process $\{X(t)\}_{t\in T}$ on topological space T separable if there is countable dense separant $D\subseteq T$ and **P**-null N, such that to $\omega\in\Omega\backslash N$ and $t\in T$, there is $\{t_n\}_{n=1}^{\infty}\subseteq D$ with $t_n\to_T t$ and $X(\omega;t_n)\to X(\omega;t)$.

Separable processes are of utmost importance since their important properties are determined by fidi's, and they possess "all nice properties" allowed by the fidi's.

Example. If separable $\{X(t)\}_{t\in[0,1]}$ has same fidi's as $\mathcal{C}$-valued r.v. Y, then

$$\begin{aligned}\mathbf{P}\{X\text{ not uniformly continuous}\}&=\mathbf{P}\big\{\textstyle\bigcup_{k=1}^{\infty}\bigcap_{\ell=1}^{\infty}\bigcup_{s,t\in[0,1]:|s-t|<\frac{1}{\ell}}\{|X(s)-X(t)|>\frac{1}{k}\}\big\}\\&=\mathbf{P}\big\{\textstyle\bigcup_{k=1}^{\infty}\bigcap_{\ell=1}^{\infty}\bigcup_{s,t\in D:|s-t|<\frac{1}{\ell}}\{|X(s)-X(t)|>\frac{1}{k}\}\big\}\\&=\mathbf{P}\{X\text{ not uniformly continuous on }D\}=0,\end{aligned}$$

since $\lim_{L\uparrow\infty}\lim_{N\uparrow\infty}\mathbf{P}\big\{\bigcap_{\ell=1}^{L}\bigcup_{i,j\in\{1,\ldots,N\}:|d_i-d_j|<\frac{1}{\ell}}\{|X(d_i)-X(d_j)|>\frac{1}{k}\}\big\}=0$, since determined by fidi's and Y uniformly continuous, so 1 for Y. (On non-continuity null-set we may change X to e.g., 0 without affecting separability or fidi's.)

Example. Process $\{X(t)\}_{t\in\mathbb{R}}$ with iid. N$(0,1)$-values is separable.

Theorem. *If separable process* $\{X(t)\}_{t\in T}$ *on second order countable space* T *is* **P**-*continuous* [i.e., $X(s)\to_{\mathbf{P}}X(t)$ if $s\to_T t$], *each dense in* T *is separant.*

Proof. Pick base $\{G_n\}_{n=1}^{\infty}$, separant $D\subseteq T$ and countable dense $S\subseteq T$. To $t\in D$, $\hat{s}_n(t)\in B_n\equiv\bigcap_{k\in\{1,\ldots,n\}:G_k\ni t}G_k$ some $\{\hat{s}_n(t)\}_{n=1}^{\infty}\subseteq S$, so $\hat{s}_n(t)\to t$ and $X(\hat{s}_n(t))\to_{\mathbf{P}}X(t)$. There is **P**-null Ω_t and $\{\hat{s}'_n(t)\}_{n=1}^{\infty}\subseteq\{\hat{s}_n(t)\}_{n=1}^{\infty}$ with $X(\hat{s}'_n(t))\to X(t)$,

$\omega \notin \Omega_t$. Take $|X(\hat{s}'_n(t)) - X(t)| \le \frac{1}{n}$. There is $\mathbf{P}$-null $\hat{\Omega}$ such that to $t \in T$ and $\omega \notin \hat{\Omega}$ there is $\{t_n^\omega(t)\}_{n=1}^\infty \subseteq D$ with $t_n^\omega(t) \to t$ and $|X(t_n^\omega(t)) - X(t)| \le \frac{1}{n}$. Put $N \equiv \hat{\Omega} \cup \bigcup_{t\in D} \Omega_t$. Let $s_n^\omega(t) \equiv \hat{s}'_n(t_n^\omega(t))$. Then $X(s_n^\omega(t)) \to X(t)$, $t \in T$, $\omega \notin N$, since

$$|X(s_n^\omega(t)) - X(t)| \le |X(\hat{s}'_n(t_n^\omega(t))) - X(t_n^\omega(t))| + |X(t_n^\omega(t)) - X(t)| \le \tfrac{2}{n}.$$

For open $T \supseteq G \ni t$, $t \in G_{n_0} \subseteq G$, some $n_0 \in \mathbb{N}$, so $t_n^\omega(t) \in G_{n_0}$, $n \ge n_1$, some $n_1 \in \mathbb{N}$, giving $s_n^\omega(t) = \hat{s}'_n(t_n^\omega(t)) \in B_n$, $n \ge n_0 \vee n_1$, by definition of B_n, so $s_n^\omega(t) \to t$. $\square$

Example (MT 51). For separable $\{X(t)\}_{t\in[0,1]}$ with separant $D \in [0,1]$, $\mathbf{P}\{X(t) \in G$ some $t \in [0,1]\} = \mathbf{P}\{X(t) \in G$ some $t \in D\}$ for open $G \subseteq \mathbb{R}$. For X $\mathbf{P}$-continuous, this holds for any dense $D \subseteq [0,1]$.

Definition. Process $\{X(t)\}_{t\in[0,1]}$ <u>bounded</u> if $\lim_{M\to\infty} \mathbf{P}\{\|X\|_{\mathcal{C}} > M\} = 0$.

Example (continued). $\mathbf{P}\{\|X\|_{\mathcal{C}} > M\} = \mathbf{P}\{X(t) > M$ some $t \in D\} = \lim_{n\to\infty} \mathbf{P}\{\max_{1\le i\le n} X(d_i) > M\}$. X bounded iff. some process with same fidi's bounded.

Definition. Processes $\{X(t)\}_{t\in T}$ and $\{Y(t)\}_{t\in T}$ on common probability space <u>indistinguishable</u> if $\mathbf{P}\{X(t) = Y(t)\} = 1$ for $t \in T$. (Cf. **MT 51**.)

Example. Indistinguishable processes have same fidi's.

Right- and leftcontinuous version of a given Poisson process indistinguishable.

If continuously distributed r.v. ξ independent of process $\{X(t)\}_{t\in\mathbb{R}}$, then $Y(t) = X(t)$ for $t \ne \xi$, $Y(t) = X(t)+1$ for $t = \xi$, is indistinguishable from X.

Theorem. (Doob) *Process $\{X(t)\}_{t\in T}$ on second order countable topological space T has separable indistinguishable version $\{Y(t)\}_{t\in T}$.*

Proof. Pick countable basis $\mathfrak{G}$ in T, with $\mathfrak{G}_t = \{B_n^t\}_{n=1}^\infty$ sets of $\mathfrak{G}$ containing t.

Enough show for each closed $F \subseteq \mathbb{R}$ and open $G \subseteq T$ (see also **MT 52**)

$$(\star) \qquad \{Y(t) \in F \text{ for } t \in G \cap D\} \setminus \{Y(t) \in F \text{ for } t \in G\} \subseteq N:$$

Take $\omega \notin N$ and $t \in T$. By $(\star)$ with $F = \{y \in \mathbb{R} : |y - Y(t)| \ge \frac{1}{n}\}$, there is $t_n \in \bigcap_{i=1}^n B_i^t \cap D$ with $|Y(t_n) - Y(t)| < \frac{1}{n}$. Now $Y(t_n) \to Y(t)$ and $t_n \to t$.

Let $\mathcal{A}_0$ be finite unions of closed intervals with rational or infinite endpoints, and $\mathcal{A}$ intersections of such, so $\mathcal{A}$ is the closed in $\mathbb{R}$. There is countable $D \subseteq T$ with

$$(\star\star) \qquad \bigcup_{A\in\mathcal{A}} \big(\{X(s) \in A \text{ for } s \in D\} \setminus \{X(t) \in A\}\big) \quad \mathbf{P}\text{-null}, \quad \text{for } t \in T:$$

Notice $D_A \equiv \operatorname{arginf}\{\mathbf{P}(\bigcap_{s\in D}\{X(s) \in A\}),\, D \subseteq T \text{ countable}\} = \bigcup_{n=1}^\infty D_n$, where $\mathbf{P}\{\bigcap_{s\in D_n}\{X(s) \in A\}\} \le \inf + \frac{1}{n}$, $A \in \mathcal{A}_0$, so $\Omega_t(A) \equiv \bigcap_{s\in D_A}\{X(s) \in A,\, X(t) \notin A\}$ $\mathbf{P}$-null, since $\mathbf{P}\{\bigcap_{s\in D_A}\{X(s) \in A\}\} = \mathbf{P}\{\bigcap_{s\in D_A\cup\{t\}}\{X(s) \in A\}\}$ by minimality of D_A. Put $D \equiv \bigcup_{A\in\mathcal{A}_0} D_A$, $\Omega_t \equiv \bigcup_{A\in\mathcal{A}_0} \Omega_t(A)$. For $B = \bigcap_k A_k \in \mathcal{A}$

$\bigcap_{s\in D}\{X(s)\in B, X(t)\notin B\} = \bigcup_k\bigcap_{s\in D}\{X(s)\in B, X(t)\notin A_k\} \subseteq \bigcup_k \Omega_t(A_k) \subseteq \Omega_t$.

By $(\star\star)$ on $\{X(t)\}_{t\in B}$, $B\in\mathfrak{G}$, for some countable $D^B\subseteq B$, to $t\in B$, there is **P**-null Ω_t^B with $\bigcap_{s\in D^B}\{X(s)\in F, X(t)\in F\} \subseteq \Omega_t^B$, closed $F\subseteq\mathbb{R}$. Let $D\equiv\bigcup_{B\in\mathfrak{G}} D^B$, $\Omega_t\equiv\emptyset$ if $t\in D$, $\Omega_t\equiv\bigcup_{B\in\mathfrak{G}_t}\Omega_t^B$ if $t\notin D$, and $N\equiv\bigcup_{t\in D}\Omega_t$. Put $Y(t)\equiv X(t)$ if $\omega\notin\Omega_t$, $Y(t)\equiv\limsup_{D\ni s\to t}X(s)$ if $\omega\in\Omega_t$, so X,Y indistinguishable.

By $(\star)$, enough show $\Delta_t(B)\equiv\bigcap_{s\in B\cap D}\{Y(s)\in F, Y(t)\notin F\}\subseteq N$, $t\in G\supseteq B\in\mathfrak{G}_t$.

Let $Y(t)\notin F$ and $\omega\notin\Omega_t$. Since $X(t)=Y(t)\notin F$ and $\omega\notin\Omega_t^B$, $X(s)\notin F$, some $s\in B\cap D=D^B$. So $Y(s)=X(s)\notin F$, $\omega\notin\Omega_s$. Thus $\Delta_t(B)\cap\Omega_t^c\subseteq\Omega_s\subseteq N$.

Let $Y(t)\notin F$ and $\omega\in\Omega_t$. If $Y(s)\in F$, $s\in D^B$, then $Y(t)=\limsup_{D\ni s\to t}X(s) = \limsup_{D\ni s\to t}Y(s)\in F$, $\omega\notin N$, by F closed and $\Omega_s\in N$. So $\Delta_t(B)\cap\Omega_t\subseteq N$. □

Theorem. (KOLMOGOROV) *Separable process $\{X(t)\}_{t\in[0,1]}$ fits into $\mathcal{C}$ if*

$$\mathbf{P}\{|X(s)-X(t)|>x\}\le c|t-s|^{1+\gamma}/x^\beta \quad \textit{for } s,t\in[0,1],\ x>0, \quad \textit{some constants } c,\beta,\gamma>0.$$

Proof. By Example, enough show there is $\mathcal{C}$-valued r.v. Y with same fidi's. Sample X at $0,\frac{1}{n},\ldots,\frac{n}{n}$ and interpolate; $X_n(t)\equiv X(\lfloor nt\rfloor/n)+(nt-\lfloor nt\rfloor)[X(\lceil nt\rceil/n)-X(\lfloor nt\rfloor/n)]$. Fidi's of X_n $\Rightarrow$ those of X, since by condition in theorem,

$$|X_n(t)-X(t)|\le(1-nt+\lfloor nt\rfloor)|X(\lfloor nt\rfloor/n)-X(t)|+(nt-\lfloor nt\rfloor)|X(\lceil nt\rceil/n)-X(t)|\to_{\mathbf{P}}0.$$

Inequality for X gives it for X_n with other c, so tight by Criterion. □

25. Lévy Processes.

Definition. $\{X(t)\}_{t\in[0,1]}$ Lévy process if **P**-continuous, $X(0)=0$ and independent stationary increments. R.v. ξ ID if $\xi=_d\sum_{i=1}^n\eta_i$ some iid. r.v.'s $\{\eta_i\}_{i=1}^n$, $n\in\mathbb{N}$.

Theorem. (LÉVY-KHINTCHINE) ξ ID *iff. for some unique constants $\mu\in\mathbb{R}$, $\sigma^2\ge 0$, and Borel measure ν on $\mathbb{R}$ with $\int_{\mathbb{R}}1\wedge|x|^2\,d\nu(x)<\infty$, denoted $\xi\sim\mathrm{ID}(\mu,\sigma^2,\nu)$,*

$$\phi_\xi(x)=\mathbf{E}\{\mathrm{e}^{ix\xi}\}=\exp\Big\{i\mu x+\int_{\mathbb{R}}(\mathrm{e}^{ixy}-1-ix\tfrac{y}{1\vee|y|})\,d\nu(y)-\tfrac{1}{2}\sigma^2x^2\Big\} \quad \textit{for } x\in\mathbb{R}.$$

Remark. $\frac{y}{1\vee|y|}$ can be replaced with any bounded measurable $y+O(y^2)$ as $y\to 0$.

Fidi's of Lévy process $\{X(t)\}_{t\in[0,1]}$ are determined by univariate marginal distributions $F_{X(t)}$, and even by $F_{X(1)}$, $X(t)=_d X(1)^{\star t}$, where $\phi_{\xi^{\star t}}(x)=(\phi_\xi(x))^t$.

By Kolmogorov's Theorem, there is Lévy process $\{Y(t)\}_{t\in[0,1]}$ with $Y(t)=_d\mathrm{ID}\,(m,\sigma^2,\nu)^{\star t}$. [There is Lévy process $\{X(t)\}_{t\in[0,1]}$ in $\mathcal{C}$ with $X(t)=_d\mathrm{ID}(m,\sigma^2,\nu)^{\star t}$ iff. $\nu=0$. Here $\boxed{\Rightarrow}$ is difficult, while $\boxed{\Leftarrow}$ is **MT 56**.]

Lemma. *For Lévy process $\{X(t)\}_{t\in[0,1]}$ with $\mathbf{E}\{X(1)^2\}<\infty$,* $\mathbf{E}\{X(t)^2\}=[\mathbf{E}\{X(1)\}]^2t^2+\mathbf{Var}\{X(1)\}t$, $t\in[0,1]$.

Proof. Recall $\mathbf{E}\{X(1)^2\}<\infty$ iff. $\phi_{X(1)}(x)$ two times (continuously) differentiable at zero, in which case $\mathbf{E}\{X(1)\} = i\phi'_{X(1)}(0)$ and $\mathbf{E}\{X(1)^2\} = -\phi''_{X(1)}(0)$, so $\phi_{X(t)}(x)=\phi_{X(1)}(x)^t$ gives $\mathbf{E}\{X(1)^2\}<\infty$ iff. $\mathbf{E}\{X(t)^2\}<\infty$, $t\in[0,1]$, and

$$\mathbf{E}\{X(t)^2\} = -\tfrac{d^2}{dx^2}\phi_{X(1)}(x)^t|_{x=0} = -t(t-1)\phi'_{X(1)}(0)^2\phi_{X(1)}(0)^{t-2} - t\phi''_{X(1)}(0)\phi_{X(1)}(0)^{t-1}$$
$$= t(t-1)[\mathbf{E}\{X(1)\}]^2 + t\mathbf{E}\{X(1)^2\} = \text{claim.} \quad \Box$$

Theorem. *To* $\xi\sim\mathrm{ID}(m,\sigma^2,\nu)$ *there is* D*-valued r.v.* $\{X(t)\}_{t\in[0,1]}$ *that is Lévy process with* $X(t)=_d\xi^{\star t}$.

Proof for $\mathbf{E}\{\xi^2\}<\infty$. $X_n(t)\equiv\sum_{i=1}^{\lfloor nt\rfloor}\eta_i^{(n)}$ has $\phi_{X_n(t)-X_n(s)}(x)\to\phi_{\xi^{\star(t-s)}}(x)$, so $X_n(t)-X_n(s)\Rightarrow\xi^{\star(t-s)}$ in $\mathcal{P}(\mathbb{R})$, $0\le s<t\le1$. Now $(X_n(t_1),\ldots,X(t_k))\Rightarrow$ in $\mathcal{P}(\mathbb{R}^k)$, to right limit, $0\le t_1<\ldots<t_k\le1$, since $(X_n(t_1),X(t_2)-X(t_1),\ldots,X(t_k)-X(t_{k-1}))$ does, by $X_n(t)\Rightarrow$ and independent increments. **(i′)** of Theorem 13.5 holds, since $\lim_{\delta\downarrow0}\mathbf{P}\{|\xi^{\star\delta}|>\varepsilon\}=0$ for $\varepsilon>0$, by **MT 56**. For **(ii′)**, notice that, by independent increments and Lemma, for $0\le r\le s\le t$, with Y as above,

$$\begin{aligned}\mathbf{E}\big\{|X_n(s)-X_n(r)|^4\wedge|X_n(t)-X_n(s)|^4\big\} &\le \mathbf{E}\{|X_n(s)-X_n(r)|^2\}\mathbf{E}\{|X_n(t)-X_n(s)|^2\}\\ &= \mathbf{E}\big\{[Y(\tfrac{\lfloor ns\rfloor}{n})-Y(\tfrac{(\lfloor nr\rfloor+1)\wedge\lfloor ns\rfloor}{n})]^2\big\}\\ &\qquad\times\mathbf{E}\big\{[Y(\tfrac{\lfloor nt\rfloor}{n})-Y(\tfrac{(\lfloor ns\rfloor+1)\wedge\lfloor nt\rfloor}{n})]^2\big\}\\ &\le K(s-r)(t-s)\le K(t-r)^2.\end{aligned}$$

This holds also for $t-r<\frac{1}{n}$, since then $|X_n(s)-X_n(r)|^4\wedge|X_n(t)-X_n(s)|^4=0$. $\Box$

Theorem. $\mathrm{ID}(\mu_n,\sigma_n^2,\nu_n)\ni\xi_n\Rightarrow\xi$ *in* $\mathcal{P}(\mathbb{R})$ *iff.* $\xi\in\mathrm{ID}(\mu,\sigma^2,\nu)$ *and the following*

(i) $\lim_n\mu_n=\mu$*;*

(ii) $\lim_n\nu_n(A)=\nu(A)$ *for* ν*-continuity Borel* $A\subseteq\mathbb{R}$ *with* $0\notin\mathrm{clos}(A)$*;*

(iii) $\lim_{\varepsilon\downarrow0}\limsup_n\big|\int_{|y|\le\varepsilon}x^2y^2\,d\nu_n(y)+(\sigma_n^2-\sigma^2)x^2\big|=0$.

Proof of $\boxed{\Leftarrow}$. By Lévy-Khintchine, as $n\to\infty$ and $\varepsilon\downarrow0$ (in that order),

$$\begin{aligned}\phi_{\xi_n}(x) &= \exp\big\{i\mu_nx+\textstyle\int_{|y|>\varepsilon}(\mathrm{e}^{ixy}-1-ix\frac{y}{1\vee|y|})\,d\nu_n(y)-\frac12\sigma^2x^2\big\}\\ &\quad\times\exp\big\{\textstyle\int_{|y|\le\varepsilon}(\mathrm{e}^{ixy}-1-ixy)\,d\nu_n(y)-\frac12(\sigma_n^2-\sigma^2)x^2\big\}\to\phi_{\mathrm{ID}(\mu,\sigma^2,\nu)}(x).\quad\Box\end{aligned}$$

Theorem. D*-valued Lévy processes* $X_n\Rightarrow X$ *in* $\mathcal{P}(D)$ *iff.* $X_n(1)\Rightarrow X(1)$ *in* $\mathcal{P}(\mathbb{R})$.

Proof. $\boxed{\Rightarrow}$ By Theorem 13.1 ($1\in T_X$). $\boxed{\Leftarrow}$ *for* $\limsup_n\mathbf{E}\{X_n(1)^2\}<\infty$. $X_n(t)=_d X_n(1)^{\star t}\Rightarrow X(1)^{\star t}=_d X(t)$, $t\in[0,1]$, gives $(X_n(t_1),\ldots,X(t_k))\Rightarrow$ in $\mathcal{P}(\mathbb{R}^k)$, by independent stationary increments. **(i′)** of Theorem 13.5 holds, since $\lim_\delta\mathbf{P}\{|X(1)^{\star\delta}|>\varepsilon\}=0$, while Lemma gives **(ii′)**, since for $0\le r\le s\le t$,

$$\mathbf{E}\big\{|X_n(s)-X_n(r)|^4\wedge|X_n(t)-X_n(s)|^4\big\}\le\big(\mathbf{E}\{X_n(1)\}^2(t-r)^2+\mathbf{Var}\{X_n(1)\}(t-r)\big)^2.\ \Box$$

Appendix. Micro Theorems (<u>MT</u>) **for Exercise** (1=EASY, 5=DIFFICULT)**.**

MT 1 (1.5). *For* D *countable dense in* $\mathcal{T}$*, shew* $\{\bigcap_{i=1}^{k}\{x\in\mathcal{C}(\mathcal{T}) : x(t_i)\in H_i\},$ $H_i\subseteq\mathcal{T}$ *open,* $t_i\in D$*,* $k\in\mathbb{N}\}$ *is separating for* $\mathcal{P}(\mathcal{C}(\mathcal{T}))$*.*

MT 2 (1.5). In Theorems 3.5-3.6, it is not desirable to require that each $\mathbf{E}\{|X_n|\}<\infty$, since convergence of $\mathbf{E}\{|X_n|\}$ do not involve all X_n. *Give characterization of when* $\mathbf{E}\{|X_n|\}\to\mathbf{E}\{|X|\}<\infty$*, using a version of uniform integrability, so that the requirement that each* $\mathbf{E}\{|X_n|\}<\infty$ *is not needed* (at all).

MT 3 (2.5). A version of Scheffe's Theorem: *For densities* $f, f_n : S\to\mathbb{R}$ *wrt. measure* μ*, with* $f_n\to f$ *a.e.* (μ)*, and* $g_n : S\to\mathbb{R}$ *uniformly bounded measurable, shew* $\int_S g_n f_n\,d\mu\approx\int_S g_n f\,d\mu$ *as* $n\to\infty$*, in sense that should be specified. Shew* $\liminf_n \int_S g_n f_n\,d\mu\geq\int_S \liminf_n g_n f\,d\mu$ *and* $\limsup_n\int_S g_n f_n\,d\mu\leq\int_S \limsup_n g_n f\,d\mu$*.*

MT 4 (3). Another: *For* X_n *r.v.'s with* $X_n\to X\in\mathbb{L}^1(\Omega)$ *a.s.,* $\limsup_n \mathbf{E}\{X_n^+\}\leq \mathbf{E}\{X^+\}$ *and* $\limsup_n\mathbf{E}\{X_n^-\}\leq\mathbf{E}\{X^-\}$*, shew* $X_n\to_{\mathbb{L}^1(\Omega)}X$*. Relation to Scheffé?*

MT 5 (3). *For* $X_n\to_{\mathbf{P}}X$*, shew* $X_n\to_{\mathbb{L}^1(\Omega)}X$ *iff.* $\{X_n\}_{n=1}^{\infty}$ *uniformly integrable.*

MT 6 (1.5). *For* $\mathbb{Z}$*-valued r.v.'s* X_n, X*, shew* $\mathbf{P}\{X_n=k\}\to\mathbf{P}\{X=k\}$ *for* $k\in\mathbb{Z}$ *implies* $X_n\Rightarrow X$ *in* $\mathcal{P}(\mathbb{R})$*.*

MT 7 (1-5). *Discuss first countability of* $\mathcal{P}(S)$ (i.e., when the open neighborhoods of each $P\in\mathcal{P}(S)$ has a countable basis).

MT 8 (3). *State general principle of correspondance between weak convergence, in the sense of convergence wrt. imposed weak topologies, and pointwise convergence.*

MT 9 (4). *For* S*-valued r.v.'s* X_n*,* S *separable complete, shew* $(X_m, X_n)\Rightarrow(\hat{X}, \hat{X})$ *in* $\mathcal{P}(S\times S)$*, some* S*-valued r.v.* $\hat{X}$*, iff.* $X_n\to_{\mathbf{P}}\tilde{X}$ *in* S*, some* S*-valued r.v.* $\tilde{X}$*.*

MT 10 (3). `Billingsley Problem 5.10.`

MT 11 (1). *Shew* $(F_{\hat{\varepsilon}})_{\tilde{\varepsilon}}\subseteq F_{\hat{\varepsilon}+\tilde{\varepsilon}}$ *and* $F\subseteq(((F_{\hat{\varepsilon}})^c)_{\varepsilon})^c$*,* $\hat{\varepsilon}>\varepsilon$*. Exemplify strict inclusion.*

MT 12 (1). *For separable* S', S'' *and* $P\in\mathcal{P}(S'\times S'')$*, shew opens in* $S'\times S''$ *countable unions of* P*-continuity* $S'_{r'}(x')\times S''_{r''}(x'')$*. How is this used in weak convergence?*

MT 13 (1). *Shew convergence determining class is separating. Shew* $\mathcal{A}_P\subseteq\mathfrak{S}$ *separating for* $P\in\mathcal{P}(S)$ *if* $P_n(A)\to P(A)$ *for* $A\in\mathcal{A}_P$ *implies* $P_n\Rightarrow P$ *in* $\mathcal{P}(S)$*.*

MT 14 (1.5). *Shew* $\lim_{\varepsilon\downarrow 0}P(A_\varepsilon)=P(A_0)$ *for* $P\in\mathcal{P}(S)$ *and* (any!) $A\subseteq S$*.*

MT 15 (2). *For $P\in\mathcal{P}(S)$ and S separable, shew opens in S are countable unions of P-continuity open balls, as well as of P-continuity closed balls. Deduce that finite intersections of open (closed) balls are convergence determining.*

MT 16 (2). *For tight $P\in\mathcal{P}(S)$, shew* $P(A)=\sup\{P(K): A\supseteq K \text{ compact}\},\ A\in\mathfrak{S}$.

MT 17 (3.5). `Billingsley Problem 1.5.`

MT 18 (1.5). *Shew $\mathbb{C}_B(S)$ Banach space.*

MT 19 (2). A real Banach space is normed linear space $(B,\|\cdot\|)$ over $\mathbb{R}$, that is complete (in metric from norm). *Shew linear $L:B\to\mathbb{R}$ continuous iff. continuous at 0 iff. bounded* i.e., $\|L\|^\star\equiv\sup\{|Lx|:\|x\|\le 1\}<\infty$. The dual $B^\star\equiv$ $\{\text{linear continuous } L:B\to\mathbb{R}\}$, with norm $\|\cdot\|^\star$. *Shew $(B^\star,\|\cdot\|^\star)$ is Banach space.*

MT 20 (2). *Shew $BL(S)$ Banach space.*

MT 21 (1.5). *Starting with $BL(S)$, explain notation $\|P\|^\star_{\mathrm{BL}(S)}$ for $P\in\mathcal{P}(S)$.*

MT 22 (4). Topological space T is metrizable iff. T_1, regular [i.e., for $x\in T$ and open $G\ni x$, there is closed $F\subseteq G$ with $x\in\mathrm{INT}(F)$], with σ-locally finite base [i.e., open base $\mathcal{O}=\bigcup_{n=1}^\infty\mathcal{O}_n$ such that, given $n\in\mathbb{N}$, for $x\in T$ there is open $G\ni x$ with $\#\{H\in\mathcal{O}_n: G\cap H\neq\emptyset\}<\infty$]. *Shew $(0)_1^\star$ is metrizable if B separable.* (It is *iff*.) [**Hint:** Take dense $\{x_i\}_{i=1}^\infty\subseteq B$ and $\mathcal{O}=\bigcup_{i,j=1}^\infty\mathcal{O}_{i,j}$, where $\mathcal{O}_{i,j}=\bigcup_{k=1}^{K_{i,j}}$ $\{f\in(0)_1^\star:(f-f_k^{i,j})(x_i)<\frac{1}{j}\}$ covers $(0)_1^\star$, some $\{f_k^{i,j}\}_{k=1}^{K_{i,j}}\subseteq(0)_1^\star$, by Alaoglu.]

MT 23 (1). *Shew T_1 can be changed to "Hausdorff" in* **MT 24**. *Why isn't it?*

MT 24 (4). *Shew $\mathcal{C}(\mathcal{T})$ separable.* [**Hint:** By Ascoli, $\{f\in\mathrm{BL}(\mathcal{T}):\|f\|_{\mathrm{BL}(\mathcal{T})}\le n\}$, and $\mathrm{BL}(\mathcal{T})$, are $\mathcal{C}(\mathcal{T})$-separable, so $\mathrm{clos}(\mathrm{BL}(\mathcal{T}))_{\mathcal{C}(\mathcal{T})}=\mathcal{C}(\mathcal{T})$ by Stone-Weierstrass.]

MT 25 (2). *Shew $\mathcal{C}[0,1]$ dense in $\mathbb{L}^1[0,1]$. Conclude $\mathbb{L}^1[0,1]$ separable. Discuss $\mathbb{L}^p[0,1]$ for general $p\in(0,\infty)$, and more general $\mathcal{T}$ in $\mathcal{C}(\mathcal{T})$ than $\mathcal{T}=[0,1]$.*

MT 26 (3). *Shew* $w^\star\text{-clos}(\{\mu\in\mathcal{M}[0,1]:\mu=f\,dx,\ f\in\mathcal{C}[0,1]\})=\mathcal{M}[0,1]$. *Argue* $\mathrm{clos}(\{f\,dx:f\in\mathcal{C}[0,1]\})\subsetneq\mathcal{M}[0,1]$.

MT 27 (5). *For $\Pi\subseteq\mathcal{P}(S)$ tight, shew* (Π relatively compact iff.) $\mathrm{clos}(\Pi)$ *sequentially compact. For $\mathcal{P}(S)$ metrizable (i.e., S separable) shew $\Pi\subseteq\mathcal{P}(S)$ relatively compact iff.* $\mathrm{clos}(\Pi)$ *sequentially compact.*

MT 28 (3). *Shew $P_n\Rightarrow P$ in $\mathcal{P}(S)$ if $\{P_n\}_{n=1}^\infty$ is relatively compact and $P_n(A)\to P(A)$ for A in class that contains a separating class of continuity sets of each $Q\in\mathcal{P}(S)$. Shew it is iff.* (and not only if) *for S separable.*

MT 29 (3). `Billingsley Problem 6.5.` [**Hint:** See `Billingsley` on ball σ-field.]

MT 30 (3). `Billingsley Problem 6.10.`

MT 31 (4). *For S separable complete shew $(\mathcal{P}(S), \|\cdot\|_{\mathrm{BL}}^{\star})$ complete.* [**Hint:** Take compacts in tightness as continuity sets of finite $\Pi \subseteq \mathcal{P}(S)$, approximate by $f \in \mathrm{BL}(S)$, use Cauchy and Prohorov.]

MT 32 (1.5). *Shew $\eta(x)=\|x\|$ some $\eta \in (0)_1^{\star}$ for $x \in B$.* [**Hint:** Hahn-Banach.]

MT 33 (3). *Shew three corollaries after* Theorem 7.3.

MT 34 (3.5). *Shew B-valued r.v. X has Bochner mean iff. $\mathbf{E}\{\|X\|\}<\infty$, and in that case X has Pettis mean $\mathbf{E}\{X\}$ and $\|\mathbf{E}\{X\}\| \leq \mathbf{E}\{\|X\|\}$. Shew $B \subseteq B^{\star\star}$.* If $B=B^{\star\star}$, B is <u>reflexive</u>. *Discuss Pettis-means for B reflexive.*

MT 35 (2.5). *Shew B-valued r.v. X has covariance if $\mathbf{E}\{(\eta(X))^2\}<\infty$, $\eta \in B^{\star}$.* [**Hint:** Closed Graph Theorem].

MT 36 (3). *Discuss correspondance between r.v. $X \sim \mathrm{N}_B(E,Q)$ and pair $E \in B^{\star\star}$, $Q: B^{\star} \rightarrow B^{\star\star}$, with Q symmetric non-negative.*

MT 37 (2). *Shew proof of* Donsker's Theorem *not uses but proves existence of W.*

MT 38 (1). *Shew version of criterion in* Section 7 *with requirements on moments.*

MT 39 (2). *Why doesn't criterion in* Section 7 *give tightness in* Donsker's Theorem*?*

MT 40 (3). *Motivate different parts of* Lemma 12.1*, except third inequality.*

MT 41 (1). *Shew $x=y$ in D if same on dense $[0,1] \supseteq T \ni 1$. Shew $x_n \rightarrow_D x \Rightarrow x_n \rightarrow_{\text{a.e.}} x$*

MT 42 (1). *Shew Cauchy $\{x_n\}_{n=1}^{\infty}$ has subsequence $\{y_n\}_{n=1}^{\infty}$ with $d(y_n, y_m) \leq 2^{-n}$ for $m \geq n$. Shew $\{x_n\}_{n=1}^{\infty}$ converges if $\{y_n\}_{n=1}^{\infty}$ does.*

MT 43 (3). *Shew $\Pi \subseteq E$ relatively compact iff. totally bounded. Shew* Ascoli's Theorem $\Pi \subseteq \mathcal{C}(\mathcal{T})$ *bounded and <u>equicontinuous</u>* [i.e., $\lim_{\delta \downarrow 0} \sup_{x \in \Pi} w_x(\delta)=0$] *iff.* clos (Π) *compact. Can bounded be replaced with $\sup_{x \in \Pi}|x(t)|<\infty$ for a single $t \in \mathcal{T}$?*

MT 44 (4). *Shew $x_n, x \in D$ with $\int_0^1 x_n f \rightarrow \int_0^1 x f$, $f \in \mathcal{C}$, $\nRightarrow$ $d_1(x_n, x) \rightarrow 0$.*

MT 45 (2.5). *Shew two corollaries in* Section 13.

MT 46 (1). *Shew* INDEPENDENCE *in* Section 23*, and* CRAMÉR-WOLD DEVICE.

MT 47 (1). *For non-negative definite* $r : T\times T\to\mathbb{R}$, *shew there is Gaussian process on* T *with this covariance.*

MT 48 (1). *Shew separable Wiener process continuous wp.* 1.

MT 49 (1). *Explain processes with same fidi'd need not be indistinguishable.*

MT 50 (1). *Shew* $\mathbf{P}$*-continuity need not imply separability.*

MT 51 (2). *For separable* $\{X(t)\}_{t\in[0,1]}$, *separant* D, *open* $G\subseteq\mathbb{R}$, *shew* $\bigcup_{t\in[0,1]}\{X(t)\in G\}=\bigcup_{t\in D}\{X(t)\in G\}$ *except on* $\mathbf{P}$*-null. How about general* $G\subseteq\mathbb{R}$*? With* $\mathbf{P}$*-continuity, shew* $\lim_{n\to\infty}\mathbf{P}\{\sup_{t\in[0,\ldots,2^n]}X(\frac{t}{2^n})>x\}=\mathbf{P}\{\sup_{t\in[0,1]}X(t)>x\}$.

MT 52 (4). *Shew* $(\star)$ *is also necessary for separability in* Doob's Theorem.

MT 53 (4). *Give missing details of proof of* Kolmogorov's Theorem, Section 24.

MT 54 (2). *Shew Kolmogorov gives existence of Lévy process* $X=_d \mathrm{ID}(\mu,\sigma^2,\nu)^{\star\cdot}$.

MT 55 (2.5). *Shew* $\mathbb{R}^k$*-valued r.v.* ξ *symmetric iff.* ϕ_ξ *real. Shew* $\mathrm{ID}(\mu,\sigma^2,\nu)^{\star\delta}\sim\mathrm{ID}(\delta\mu,\delta\sigma^2,\delta\nu)$ *and* $\lim_{\delta\downarrow 0}\mathbf{P}\{|\mathrm{ID}(\mu,\sigma^2,\nu)^{\star\delta}|>\varepsilon\}=0$ *for* $\varepsilon>0$. *Shew* $\mathrm{ID}(\mu,\sigma^2,\nu)$ *symmetric iff.* $\mu=0$ *and* ν *symmetric iff.* $\phi_{\mathrm{ID}(\mu,\sigma^2,\nu)}$ *strictly positive.*

MT 56 (1.5). *Shew there is* $\mathcal{C}$*-valued Lévy process* $X=_d\mathrm{ID}(m,\sigma^2,\nu)^\star$ *if* $\nu=0$.