Nobel prize 2996 to John C. Mather George F. Smoot “for discovery of the blackbody form and anisotropy of the cosmic microwave background radiation”

Ocean waves
The oceans cover 72% of the earth’s surface. Essential for life on earth, and huge economic importance through fishing, transportation, oil and gas extraction
Slides for ATW ch 2, p 60-71

Exercises: 2.8.16, 2.8.17, 2.8.19, 2.8.20 + exercises on slides
Continuity of random fields: (ATW p. 60-65)

Recall:

• $f$ is continuous in probability if

$$\lim_{s \to t} P\{|f(t) - f(s)| \geq \epsilon\} = 0,$$

for each $t \in T$ and $\epsilon > 0$

• $f$ is continuous in mean square (or $L^2$ continuous) if

$$\lim_{s \to t} E\{|f(t) - f(s)|^2\} = 0,$$

for each $t \in T$

• $f$ is continuous with probability 1 (or a.s continuous) if

$$P\left\{\lim_{s \to t}|f(t) - f(s)| = 0, \text{ for all } t \in T\right\} = 1$$
Intuition: $f$ a real centered stationary Gaussian field. Then

- $f(t) - f(s) \sim N(0, E\{|f(t) - f(s)|^2\})$
  $$= N(0, 2\{C(0) - C(t - s)\})$$
  and hence $f(t) - f(s)$ is "small" if $C(0) - C(t - s)$ is "small"

- Karhuen-Loeve:
  $$f(t) = a \sum_{n=1}^{N} \xi_n \sqrt{\lambda_n} \psi(t) + f(t) + \sum_{n=N}^{\infty} \xi_n \sqrt{\lambda_n} \psi(t)$$
  The first term is a sum of continuous functions, and hence, continuous, so if the last term is "small" for large $N$, then $f$ is continuous.

- $f(t) = \int_{\text{cube}} e^{i\langle t, \lambda \rangle} W(d\lambda)) + \int_{\mathbb{R}^N - \text{cube}} e^{i\langle t, \lambda \rangle} W(d\lambda)$
  the first integral is continuous, so if the last integral is "small" for large cubes, then $f$ is continuous.
$f$ a real centered stationary Gaussian field throughout

Canonical metric:

$$d(s, t) \triangleq \left[ E \left\{ (f(s) - f(t))^2 \right\} \right]^{1/2}$$

Ball of radius $\epsilon$ centered at $t$:

$$B_d(t, \epsilon) \triangleq \{ s \in T: d(s, t) \leq \epsilon \}$$

Regions $a$ and $b$ local isotropy, though not the same, regions $c$ and $d$ no local isotropy
$T$ is $d$-compact if

$$\text{diam}(T) \triangleq \sup_{s,t \in T} d(s, t) < \infty$$

**Entropy:**

$N(\varepsilon) = N(T, d, \varepsilon)$ the smallest number of $d$-balls of radius $\varepsilon$ whose union covers $T$

**Log entropy:**

$$H(\varepsilon) = H(T, d, \varepsilon) = \log N(\varepsilon)$$

If $T$ is $d$-compact then $N(\varepsilon) < \infty$ for $\varepsilon > 0$, but typically $N(\varepsilon) \to \infty$ as $\varepsilon \downarrow 0$. The growth rate determines smoothness
General boundedness theorem: (ATW p. 64)

Modulus of continuity:

\[
\omega_f(\delta) = \omega_{f,d}(\delta) = \sup_{d(s,t) \leq \delta} |f(t) - f(s)|
\]

If \( T \) is \( d \)-compact then there exist constant \( K \) and a random variable \( \eta \) such that

\[
E\{\sup_{t \in T} f(t)\} \leq K \int_0^{\frac{\text{diam}(T)}{2}} H(\epsilon)^\frac{1}{2} d \epsilon,
\]

\[
E\{\omega_f(\delta)\} \leq K \int_0^{\delta} H(\epsilon)^\frac{1}{2} d \epsilon,
\]

\[
\omega_f(\delta) \leq K \int_0^{\delta} H(\epsilon)^\frac{1}{2} d \epsilon,
\]

for all \( \delta \leq \eta \).
Continuity of Gaussian processes on $\mathbb{R}^N$:
(ATW p. 65-67)

\[ p^2(u) \triangleq \sup_{|s-t| \leq u} E \left\{ (f(t) - f(s))^2 \right\} \]
\[ = \sup_{|s-t| \leq u} \{ C(t, t) - 2C(s, t) + C(s, s) \} \]

If $f$ is stationary, then $p^2(u) = 2 \sup_{|t| \leq u} \{ C(0) - C(t) \}$
$T \subset \mathbb{R}^N$ compact, $C$ continuous, $\omega_{f,e}(\delta)$ the modulus of continuity with respect to the Euclidian metric

- If for some $\delta > 0$

$$\int_0^{\delta} (-\log u)^{1/2} dp(u) < \infty \text{ or } \int_{\delta}^{\infty} p(e^{-u^2}) du < \infty$$

then $f$ is bounded and continuous on $T$ with probability 1

- A sufficient condition for the inequalities is that

$$E \left\{ (f(t) - f(s))^2 \right\}^{1/2} \leq \frac{K}{|\log|s-t||^{1/2+\alpha}} \text{ for all } |s - t| \leq \delta_0$$

for some constants $K, \alpha, \delta_0 > 0$

- There exists a constant $K'$ and a random $\eta > 0$ such that

$$\omega_{f,e}(\delta) \leq K' \int_0^p (-\log u)^{1/2} dp(u) \text{ for all } \delta \leq \eta$$
proof (for $N = 2$):

- $D(s, t) \leq p(|s - t|) \leq \epsilon \text{ if } |s - t| \leq p^{-1}(\epsilon)$, so an Euclidian ball of radius $p^{-1}(\epsilon)$ is contained in some $d$-ball of radius $\epsilon$.

- An Euclidean square with sidelength $\sqrt{2}p^{-1}(\epsilon)$ is contained in the Euclidean ball with radius $\epsilon$.

- Since $T$ is compact there is a "square" $L$ with sidelength $L$ such that $T \subset L$.

- $L$ can be covered by $\left(1 + \frac{L}{\sqrt{2}p^{-1}(\epsilon)}\right)^2$ squares with sidelength $\sqrt{2}p^{-1}(\epsilon)$, so $L$ can be covered by $\left(1 + \frac{L}{\sqrt{2}p^{-1}(\epsilon)}\right)^2$ $d$-balls of radius $\epsilon$. 

\[ \Rightarrow \int_0^{\delta} H_{\epsilon}^2 d\epsilon \leq \sqrt{2} \int_0^{\delta} \left\{ \log \left( 1 + \frac{L}{\sqrt{2} p^{-1}(\epsilon)} \right) \right\}^{1/2} d\epsilon \]

\[ \leq \sqrt{2} \int_0^{p^{-1}(\delta)} \left\{ \log \left( 1 + \frac{L}{\sqrt{2} u} \right) \right\}^{1/2} d\frac{1}{p} (u) \]

\[ \leq 2\sqrt{2} \int_0^{p^{-1}(\delta)} \left\{ -\log(u) \right\}^{1/2} d\frac{1}{p} (u), \]

where the last inequality comes from

\[ \log \left( 1 + \frac{L}{\sqrt{2} u} \right) = \log \left( \frac{1}{u} \right) + \log(u + \frac{L}{\sqrt{2}}) \]

plus \( \log \left( \frac{1}{u} \right) \geq \log \left( \frac{1}{p^{-1}(\delta)} \right) \to \infty \) and \( \log(u + \frac{L}{\sqrt{2}}) \to \log \left( \frac{L}{\sqrt{2}} \right) \)

so that

\[ \log(u + \frac{L}{\sqrt{2}}) \leq \log \left( \frac{1}{u} \right), \]

for sufficiently small \( \delta \). The result now follows from the general boundedness theorem.
Differentiability of Gaussian processes on $\mathbb{R}^N$ : (ATW p. 67-69)

Conditions for a.s. differentiability follows at once from the conditions for conditions for continuity. For example:

A $(1,1)$ field is a.s. differentiable at a point $t$ if

$$\frac{1}{hh'} \{ C(t + h, t + h') - C(t + h, t) - C(t, t + h') \\
+ C(t + h', t + h') \} \leq \frac{K}{|\log|h - h'||}$$
Pf: For $t$ fixed define a Gaussian field $g(h)$ by

$$g(h) = \begin{cases} f'(t) & \text{if } h = 0 \\ \frac{1}{h} \{f(t + h) - f(t)\} & \text{if } h \neq 0 \end{cases}$$

where $f'(t)$ is the mean square derivative of $f$. Then $f$ is a.s. differentiable at $t$ if $g$ is continuous at zero. But

$$E\{(g(h)g(h'))^2\} = \frac{1}{hh'}\{C(t + h, t + h') - C(t + h, t) - C(t, t + h') + C(t + h', t + h')\},$$

And hence continuity of $g$ follows from the assumption on the previous slide and the continuity theorem.