Exercises for ATW ch 2, p 39-70

2.8.16, 2.8.17, 2.8.19 (x2), 2.8.21, 2.8.22, 2.8.25 (x2), RF2, p.22, RF2
SOLUTION FOR EXERCISE 2.8.16

ALEXEY LINDO

1. Continuity of Poisson process

1.1. Continuity in probability

Definition 1. [2] We say \( \{X(t)\} \) is continuous in probability if, for every \( t \) and positive \( \epsilon \),

\[
\lim_{s \to t} P(|X(s) - X(t)| > \epsilon) = 0.
\]

Unit rate Poisson process \( \{N(t)\}_{t \geq 0} \) is an integer valued process, thus we have

\[
P(|N(s) - N(t)| = 0) = 1 - P(|N(s) - N(t)| > \epsilon) = e^{-(t-s)} \to 1, \quad s \to t.
\]

Another way of proving this property of Poisson process is to apply the Chebyshev's inequality

\[
P(|X(s) - X(t)| > \epsilon) \leq \frac{1}{\epsilon^2} \mathbb{E}(|X(s) - X(t)|^2), \quad \epsilon \geq 0.
\]

Continuity in probability then follows from continuity in mean square, which is proved below.

1.2. Continuity in mean square

Definition 2. [2] We say \( \{X(t)\} \) is continuous in mean square if, for every \( t \),

\[
\lim_{s \to t} \mathbb{E}(|X(s) - X(t)|^2) = 0.
\]

Continuity in mean square follows from continuity of the covariance function \( C \) at diagonal points \((t, t)\), see exercise 2.8.17 [1]. Covariance function of unit rate Poisson process is

\[
C(s, t) = \min(s, t),
\]

which is clearly continuous at diagonal points, \( C(t, t) = t \).
1.3. Continuity with probability one

**Definition 3.** We say \( \{X(t)\} \) is continuous with probability one (or a.s. continuous) if,

\[
P\left( \lim_{s \to t} |X(s) - X(t)| = 0, \text{ for all } t \geq 0 \right) = 1.
\]

We have

\[
P\left( \lim_{s \to t} |X(s) - X(t)| = 0, \text{ for all } t \in [0, t_0] \right) =
\]

\[
P\{\text{There were no jumps in the interval } [0, t_0]\} = e^{-t_0} \to 0, \quad t_0 \to \infty,
\]

therefore unit rate Poisson process is not continuous almost surely.

**Remark 1.** An alternative pointwise definition of almost surely continuity can be given akin to the definitions of continuity in probability and in mean square.

**Definition 4.** [2] We say \( \{X(t)\} \) is continuous with probability one (or a.s. continuous) at point \( t \) if,

\[
\lim_{s \to t} P(|X(s) - X(t)| = 0) = 1.
\]

Poisson process is continuous according to this definition, because waiting time till the next jump has an exponential distribution and thus we have

\[
\lim_{s \to t} P(|X(s) - X(t)| \neq 0) = P(\{\text{Jump occurred at time } t\}) = 0.
\]

Note that according to the pointwise definition of a.s. continuity, both Poisson process and Brownian motion are continuous. This can be seen as a drawback of the latter definition and therefore pathwise definition is preferable.

2. Process that is continuous in probability but not in mean square

Consider a trivial stochastic process \( X(t) = \xi \), for \( t \geq 0 \), where \( \xi \) is a Cauchy distributed random variable. Clearly \( \{X(t)\}_{t \geq 0} \) is continuous in probability and even a.s. pathwise, but not in mean square, because \( E(\xi) = \infty \).

Every process that is continuous in mean square is continuous in probability, this follows from the Chebyshev’s inequality in a form cited above.
Remark 2. Another family of non-trivial examples are given by a compound Poisson process with a compounding distributions that have infinite moments.

3. Differentiability of an integral of Poisson process

An integral of unit rate Poisson process $M(t) = \int_0^t N(s)ds$ can be defined at least in two different ways.

Definition 5. First we can consider a mean square limit of Riemann sums

$$M(t) = \lim_{\delta \to 0} \frac{1}{\delta} \sum_{i} N(s_i)(t_{i+1} - t_i), \quad 0 = t_0 < \ldots < t_n = t,$$

where $\delta = \max_{i}(t_{i+1} - t_i)$ and $s_i \in [t_i, t_{i+1}]$.

Secondly for every $\omega \in \Omega$ we can consider a Riemann integral over trajectory

$$M(t, \omega) = \int_0^t N(s, \omega)ds.$$

Remarks 3. Actually for Poisson process both of this constructions are valid and define a.s. the same stochastic process $[3]$.

Convergence in probability does not lead to a meaningful definition of integrals, because for this mode of convergence integrability does not follow from continuity.

3.1. Differentiability in mean square

One can show that from continuity in mean square follows a uniform continuity and integrability. The Fundamental Theorem of Calculus (FTC) holds as well for a mean square integrals. Therefore $\{M(t)\}_{t \geq 0}$ is mean square differentiable and $M'(t) = N(t)$ by FTC.

Another way of proving that $\{M(t)\}_{t \geq 0}$ is differentiable in mean square is to find its mean and covariance functions and to show that they are differentiable,
see again exercise 2.8.17 [1]. These functions for \(\{M(t)\}_{t \geq 0}\) are given by

\[
E(M(t)) = \int_0^t E(N(x)) \, dx
\]

\[
C_M(t, s) = \int_0^t \int_0^s C_{N}(x, y) \, dx \, dy,
\]

where \(C_m\) and \(C_n\) are covariance functions of \(\{M(t)\}_{t \geq 0}\) and Poisson process \(\{N(s)\}_{t \geq 0}\) respectively. For an integral of unit rate Poisson process we have \(E(M(t)) = t\) and \(C_M(t, t) = \frac{t^2}{2}\), therefore \(\{M(t)\}_{t \geq 0}\) is differentiable in mean square.

3.2. Differentiability with probability one

The trajectory of \(\{M(t)\}_{t \geq 0}\) is a.s. differentiable over interval \([0, t_0]\) if and only if no jumps of Poisson process \(\{N(s)\}_{t \geq 0}\) occur in this interval, this event has probability \(e^{-t_0}\), which turns to zero, when \(t_0\) goes to infinity. Therefore trajectories of process \(\{M(t)\}_{t \geq 0}\) are a.s. non-differentiable. Note that at the same time \(\{M(t)\}_{t \geq 0}\) is a.s. differentiable and every point \(t > 0\).

References


2-8.17 If any stochastic process on \( \mathbb{R}^n \)

(i) Show that \( f \) is mean square continuous iff its covariance function

is continuous on \( \mathbb{T} \times \mathbb{T} \).

If \( f \) is m.s. continuous \( \implies \) \( E[(f(t + h) - f(t))^2] \to 0 \) \( \forall t, h, 0, t + t' \in \mathbb{T} \).

For any \( t, t', s, s' \in \mathbb{T} \), this implies

\[
E[(f(t + h) - f(s))^2] \to 0
\]

by Cauchy-Schwartz -> 0

\[
= \left| E[(f(t + h) - f(s))^2] - E[(f(t + h) - f(s))^2 + E[(f(t + h) - f(s))^2] + E[(f(t + h) - f(s))^2] \right|
\]

\[
\implies 0, h, s \to 0, \text{ so } C \text{ is continuous.}
\]

If \( f \) does not have zero mean we also need that

\[
E[f(s, s', t + h) - E[f(s, t + h)]] \to 0
\]

which follows since if \( Z_n \to Z \) and \( Y_n \to Y \). Then

\[
E[Z_n] - E[Z] = |E[Z_n] - E[Z]| \to 0, h, s \to 0, \text{ as } s, s', t + t' \in \mathbb{T}
\]

by Cauchy-Schwartz.

\( C \) is continuous on \( \mathbb{T} \times \mathbb{T} \) if

\[
E[(f(t + h) - f(s))^2] - 2E[(f(t + h) - f(s))f(s)] + E[f(s)^2] \to 0
\]

\[
\implies C(t + h, s) - C(t, s) \to 0, h, s \to 0, \text{ as } s, s', t + t' \in \mathbb{T}
\]

so \( f \) is m.s. continuous.

If \( f \) does not have zero mean then

\[
E[(f(s, t + h) - f(s))^2] \to C(t + h, t + h) - 2C(t + h, t) + C(t, t) + E[f(t + h)^2] - E[f(t)^2]
\]

so we need the extra condition that the mean is continuous.
Show that if \( C \) is continuous at diagonal points \((t,t)\), then \( D \) is continuous everywhere on \( T \times T \).

\( C \) continuous at diagonal points if

\[
\lim_{t \to t_0} C(t, t_0) = C(t_0, t_0)
\]

\[
|C(t_{n+1}, t_{n+1}) - C(t_n, t_n)| = |E| \frac{(F(0, s') - F(0, t'))}{s' - t'} dt' + |E| \frac{(F(0, t') - F(0, t'))}{t' - t} dt + |E| \frac{(F(0, s) - F(0, t'))}{s - t'} ds
\]

\[
\leq \sqrt{|E|} \frac{(F(0, s') - F(0, t'))^2}{s' - t'} dt' + \sqrt{|E|} \frac{(F(0, t') - F(0, t'))^2}{t' - t} dt + \sqrt{|E|} \frac{(F(0, s) - F(0, t'))^2}{s - t'} ds
\]

\[
\rightarrow 0 \quad \text{as} \quad (s, t) \rightarrow 0, \quad \text{for} \quad s, t \in T
\]

So, \( C \) is continuous everywhere on \( T \times T \).
2.8.19 Take $T = \{0,1\}$, $f, g$ standard normal with $f \sim N(0, 1)$, $g \sim N(0, 1)$.

Then $E[f_i^2] = E[g_i^2]$, $f, g$ a.s. hold

$E[(f_i - f_0)^2] = 2(1 - \Phi) = 2$

$E[(g_0 - g_i)^2] = 2(1 - \Phi) = 2$

Pick your favourite $\omega > 0$, e.g. $\omega = 1.96$:

$P(\sup_t |f_t| > \omega) = 1 - P(\{1|f_t| < \omega\} \cap \{1|f_t| < \omega\}) = 1 - 0.95^2 > 0.05$

$P(\sup_t |g_t| > \omega) = 1 - P(\{1|g_t| < \omega\} \cap \{1|g_t| < \omega\}) = 1 - 0.95^2 = 0.05$

Slepian's inequality holds, since

$P(\sup_t |f_t| > \omega) = 1 - P(\{|f_t| < \omega\} \cap \{|f_t| < \omega\}) = 1 - 0.95^2$

$P(\sup_t |g_t| > \omega) = 1 - P(\{|g_t| < \omega\} \cap \{|g_t| < \omega\}) = 1 - 0.95^2 = 0.05$
Exercise 2.8.19

Formulation

Prove:
\[ \int_0^\delta (-\log(u))^{1/2} dp(u) < \infty \iff \int_\delta^\infty p(e^{-t^2}) dt \]

Solution

By integration theory we have:
\[ \int G \, dF = GF - \int F \, dG \]

since all functions are non-decreasing absolutely continuous functions. We have
\[
\int_0^\delta (-\log(u))^{1/2} dp(u) = (-\log(\delta))^{1/2} p(\delta) - \int_0^\delta p(u) \frac{1}{2(-\log(u))^{1/2}u} \, du =
\]
\[ = \left[ u = e^{-t^2}, \, du = -2tu \, dt \right] = (-\log(\delta))^{1/2} p(\delta) + \int_{(-\log(\delta))^{1/2}}^\infty p(e^{-t^2}) dt \]

The second part is immediate since if \( \mathbb{E}(f_t - f_s)^2 \leq \frac{K}{\log|t-s|^{1+\alpha}} \) then
\[
p^2(u) \leq \frac{K}{\log(|t-s|)^{1+\alpha}} \Rightarrow \int_\delta^\infty p(e^{-t^2}) dt \leq \int_\delta^\infty \frac{K'}{u^{1+\alpha}} du \]
\[ T = \{ \omega_0, \theta_j \} \]
\[ \text{Cov}(\theta, \theta) = 0 \]
\[ \text{Cov}(\theta, \phi) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \]
\[ E\left[ (\theta - \xi)^2 \right] = 0 \]
\[ E\left[ (\phi - \eta)^2 \right] = 0 \]

\[ P(\sup |\theta| > 0.5) = 1 \]
\[ P(\sup |\phi| > 0.5) < 1 \]

So, Slepian's far absolute values do not hold.

\[ 2.8.22 \]
\[ g_{b} = \frac{1}{N^2} \sum_{k=1}^{N} \left[ \frac{1}{2N} \sum_{k=1}^{N} (\delta_k \xi_k) - (\delta_k \xi_k) \right] \]
\[ G_{b} = \frac{1}{N} E[\theta^2] \sum_{k=1}^{N} (\delta_k \xi_k) (\delta_k \xi_k - \xi_k^2) \]
\[ C_{b}(b) = C(0) + \frac{1}{2} \tau^T A \tau + o(b^2) \]

Let \( f \sim N(0, C(0)) \Rightarrow C_{b}(0,0) = C(0) \)

\( g_{b} \) is centered and \( f_{b} \) is centered.

\[ E[|g_{b}|^2] = E[|f_{b}|^2] \quad \text{for assumptions in Slepian's inequality} \]
\[ E[|g_{b}|^2] = C(0) \]

\[ E[|f_{b}|^2] = \frac{4}{2} (t-s) \Lambda (t-s)^T + o(1 t-s^T) \]
\[ E \left[ \left( g_t - g_s \right)^2 \right] = 2 \ E \left[ 3^2 \right] - 2 \ E \left[ g_t g_s \right] = \]
\[ 2 \ E \left[ 3^2 \right] \left( 1 - \frac{1}{N} \sum_{k=1}^{N} \cos(\lambda_k \left| t_k - s_k \right|) \right) = \]
\[ \frac{2 E[3^2]}{N} \left( \sum_{k=1}^{N} \left( 1 - \cos(\lambda_k \left| t_k - s_k \right|) \right) \right) = \left( \sigma^2 \right) \left( \text{trace} \right) = \]
\[ \frac{1}{N} \left( \sum_{k=1}^{N} \left| t_k - s_k \right|^2 \right) + o \left( \left| t - s \right|^2 \right) = \]
\[ \frac{1}{2} \left( t - s \right) \Lambda^* \left( t - s \right) + o \left( \left| t - s \right|^2 \right) \text{, where} \]
\[ \Lambda^* = \frac{2 \left( \sigma^2 \right)}{N} \begin{bmatrix} \chi_1^2 & 0 \\ 0 & \chi_N^2 \end{bmatrix} \]

We want the inequality
\[ E \left[ \left| t_t - t_s \right|^2 \right] \leq E \left[ \left| g_t - g_s \right|^2 \right] \]

Therefore, we want to choose \( \lambda_1, \ldots, \lambda_N \) s.t.
\[ (t - s) \Lambda (t - s) \leq (t - s) \Lambda^* (t - s) \]
in a neighborhood of origin.

This is true if we choose
\[ \lambda_1 = \lambda_2 = \ldots = \lambda_N = \frac{N}{2 \left( \sigma^2 \right)} \max \left( \text{eig}(\Lambda) \right) \]

Similar for the lower bound but with \( \max \) replaced by \( \min \).
$f \in C(\mathbb{R})$

$N_k(T)$ is the number of crossings by $f$ of the level $u$ on the interval $[0,T]$

(i) Use Theorem 2.7.1 to derive

$$E[N_k(T)] = \int_0^T E\left[ f(b) I(f(b) > 0) \mid f(b) = u \right] P_b(u) \, db$$

Since we are looking at crossings we want the derivative to be positive $\Rightarrow g(b) = f'(b)$, $B = [0,\infty]$.

$$f(b) = f'(b) \quad \text{let} \quad \nabla f(b) = \dot{f}(b)$$

$$\Pi_B(g(b)) = \Pi_B(f'(b) > 0)$$

Using 2.7.1 (and assuming conditions hold)

$$E[N_k(T)] = \int_0^T E\left[ f'(b) I(f'(b) > 0) \mid f(b) = u \right] P_b(u) \, db$$

(ii)

Assume that $f$ is stationary and Gaussian with zero mean and unit variance and show that

$$E[N_k(T)] = \frac{T \sqrt{\lambda}}{2 \pi}$$

if $f$ is a stationary, standard Gaussian this means that $f(t)$ and $f(t)$ are independent and $E[f(t)] = 0$, $E[f(t)^2] = \lambda$.

$$\Rightarrow E[N_k(T)] = \int_0^T E\left[ f(b) I(f(b) > 0) \right] P_b(u) \, db$$
Since \( f_r \) is stationary \( \Rightarrow f(t) = t \), \( f'(t) = 1 \\

\[ E[N_0(t)] = \int_0^T P(t > 0) P(W) \] \\
\[ \int_0^T db = T \hspace{1cm} P_b = \frac{e^{-\frac{t^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} \] \\
\[ P(t > 0) = \int_0^\infty \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}} dx = \frac{-\lambda_2}{\sqrt{2\pi} \sigma_2} \left[ e^{-\frac{x^2}{2\sigma^2}} \right]_0^\infty = \frac{\lambda_2}{\sqrt{2\pi} \sigma_2} \]

\[ E[N_0(t)] = T \frac{e^{-\frac{t^2}{2\sigma^2}}}{\sqrt{2\pi} \sigma_2} \frac{\lambda_2}{\sqrt{2\pi} \sigma_2} = T \frac{\sqrt{\lambda_2}}{\sqrt{2\pi} \sigma_2} e^{-\frac{t^2}{2\sigma^2}} \]

(iii) 

Now \( f(t) \) is not assumed stationary anymore. We should show that \( E[N_0(t)] = \frac{e^{-\frac{t^2}{2\sigma^2}}}{2\pi} \int b^2 \sigma_b^2 db \)

\[ \lambda_2 = E[f(t)^2] = \sum_{t} E[f(t)] = 1 \]

Since \( f_b \) is Gaussian \( \Rightarrow f_b \) Gaussian.

\( f_0 \) is centered and therefore \( f \) is centered as well.

\[ \text{Cor}(f_0, f_b) = E[f_0 f_b] = \frac{\partial}{\partial b} C(f_0, f_b) \frac{\partial}{\partial b} \int b \sigma_b^2 db = 0 \]

Since both \( f_0 \) and \( f_b \) are Gaussian this means that they are independent of each other.

\[ P_b(f_0, f_b) = P_b(f_b) P_b(f_0) \]

\[ E[f_b^2 I(f_b > 0) / f_b = u] = E[f_b^2 I(f_b > 0)] \]

\[ E[N_0(t)] = \int_0^T E[f_b^2 I(f_b > 0) / f_b = u] P_b f_b db = \int_0^T \frac{e^{-\frac{t^2}{2\sigma^2}}}{\sqrt{2\pi} \sigma_2} db = \frac{e^{-\frac{t^2}{2\sigma^2}}}{\sqrt{2\pi} \sigma_2} \int_0^\infty \sqrt{2\pi \sigma_2} \sigma_2 db \]
Exercise 2.8.25

Formulation

Let $f \in C^2(\mathbb{R})$ almost surely. Define $M_u(T) = \# \{ t \in T : f(t) \geq u, f'(t) = 0, f''(t) < 0 \}$, and let $M(T) = M_{-\infty}(T)$

1. Show that $E[M(T)] = T \lambda_4^{1/2}/(2 \pi \lambda_2^{1/2})$ if $f$ is stationary.

2. Show that for general $f$ we have:

$$E[M_u(T)] = \int_0^T E[-f''(t)1_{f'(t) < 0}1_{f(t) \geq 0}|f'(t) = 0]p(t)dt.$$

3. Show that $EM_u(T) = T \lambda_4^{1/2}/2 \pi \lambda_2^{1/2} \Psi(\lambda_4^{1/2}u/\Delta^{1/2}) - T \lambda_4^{1/2}/2 \pi \Phi(u)\Phi(\lambda_2u/\Delta^{1/2})$, for $f$ centered, unit variance stationary Gaussian process, where $\Delta = \lambda_4 - \lambda_2^2$.

4. Show that $\lim_{u \to \infty} EM_u(T)/E_M(T) = 1$.

Solution

Part 1

Writing out $M(T) = \# \{ t \in T : f'(t) = 0, f''(t) > 0 \}$ we see that we can apply Rice formula/the previous exercise:

$$EM(T) = \int_0^T E[-f''(t)1_{f'(t) < 0}|f'(t) = 0]p(t)dt = \frac{1}{\sqrt{2\pi\lambda_2}} \int_0^T E[-f''(t)1_{f'(t) < 0}] =$$

$$= \frac{1}{\sqrt{2\pi\lambda_2}} \int_0^T \int_{-\infty}^0 -y \frac{1}{\lambda_4} \Phi(y/\lambda_4^{1/2})dy dt = \frac{T}{2\pi} \sqrt{\frac{\lambda_4}{\lambda_2}}.$$

Part 2

For this part one only notes that using the general Rice formula with $g(t) = (f''(t), f(t))$ and $B = (-\infty, 0) \times [u, \infty)$ and $f(t) = f'(t)$ we get the sought result.
Part 3

This result follows a series of tedious computations. First of all since \( f \) is Gaussian and stationary, \( f' \) and \( f'' \) are independent, and so is \( f \) and \( f' \). Using the formula above, noting that \( p'_f(0) = \frac{1}{\sqrt{2\pi\lambda_0}} \) and

\[
E[-f''(t)1_{f''(t)<0}1_{f(t)>0}f'(t) = 0] = E[-f''(t)1_{f''(t)<0}1_{f(t)>0}] = 
\]

\[
= \int_{-\infty}^{0} \int_{-\infty}^{0} \int_{-\infty}^{\infty} \int_{-\infty}^{0} \frac{1}{2\pi\sqrt{\det(C)}} e^{-\frac{1}{4}(x,y)^T C^{-1}(x,y)} dx dy 
\]

Now we have \( E_{f_t}^2 = 1, E(f_t'')^2 = \lambda_4, E_{f_t} f_t'' = -\lambda_2 \) so that

\[
C = \begin{bmatrix} 1 & -\lambda_2 \\ -\lambda_2 & \lambda_4 \end{bmatrix}, \quad C^{-1} = \frac{1}{\Delta} \begin{bmatrix} \lambda_4 & \lambda_2 \\ \lambda_2 & 1 \end{bmatrix} 
\]

so that the integral reads

\[
\int_{-\infty}^{0} \int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{\Delta}} (-y) \exp\left\{ \frac{-\lambda_4 x^2 + 2\lambda_2 xy + y^2}{2\Delta} \right\} dx dy = 
\]

\[
= \frac{1}{2\pi\sqrt{\Delta}} \int_{-\infty}^{0} \int_{-\infty}^{\infty} (-y) \exp\left\{ \frac{-(y + \lambda_2 x^2)^2 + \Delta x^2}{2\Delta} \right\} dx dy 
\]

The first integral we can compute now is

\[
\int_{-\infty}^{0} -y e^{-\frac{(y + \lambda_2 x^2)^2}{2\Delta}} dy = \Delta e^{-\frac{\lambda_2 x^2}{2\Delta}} + \lambda_2 x \int_{-\infty}^{0} e^{-\frac{(y + \lambda_2 x^2)^2}{2\Delta}} dy = 
\]

\[
= \Delta e^{-\frac{\lambda_2 x^2}{2\Delta}} + \sqrt{2\pi\Delta} \lambda_2 x \Phi(\lambda_2 x / \Delta^{1/2}) 
\]

Hence we obtain the following expression

\[
\int_{-\infty}^{\infty} \frac{1}{2\pi\sqrt{\Delta}} \int_{u}^{\infty} e^{-x^2/2} \left( \Delta e^{-\frac{\lambda_2 x^2}{2\Delta}} + \sqrt{2\pi\Delta} \lambda_2 x \Phi(\lambda_2 x / \Delta^{1/2}) \right) 
\]

\[
= \frac{\Delta}{\sqrt{2\pi\lambda_4}} \Phi(\lambda_4^{1/2} x / \Delta^{1/2}) - \frac{\lambda_2}{\sqrt{2\pi}} \int_{u}^{\infty} \frac{d}{dx} e^{-x^2/2} \Phi(\lambda_2 x / \Delta^{1/2}) dx 
\]

...
So it only remains to compute the last integral:

$$
- \frac{\lambda_2}{\sqrt{2\pi}} \int_{u}^{\infty} \frac{d}{dx} e^{-x^2/2} \Phi(\lambda_2 x / \Delta^{1/2}) =
$$

$$
= - \left[ \lambda_2 \Phi(x) \Phi(\lambda_2 x / \Delta^{1/2}) \right]_{u}^{\infty} + \frac{\lambda_2}{\sqrt{2\pi}} \int_{u}^{\infty} e^{-x^2/2} \phi(\lambda_2 x / \Delta^{1/2}) \frac{\lambda_2}{\Delta^{1/2}} \, dx =
$$

$$
= \lambda_2 \Phi(u) \Phi(\lambda_2 u / \Delta^{1/2}) + \frac{\lambda_2}{\sqrt{2\pi} \lambda_4} \Psi(\lambda_4^{1/2} u / \Delta^{1/2})
$$

which finally yields:

$$
\mathbb{E}[U_u(T)] = \frac{1}{\sqrt{2\pi}} \int_{0}^{T} \frac{\Delta}{\sqrt{2\pi} \lambda_4} \Psi(\lambda_4^{1/2} u / \Delta^{1/2}) + \lambda_2 \Phi(u) \Phi(\lambda_2 u / \Delta^{1/2}) + \frac{\lambda_2^2}{\sqrt{2\pi} \lambda_4} \Psi(\lambda_4^{1/2} u / \Delta^{1/2}) \, dt -
$$

$$
= T \frac{\lambda_4^{1/2}}{2\pi \lambda_2^{1/2}} \Psi(\lambda_4^{1/2} u / \Delta^{1/2}) + T \frac{\lambda_2}{2\pi} \Phi(u) \Phi(\lambda_2 u / \Delta^{1/2}).
$$

Part 4

Computing the limit is a simple matter. We have \( \mathbb{E}[N_u(T)] = T \sqrt{\frac{2\pi}{2\pi}} \phi(u) \) and using the elementary estimate \( \Psi(u) \leq \frac{1}{\sqrt{2\pi}} \phi(u) \) we get:

$$
\lim_{u \to \infty} \frac{C_1(T) \Psi(\lambda_4^{1/2} u / \Delta^{1/2}) + C_2(T) \phi(u) \Phi(\lambda_2 u / \Delta^{1/2})}{C_2(T) \phi(u)} = 0 + \lim_{u \to \infty} \Phi(\lambda_2 u / \Delta^{1/2}) = 1
$$

since

$$
\lim_{u \to \infty} \frac{C_1(T) \Psi(\lambda_4^{1/2} u / \Delta^{1/2})}{C_2(T) \phi(u)} \leq \lim_{u \to \infty} \frac{C_1}{u} = 0.
$$
So it only remains to compute the last integral:

\[- \frac{\lambda_2}{\sqrt{2\pi}} \int_{u}^{\infty} \frac{d}{dx} e^{-x^2/2} \Phi(\lambda_2 x / \Delta^{1/2}) = \]

\[- \left[ \lambda_2 \Phi(x) \Phi(\lambda_2 x / \Delta^{1/2}) \right]_{u}^{\infty} + \frac{\lambda_2}{\sqrt{2\pi}} \int_{u}^{\infty} e^{-x^2/2} \Phi'(\lambda_2 x / \Delta^{1/2}) = \]

\[= \lambda_2 \Phi(u) \Phi(\lambda_2 u / \Delta^{1/2}) + \frac{\lambda_2}{\sqrt{2\pi}} \int_{u}^{\infty} e^{-x^2/2} \phi(\lambda_2 x / \Delta^{1/2}) \frac{\lambda_2}{\Delta^{1/2}} dx = \]

\[= \lambda_2 \Phi(u) \Phi(\lambda_2 u / \Delta^{1/2}) + \frac{\lambda_2^2}{\sqrt{2\pi} \lambda_4} \Psi(\lambda_4^{1/2} u / \Delta^{1/2}) \]

which finally yields:

\[\mathbb{E}[M_n(T)] = \frac{1}{\sqrt{2\pi}} \int_{0}^{T} \frac{\Delta}{\sqrt{2\pi} \lambda_4} \Psi(\lambda_4^{1/2} u / \Delta^{1/2}) + \lambda_2 \Phi(u) \Phi(\lambda_2 u / \Delta^{1/2}) + \frac{\lambda_2^2}{\sqrt{2\pi} \lambda_4} \Psi(\lambda_4^{1/2} u / \Delta^{1/2}) \, dt = \]

\[= T \frac{\lambda_4^{1/2}}{2\pi \lambda_2} \Phi(\lambda_4^{1/2} u / \Delta^{1/2}) + T \sqrt{\frac{\lambda_2}{2\pi}} \Phi(u) \Phi(\lambda_2 u / \Delta^{1/2}). \]

**Part 4**

Computing the limit is a simple matter. We have \( \mathbb{E}[N_n(T)] = T \sqrt{\frac{\lambda_2}{2\pi}} \Phi(u) \) and using the elementary estimate \( \Psi(u) \leq \frac{1}{\sqrt{\pi}} \phi(u) \) we get:

\[\lim_{u \to -\infty} \frac{C_1(T) \Psi(\lambda_4^{1/2} u / \Delta^{1/2}) + C_2(T) \phi(\lambda_2 u / \Delta^{1/2})}{C_2(T) \phi(u)} = 0 + \lim_{u \to -\infty} \Phi(\lambda_2 u / \Delta^{1/2}) = 1 \]

since

\[\lim_{u \to -\infty} \frac{C_1(T) \Psi(\lambda_4^{1/2} u / \Delta^{1/2})}{C_2(T) \phi(u)} \leq \lim_{u \to -\infty} C \frac{1}{u} = 0.\]
2.8.25

\[ M_n(T) = \# \{ b \in (0,T) : \beta_b = 0, \quad \beta_b < 0, \quad \beta_b > T \} \]

\[ t_b \text{ is stationary and standard gaussian}\Rightarrow t_b \sim \mathcal{N}(0,1) \]
\[ \beta_b \sim \mathcal{N}(0, \sigma^2) \]

Rice's formula we set

\[ g(t) = \begin{bmatrix} t_b \\ \beta_b \end{bmatrix}, \quad B = \begin{bmatrix} \mu b^2 & \sigma \beta_b x \\ \sigma \beta_b x & \sigma^2 \end{bmatrix} \]

\[ \Rightarrow E \left[ M_n(T) \right] = \int_0^T E \left[ \mathbb{I} \left( \beta_b > T \right) \mathbb{I} \left( x_b < 0 \right) \right] \rho(x_b) \, dx_b = \int_0^T \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{x_b^2}{2\sigma^2}} \, dx_b = \frac{1}{\sqrt{2\pi} \sigma} \left[ \frac{x_b e^{\frac{x_b^2}{2\sigma^2}}}{\sqrt{\pi} \sigma} \right]_0^T \]

Since \( t_b \) is stationary (random) gaussian \( \Rightarrow t_b \) is independent of both \( t_0 \) and \( \beta_0 \).

\[ \Rightarrow E \left[ M_n(T) \right] = \frac{1}{\sqrt{2\pi} \sigma} \int_0^T e^{-\frac{x_b^2}{2\sigma^2}} \, dx_b = \frac{1}{\sqrt{2\pi} \sigma} \left[ \frac{x_b e^{\frac{x_b^2}{2\sigma^2}}}{\sqrt{\pi} \sigma} \right]_0^T \]

\[ = \frac{T \sqrt{\pi \sigma}}{2 \sigma \sqrt{\pi} \sigma} \quad \#
\]

\[ E \left[ M_{\infty}(T) \right] = \frac{T}{\sqrt{2\pi} \sigma} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{x^2}{2\sigma^2}} \, dx = \frac{T}{\sqrt{2\pi} \sigma} \left[ \frac{x e^{\frac{x^2}{2\sigma^2}}}{\sqrt{\pi} \sigma} \right]_{-\infty}^{\infty} \]

\[ = \frac{T \sqrt{2\pi \sigma}}{2 \sigma \sqrt{2\pi} \sigma} \quad \#
\]

\[ \Rightarrow E \left[ \mathbb{I} \left( \beta_b > T \right) \mathbb{I} \left( x_b < 0 \right) \right] \rho(x_b) \, dx_b = \int_0^T E \left[ \mathbb{I} \left( \beta_b > T \right) \mathbb{I} \left( x_b < 0 \right) \right] \rho(x_b) \, dx_b = \int_0^T \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{x_b^2}{2\sigma^2}} \, dx_b = \frac{1}{\sqrt{2\pi} \sigma} \left[ \frac{x_b e^{\frac{x_b^2}{2\sigma^2}}}{\sqrt{\pi} \sigma} \right]_0^T \]

\[ = \frac{T \sqrt{\pi \sigma}}{2 \sigma \sqrt{2\pi} \sigma} \quad \#
\]

(II) From before we know

\[ E \left[ \mathbb{I} \left( \beta_b > T \right) \mathbb{I} \left( x_b < 0 \right) \right] \rho(x_b) \, dx_b = \int_0^T \frac{1}{\sqrt{2\pi} \sigma} e^{-\frac{x_b^2}{2\sigma^2}} \, dx_b = \frac{1}{\sqrt{2\pi} \sigma} \left[ \frac{x_b e^{\frac{x_b^2}{2\sigma^2}}}{\sqrt{\pi} \sigma} \right]_0^T \]

\[ = \frac{T \sqrt{\pi \sigma}}{2 \sigma \sqrt{2\pi} \sigma} \quad \#\]
Again assume $f_0$ is Gaussian stationary with mean 0 and $\sigma^2$.

$$E[M_u(T)] = \{ \text{from (i)} \} = \frac{T}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\frac{1}{2} \left[ x^2 + \frac{y^2}{\sigma^2} \right]} \, dx \, dy$$

$$\Xi = \begin{bmatrix} E[f^2] & E[f^4] \\ E[f^2] & E[f^4] \end{bmatrix} = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \Rightarrow \Xi \sim \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix}$$

$$\begin{bmatrix} a_x & 0 & 0 \\ 0 & a_x & a_y \\ 0 & a_y & a_y \end{bmatrix} \sim \begin{bmatrix} 1 + \frac{a_y^2}{a_x} \\ \frac{a_y}{a_x} \\ \frac{a_y}{a_x} \end{bmatrix}$$

$$\Xi^{-1} \Xi = \begin{bmatrix} 1 + \frac{a_y^2}{a_x} & 0 \\ \frac{a_y}{a_x} & 1 + \frac{a_y^2}{a_x} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix}$$

$$E[M_u(T)] = \frac{T}{\sqrt{2\pi \sigma^2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (-x) e^{-\frac{1}{2} \left[ x^2 + \frac{y^2}{\sigma^2} \right]} \, dx \, dy$$

$$\varphi(x) = \frac{1}{\sqrt{2\pi \sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$D = \int \left( \frac{3\sigma^2}{2} \right) e^{-\frac{3\sigma^2}{2}} \, dz = \frac{2\sigma^2}{\sqrt{2\pi \sigma^2}} \varphi \left( \frac{\sigma^2}{\sqrt{2\pi \sigma^2}} \right) + \Delta \varphi \left( \frac{\sigma^2}{\sqrt{2\pi \sigma^2}} \right)$$

$$E[M_u(T)] = \frac{T}{\sqrt{2\pi \sigma^2}} \varphi \left( \frac{\sigma^2}{\sqrt{2\pi \sigma^2}} \right) + \Delta \varphi \left( \frac{\sigma^2}{\sqrt{2\pi \sigma^2}} \right)$$
\[ \begin{align*}
\text{(a)} & \quad \int_{-\infty}^{\infty} v_2(x) \varphi \left( \frac{u v_1(x)}{\sqrt{\Delta}} \right) \varphi \left( \frac{u v_2(x)}{\sqrt{\Delta}} \right) \, dx = a_2 \int_{-\infty}^{\infty} \frac{u v_1(x)}{\sqrt{\Delta}} \varphi \left( \frac{u v_2(x)}{\sqrt{\Delta}} \right) \, dx - u \\
& \quad - a_2 \int_{-\infty}^{\infty} \varphi(v_1(x)) \varphi \left( \frac{u v_2(x)}{\sqrt{\Delta}} \right) \left( \frac{a_2}{\sqrt{\Delta}} \right) \, dx = \\
& \quad = \sqrt{a_2} \sigma_2 \varphi \left( \frac{u v_1}{\sqrt{\Delta}} \right) + a_2 \int_{-\infty}^{\infty} e^{-\frac{u^2}{2\Delta}} \, dx = \\
& \quad = \sqrt{a_2} \sigma_2 \varphi \left( \frac{u v_1}{\sqrt{\Delta}} \right) + a_2 \varphi \left( u \frac{\sigma_2}{\sqrt{\Delta}} \right) \frac{\sqrt{\Delta}}{\sigma_2} \\
\text{(b)} & \quad \Delta Y(u \frac{\sigma_2}{\sqrt{\Delta}}) \varphi \left( \frac{u v_1}{\sqrt{\Delta}} \right) \, dx = \Delta Y(u \frac{\sigma_2}{\sqrt{\Delta}}) \frac{\sigma_2}{\sqrt{\Delta}} \\
\text{(c)} & \quad E[M_n(t)] = \frac{T}{\sqrt{2\pi} \Delta} \left[ x \varphi (u \frac{\sigma_2}{\sqrt{\Delta}}) \frac{\sigma_2}{\sqrt{\Delta}} + x \varphi (u \frac{\sigma_2}{\sqrt{\Delta}}) \frac{\sigma_2}{\sqrt{\Delta}} \right] + \\
& \quad + \frac{\sigma_2}{\sqrt{2\pi} \Delta} \left( 3u^2 + \Delta \right) \varphi \left( u \frac{\sigma_2}{\sqrt{\Delta}} \right) = \\
& \quad = T \frac{\sqrt{a_2}}{\sqrt{2\pi} \Delta} \varphi \left( \frac{a_2 u}{\sqrt{\Delta}} \right) + T \frac{\sqrt{a_2}}{\sqrt{2\pi} \Delta} \\
\end{align*} \]
\[ u(MA) = u(A), \quad \forall A \in \mathcal{B}(\mathbb{R}^2) \]

\[ u(A) = \frac{1}{2} \int_{\mathbb{R}^2} f(\lambda) \, d\lambda \]
\[ u(MA) = \int_{MA} f(\lambda) \, d\lambda = \int_{MA} f(\lambda) \, d\lambda \]

where \( MA = \{\lambda': \lambda' = MA, \lambda \in A\} \) is the matrix multiplication matrix \((MTM = I, \det M = 1)\).

This implies that \( f(\lambda') = f(\lambda) \) since if \( \lambda' = \lambda \) there exists \( M : \lambda' = MA \)

\[ C(\lambda') = \frac{1}{2} \int_{\mathbb{R}^2} e^{-i \theta(\lambda') r} \, d\lambda' \]

\[ C(\lambda') = \frac{1}{2} \int_{\mathbb{R}^2} e^{-i \theta(\lambda') r} \, d\lambda' = \frac{1}{2} \int_{\mathbb{R}^2} e^{-i \theta(\lambda') r} \, d\lambda' \]

\[ = \frac{1}{2} \int_0^{2\pi} e^{-i \theta} \, d\theta \]

\[ = \frac{1}{2} \int_0^{2\pi} \cos \theta \, d\theta = \frac{
}{n} \int_0^{2\pi} \cos \theta \, d\theta \]

\[ = 2 \int_0^{2\pi} \cos \theta \, d\theta = \frac{1}{n} \int_0^{2\pi} \cos \theta \, d\theta \]

where \( \frac{1}{n} \int_0^{2\pi} \cos \theta \, d\theta = \frac{\sin \theta}{\theta} \)

and \( \mu(\mathbf{t}, \mathbf{x}) = \int_0^{2\pi} \int_{\mathbb{R}^2} f(\mathbf{t}, \mathbf{x}) \, d\mathbf{x} = 2\pi \int_{\mathbb{R}^2} f(\mathbf{t}, \mathbf{x}) \, d\mathbf{x} \)
Exercise on Slide 2

Formulation

Let $W(t)$ be a centered Gaussian process with covariance function $C(t,s) = \delta(t-s) = 1_{t=s}$. Show that this is a well defined stochastic process and discuss some of its properties.

Solution

Showing that this is a well defined process is immediate since for any collection of times, $t_1, t_2, \ldots, t_k \in T$ the covariance matrix is just the identity matrix on $\mathbb{R}^k$ which immediately implies that Kolmogorovs consistency conditions are satisfied since all the random variables are independent. The interpretation of this process is that this is what could be thought of as a white noise process however this process lacks several important properties. First of all it is not separable, since $W(t)$ consists of independent variables for every $t$ so it is not sufficient to know countably many of these for the process to be well defined, which implies that it is not measurable either.