Nobel prize 2996 to John C. Mather
George F. Smoot
“for discovery of the blackbody form and anisotropy of the cosmic microwave background radiation”

Ocean waves
The oceans cover 72% of the earth’s surface. Essential for life on earth, and huge economic importance through fishing, transportation, oil and gas extraction

fMRI brain scan

PET brain scan
Slides for ATW ch 2, p 39-60

Exercises: 2.8.7, 2.8.9, 2.8.11, 2.8.12 (i) and (2), 2.8.15 + exercises on slides
Complex random fields (ATW p. 39-40)

Convenient for spectral representations, even if one only is interested in real fields. Assume

- \( f(t) = f_R(t) + i f_I(t) \), where \( f_R \) and \( f_I \) are \((N,1)\) fields
- \( E(\|f(t)\|^2) = E(f_R(t)^2 + f_I(t)^2) < \infty \)
- \( m(t) \triangleq E(f_R(t)) + i E(f_I(t)) \)
- \( C(s,t) \triangleq E\left\{ (f(s) - m(s))' (f(t) - m(t)) \right\} \)
- \( C'(s,t) \triangleq E\left\{ (f(s) - m(s))' (f(t) - m(t)) \right\} \)

Properties:

- \( C(t,s) = \overline{C(s,t)} \)
- \( C \) is (Hermitian) non-negative definite
- Weak stationarity and isotropy as for real fields
- A weakly stationary complex Gaussian field is strictly stationary if \( C'(s,t) \) only depends on \( t - s \)
Spectral distribution theorem (ATW p. 41)

(Bochner’s theorem): A continuous function \( C: \mathbb{R}^N \rightarrow \mathbb{C} \) is non-negative definite (i.e. a covariance function of a weakly stationary random field) iff there exists a finite measure \( \nu \) on the Borel sigma-field of \( \mathbb{R}^N \) such that

\[
C(t) = \int_{\mathbb{R}^N} e^{i\langle t, \lambda \rangle} \nu(d\lambda)
\]

- \( \sigma^2 \triangleq C(0) = \nu(\mathbb{R}^N) \)
- \( F(\lambda) \triangleq \nu(\prod_{i=1}^{N} (-\infty, \lambda_i]) \) is the spectral distribution function
- If \( \{f(t)\} \) is real, then \( C \) is real, and from \( e^{i\langle t, \lambda \rangle} = \cos\langle t, \lambda \rangle + i \sin\langle t, \lambda \rangle \) it can be seen that \( C \) is real iff \( \nu \) is symmetric, i.e. \( \nu(A) = \nu(-A) \) for all Borel sets \( A \)
- If \( \{f(t)\} \) is isotropic then \( \nu \) is spherically symmetric
Spectral moments and mean square derivatives (ATW p. 41)

The spectral moments are the "moments" of $\nu$:

$$\lambda_{i_1,\ldots,i_N} \triangleq \int_{\mathbb{R}^N} \lambda_1^{i_1} \ldots \lambda_N^{i_N} \nu(d\lambda)$$

If the field is real then the odd spectral moments are zero:

$$\lambda_{i_1,\ldots,i_N} = 0 \text{ if } \sum_{j=1}^N i_j \text{ is odd}$$

Later we will give a spectral representation of the process itself.
Partial derivatives (ATW p. 42-42)

A \((N, 1)\) field \(f\) has a first order \(L^2\) partial derivative at \(t\) in direction \(t'\) if

\[
D_{L^2}^{1} f(t, t') \equiv \lim_{h \to 0} \frac{1}{h} (f(t + ht') - f(t))
\]

exists as a mean square limit. A sufficient condition for existence of the partial derivative is that

\[
\lim_{h, \hat{h} \to 0} \frac{1}{h \hat{h}} E\{(f(t + ht') - f(t)) (f(t + \hat{h}t') - f(t))\}
\]

exists. This in turn is just a condition on the differentiability of the covariance function \(C\).
Excercises:

• Show that a sequence $X_n$ of random variables converges in mean square iff $E(X_n X_m)$ converges to a constant as $n, m \to \infty$

• What does the last formula on the previous slide mean in terms of differentiability of $C$?

• Write out the formulas for a second order partial derivative
In particular, for $e_k$ the $k$-th unit vector which has 1 as $k$-th element, and all other elements zero, gives the mean square partial derivative

$$\frac{\partial f(t)}{\partial t_k} \triangleq D_{L2}^1 f(t, e_k)$$

Properties:

- $E\{f(s) \frac{\partial f(t)}{\partial t_k}\} = \frac{\partial C(s,t)}{\partial t_k}$
- $E\{\frac{\partial f(s)}{\partial s_j} \frac{\partial f(t)}{\partial t_k}\} = \frac{\partial^2 C(s,t)}{\partial s_j \partial t_k}$
- $E\{\frac{\partial f(t)}{\partial t_k}\} = 0$ if the field has constant mean
- $E\{f(s) \frac{\partial f(t)}{\partial t_k}\} = \frac{\partial C(t-s)}{\partial t_k}$ and $E\{\frac{\partial f(s)}{\partial s_j} \frac{\partial f(t)}{\partial t_k}\} = -\frac{\partial^2 C(t-s)}{\partial t_j \partial t_k}$ if the field is weakly stationary
Properties, continued:

• In particular, $f(t)$ and $\frac{\partial f(t)}{\partial t_k}$ are uncorrelated if the field is weakly stationary, since $\frac{\partial C(t-s)}{\partial t_k}|_{s=t} = 0$

• $E\left\{ \frac{\partial f(t)}{\partial s_j} \frac{\partial f(t)}{\partial t_k} \right\} = \int_{R^N} \lambda_j \lambda_k \nu(d\lambda)$ if the field is weakly stationary

• In particular, $E\left\{ \left( \frac{\partial f(t)}{\partial t_k} \right)^2 \right\} = \int_{R^N} \lambda_k^2 \nu(d\lambda)$ if the field is weakly stationary

• If the field is isotropic

$$E\left\{ \frac{\partial f(t)}{\partial t_j} \frac{\partial f(t)}{\partial t_k} \right\} = -E\left\{ f(t) \frac{\partial^2 f(t)}{\partial t_j \partial t_k} \right\} = \lambda_2 \delta_{j,k},$$

where $\delta_{i,j} = 1$ for $i = j$ and 0 otherwise, and $\lambda_2 \triangleq \int_{R^N} \lambda_i^2 \nu(d\lambda)$ is the second spectral moment (because of isotropy, it doesn’t matter which $i$ one uses)
Properties, continued:

Suppose the field has constant variance, $\sigma^2 = C(t, t)$ but isn’t necessarily stationary. Since

$$E \left\{ f(s) \frac{\partial f(t)}{\partial t_j} \right\} = \frac{\partial C(t, s)}{\partial t_j} \bigg|_{s=t} = \frac{\partial C(t, s)}{\partial s_j} \bigg|_{s=t}$$

and

$$0 = \frac{dC(t, t)}{dt} = \frac{\partial C(t, s)}{\partial t_j} \bigg|_{s=t} + \frac{\partial C(t, s)}{\partial s_j} \bigg|_{s=t}$$

it follows that

$$E \left\{ f(s) \frac{\partial f(t)}{\partial t_j} \right\} = 0,$$

and hence $f(t)$ and $\frac{\partial f(t)}{\partial t}$ again are uncorrelated
Properties, continued:

If the field is Gaussian then also all derivatives and the field itself has a simultaneous Gaussian distribution (why?)

In particular, If the field is Gaussian and has constant variance, then $f(t)$ and $\frac{\partial f(t)}{\partial t}$ are independent (but $f(s)$ and $\frac{\partial f(t)}{\partial t}$ are usually dependent if $s \neq t$)
White noise integrals (ATW p. 45-48)

\( \nu \) sigma-finite positive measure on \( \mathbb{R}^d \)

\( \nu \) has density \( p \) if \( \nu(A) = \int_A p(x)dx \) for all Borel sets \( A \subset \mathbb{R}^N \)

*Noise directed by* \( \nu \) (or "\( \nu \)-noise") is a collection of random variables \( \{W(A); A \text{ Borel set in } \mathbb{R}^N\} \) such that

\[
E(W(A)) = 0, \quad V(W(A)) = \nu(A)
\]

\( A \cap B = \emptyset \Rightarrow W(A \cup B) = W(A) + W(B) \text{ a.s.} \)

\( A \cap B = \emptyset \Rightarrow W(A) \text{ and } W(B) \text{ are uncorrelated} \)

The last property is that \( W \) has *uncorrelated increments*. The noise is Gaussian if \( W(A) \) has a Gaussian distribution for all \( A \). If \( \nu \) is Lebesgue measure, then \( W \) is *white noise*. (Not obvious that such \( W \) exist. See ATW.)
For \( \varphi(t) = \sum_{i=1}^{n} a_i 1_{A_i}(t) \) a deterministic step function with \( A_i \cap A_j = \emptyset \) for all \( i, j \) define

\[
W(\varphi) \triangleq \int_{T} \varphi(t) \nu(dt) \triangleq \sum_{i=1}^{n} a_i W(A_i)
\]

Then

- \( E(W(\varphi)) = 0 \)
- \( V(W(\varphi)) = \sum_{i=1}^{n} a_i^2 \nu(A_i) \)
- For \( \psi(t) = \sum_{i=1}^{n} b_i 1_{A_i}(t) \) another deterministic step function

\[
E(W(\varphi)W(\psi)) = E \left\{ \sum_{i=1}^{n} a_i W(A_i) \sum_{i=1}^{n} b_i 1_{A_i}(t) \right\}
\]

\[
= \sum_{i=1}^{n} a_i b_i \nu(A_i) = \int_{\mathbb{R}^N} \varphi(t)\psi(t)\nu(dt)
\]
The integral $W(\varphi) \triangleq \int_T \varphi(t)W(dt)$ can be extended to general deterministic functions $\varphi$ which are square integrable with respect to $\nu$, using a standard mean square approximation procedure. Easy to see that also for general $\varphi$

• $E(W(\varphi)) = 0$
• $E(W(\varphi)W(\psi)) = \int_{\mathbb{R}^N} \varphi(t)\psi(t)\nu(dt)$
• If the noise is Gaussian then $W(\varphi)$ is Gaussian, jointly with all other such integrals
The engineering way of thinking of white noise:

"a field \( \{ n_t \} \) where the values \( n_t \) and \( n_s \) at different points \( t \) and \( s \) are independent"

The way to formalize this is through integrals, by defining

\[
\int_A n_s ds \triangleq W(A),
\]

\[
\int_A \varphi(t)n(t) dt \triangleq W(\varphi),
\]

for deterministic integrands \( \varphi \). Things are more complicated for stochastic integrands – this leads to Stochastic Calculus.

**Exercise:** Does there exist a Gaussian random process (a \((1,1)\) field) with mean zero and covariance \( \delta(s - t) \), where \( \delta(t) = 1 \) if \( t = 0 \) and \( 0 \) if \( t \neq 0 \)? Can you say anything about its properties? Do you think it is a useful process? Why?
Spectral representation theorem (ATW p. 47-52)

*Complex \( \nu \)-noise* is a collection of complex random variables \( \{W(A); A \ \text{Borel set in} \ \mathbb{R}^N\} \) which satisfies

\[
E\{W(A)\} = 0, \quad E\{W(A) \overline{W(A)}\} = \nu(A)
\]

\( A \cap B = \emptyset \Rightarrow W(A \cup B) = W(A) + W(B) \) a.s.

\( A \cap B = \emptyset \Rightarrow E\{W(A) \overline{W(A)}\} = 0 \)

It does not follow, for \( W_R \) and \( W_I \) the real and imaginary parts of the noise that \( A \cap B = \emptyset \Rightarrow E\{W(A) W(B)\} = 0 \) or that \( E\{W_R(A) W_R(B)\} = 0 \) or that \( E\{W_I(A) W_I(B)\} = 0 \) or that \( E\{W_I(A) W_R(B)\} = 0 \) or that \( E\{W_I(A) W_I(B)\} = 0 \), and the \( W \) in the spectral representation theorem typically doesn’t satisfy these relations.
Integration of a complex function with respect to complex $\nu$-noise is defined just as in the real case,

$$W(\varphi) \triangleq \int_T \varphi(t)W(dt),$$

and

$$E\{W(\varphi)\overline{W(\psi)}\} = \int_T \varphi(t)\overline{\psi(t)}\nu(dt)$$
(Spectral representation theorem): Let $W$ be complex $\nu$-noise. Then the complex valued random field

$$f(t) = \int_{\mathbb{R}^N} e^{i\langle t, \lambda \rangle} \nu(d\lambda)$$

has covariance

$$C(s, t) = \int_{\mathbb{R}^N} e^{i\langle s-t, \lambda \rangle} \nu(d\lambda),$$

and hence is stationary. Conversely, any mean square continuous stationary field can be represented in this way.

$W$ is the spectral process of the field.
The spectral representation of the process can also be written in real form (but the complex is easier to handle) as

$$f(t) = \int_{\mathbb{R}^N} \cos(\langle t, \lambda \rangle) W_1(d\lambda) + \int_{\mathbb{R}^N} \sin(\langle t, \lambda \rangle) W_2(d\lambda).$$

Thus, approximatively,

$$f(t) = \sum_i \{\cos(\langle t, \lambda_i \rangle) W_1(\Lambda_i) + \sin(\langle t, \lambda_i \rangle) W_1(\Lambda_i)\},$$

for some partition $\{\Lambda_i\}$ of $\mathbb{R}^N$ and $\lambda_i \in \Lambda_i$.

Acos($\lambda_1 t_1 + \lambda_2 t_2$) is shown in picture. If you add a sine wave the figure ”looks the same”, cf. calculations for cosine process.
Left: realization and right: contour plot, of sum of 10 cosine waves, with independent $N(0,1)$ amplitudes and frequencies $\lambda_i$ chosen uniformly in $(-\pi, \pi]^2$

ATW argues that the spectral representation is pretty close to the Karhunen-Loeve expansion
Spectral representation of isotropic fields (ATW p. 45-48)

An isotropic random field (with $N > 1$) must satisfy

- The spectral measure is spherically symmetric
- The spectral measure cannot be concentrated on a point or, say, a sector, but only on an annulus,
  \[ \{ \lambda \in \mathbb{R}^N : a \leq |\lambda| \leq b \} \]
- The covariance must satisfy \( C(t) \geq -\frac{C(0)}{N} \)
$C$ is the covariance function of a mean square continuous random field iff

$$C(t) = \int_0^\infty \frac{J_{N-2}(\lambda|t|)}{(\lambda|t|)^{\frac{N-2}{2}}} \frac{2}{\mu(d\lambda)}$$

with $\mu$ a finite measure on $\mathbb{R}^+$ and $J_m$ the Bessel function of the first order,

$$J_m(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(m\theta - xsin(\theta))} d\theta .$$

There are many ways to compute $J_m$, including series expansions, and they are available in standard mathematical software.
Sketch of proof for $N = 2$:

By the spectral distribution theorem and isotropy,

$$C(t) = C((t_1, t_2)) = C((|t|, 0)) = \int_{\mathbb{R}^2} e^{i|t|\lambda_1} \nu(d\lambda)$$

with $\nu$ spherically symmetric. Changing variables to polar coordinates, $T: (\lambda_1, \lambda_2) \to (\lambda \sin(\theta), \lambda \cos(\theta))$, and defining the measure $\mu$ by $\mu[0, \lambda] = \nu\{\lambda: \sqrt{\lambda_1^2 + \lambda_2^2} \leq \lambda\}$ gives that

$$\mu T^{-1}(d\theta, d\lambda) = \frac{1}{2\pi} d\theta \mu(d\lambda),$$

and hence

$$C(t) = \frac{1}{2\pi} \int_0^\infty \int_{-\pi}^\pi e^{i|t|\lambda \sin(\theta)} d\theta \mu(d\lambda)$$

$$= \int_0^\infty J_0(|t|\lambda) \mu(d\lambda).$$
Excercise:

Assume that $N = 2$ and $\nu$ is normalized so that $\mu(R^2) = 1$ and has density $f(\lambda_1, \lambda_2)$.

(i) What does $f$ look like if $\nu$ is spherically symmetrical?

(ii) Write down the spectral representation of the covariance function for this case.

(iii) Prove it.
Spectral representation of space-time fields (ATW p. 56)

- Space-time fields are \((N + 1, d)\) fields, with coordinates \((t, x)\), where \(t\) is a “time-coordinate” and \(x\) is a “space coordinate”.

- Such processes are often stationary in time and isotropic in space, so that

\[
C(t, x) = E\{f(s, u)f(s + t, u + x)\} \triangleq C(t, |x|)
\]

- Then for real \(x\),

\[
C(t, x) = \int_{-\infty}^{\infty} \int_{0}^{\infty} e^{itv} G_N(\lambda x) \mu(dv, d\lambda),
\]

for

\[
G_N(x) = \left(\frac{2}{x}\right)^{\frac{N-2}{2}} \Gamma\left(\frac{N}{2}\right) J_{(N-2)/2}(x)
\]