Nobel prize 2996 to John C. Mather George F. Smoot “for discovery of the blackbody form and anisotropy of the cosmic microwave background radiation”

Ocean waves
The oceans cover 72% of the earth’s surface. Essential for life on earth, and huge economic importance through fishing, transportation, oil and gas extraction

fMRI brain scan

PET brain scan
Slides for ATW Ch 2, pp. 78-82, Ch 4, pp. 194-203

Exercises: 2.8.23, 2.8.25, 6.7.5
Rice’s formula: (ATW p. 78-82)

\( f, g \) are \((1,1)\) random fields, \( T, B \) intervals in \( \mathbb{R}^1 \)

\[ N_u = N_u(f, g; T, B) \]

\( \triangleq \) number of points \( t \in T \) such that \( f(t) = u, g(t) \in B \)

Under (many) conditions given in ATW p. 79-81

\[ E(N_u) = \int_T \int_{\mathbb{R}^2} |y|1_B(v) p_t(u, y, v) dy dv dt, \]

where \( p_t(u, y, v) \) is the joint density of \( f_t, f'_t, g_t \).

ATW gives this formula for much more general \( T, B \), and \( f_t \) a \((T, N)\) field and \( g_t \) a \((T, K)\) field. It will come later.
Special cases:

• If $f_t, g_t$ are stationary $(1,1)$ random fields, and $T$ is the interval $[0, T]$, then $p_t = p$ doesn’t depend on $t$ and

$$E(N_u) = T \int_{R^2} |y| 1_B(v) p(u, y, v) dy dv,$$

(explain this!)
If \( g_t = f_t' \) and \( T = [0, T] \), \( B = (0, \infty) \) then \( N_u \) is the number of upcrossings of the level \( u \) and

\[
E(N_u) = \int_0^T \int_0^\infty yp(u, y)dy.
\]

where \( p_t(u, y) \) is the density of \( f_t, f_t' \).

(Doesn’t quite follow from previous formula, but does by a limiting argument.)
• If $f_t$ is stationary, the expected number of upcrossings is

$$E(N_u) = T \int_0^\infty y p(u, y) dy,$$

• If $f_t$ is a stationary standard Gaussian process, then $f_t$ and $f_t'$ are independent, $f_t'$ has mean 0 and variance $\lambda_2 = \int \lambda^2 \nu(d\lambda)$, so that

$$p(u, y) = \frac{1}{\sqrt{2\pi}} e^{-u^2/2} \frac{1}{\sqrt{2\pi \lambda_2}} e^{-y^2/(2\lambda_2)},$$

$$E(N_u) = \frac{T}{2\pi \sqrt{\lambda_2}} e^{-u^2/2} \int_0^\infty y e^{-y^2/(2\lambda_2)} dy = \frac{T \sqrt{\lambda_2}}{2\pi} e^{-u^2/2}$$

Two-point spectrum:

- $\lambda_2 = 200$
- $\lambda_2 = 1000$
For $M_T = \max_{0 \leq t \leq T} f_t$

$$P(M_T > u) \leq P(f_0 > u) + E(N_u(T))$$

$$\leq \frac{1}{\sqrt{2\pi u}} e^{-u^2/2} + \frac{T\sqrt{\lambda_2}}{2\pi} e^{-u^2/2}$$

$$\sim \frac{T\sqrt{\lambda_2}}{2\pi} e^{-u^2/2}, \quad \text{as } u \to \infty \text{ for fixed } T,$$

and this bound is asymptotically sharp. Lower bounds can be obtained via Rice formulas for factorial moments.

If $f_t$ is a stationary Gaussian process with mean $\mu$ and variance $\sigma^2$, then $h_t \equiv (f_t - \mu)/\sigma$ is a stationary standard Gaussian process with $\lambda_2(h_t) = \lambda_2(f_t)/\sigma^2$ and

$$E(N_u(f_t; T)) = E(N_{(u-\mu)/\sigma}(h_t; T))$$

$$= \frac{T\sqrt{\lambda_2}}{2\pi \sigma} e^{-\left(\frac{u-\mu}{\sigma}\right)^2/2}$$
“Proof” of Rice’s formula:

\( f, g \) are \((1,1)\) random fields, \( T, B \) intervals in \( \mathbb{R}^1 \)

\[ N_u = N_u(f, g; T, B) \]
\[ \triangleq \text{number of points } t \in T \text{ such that } f(t) = u, g(t) \in B \]

Rice’s formula:

\[ E(N_u) = \int_T \int_{\mathbb{R}^2} |y|1_B(v) p_t(u, y, v) dy dv dt, \]

with \( p_t(u, y, v) \) the joint density of \( f_t, f_t', g_t \):
• Define $\delta_\varepsilon$ by $\delta_\varepsilon(x) = 1/(2\varepsilon)$ if $|x| \leq \varepsilon$, and $\delta_\varepsilon(x) = 0$ otherwise. Then

$$N_u = \lim_{\varepsilon \to 0} \int \delta_\varepsilon(f_t - u)1_B(g_t)|f'_t|dt$$

\[ \int_{\text{region above}} \delta_\varepsilon(f_t - u)1_B(g_t)|f'_t|dt \approx \frac{1}{2\varepsilon} \frac{2\varepsilon}{f'_t} f'_t = 1 \]
\[ E(N_u) = E \left( \lim_{\epsilon \to 0} \int_T \delta_\epsilon(f_t - u)1_B(g_t)|f'_t|dt \right) \]

\[ = \lim_{\epsilon \to 0} \int_T E\{\delta_\epsilon(f_t - u)1_B(g_t)|f'_t|\}dt \]

\[ = \lim_{\epsilon \to 0} \int_T \int_{\mathbb{R}^3} \delta_\epsilon(x - u)1_B(v)|y|p_t(x, y, v)dx dy dv dt \]

\[ = \int_T \int_{\mathbb{R}^2} 1_B(v)|y| \left\{ \lim_{\epsilon \to 0} \int \delta_\epsilon(x - u)p_t(x, y, v)dx \right\} dy dv dt \]

\[ = \int_T \int_{\mathbb{R}^2} 1_B(v)|y|p_t(u, y, v)dy dv dt \]
Rice’s formula for Gaussian processes: (ATW p. 80-81)

• $f, g$ centered $(T, N)$ and Gaussian fields, $f, \nabla f, g$ a.s continuous with bounded variances, the joint distribution of $f_t, \nabla f_t, g_t$ is nondegenerate for all $t$

• The $N-1$ and $K-1$ dimensional boundaries of $T$ and $B$ have finite Lebesgue measure

• $\max_{i,j} |C^i_{f_j}(t; t) + C^i_{f_j}(s; s) - 2C^i_{f_j}(s; t)| \leq K |\log |t - s||^{-(1+\alpha)}$

• $\max_{i,j} |C^i_{g_j}(t; t) + C^i_{g_j}(s; s) - 2C^i_{g_j}(s; t)| \leq K |\log |t - s||^{-(1+\alpha)}$

Then, for $D = N^2 + K$,

$$E(N_u) = \int_T \int_{R^D} |\det \nabla y| 1_B(v) p_t(u, \nabla y, v) d(\nabla y) dv dt$$
Palm distributions (conditioning on events which have probability zero): (ATW p. 194-196)

• A point process is a non-negative integervalued random measure, \( N \), with the interpretation that the random variable \( N(B) \) is the (random) number of points in \( B \). Its intensity measure \( \mu \) is defined by

\[
\mu(B) = E(N(B))
\]

• A \((N, 1)\) field \( f_t \) and a point process \( N \) on \( \mathbb{R}^N \) are jointly stationary if

\[
\theta_\tau (f, N) \triangleq (\theta_\tau f, \theta_\tau N) = \mathbb{L}_\tau (f, N)
\]

for \( \theta_\tau f(t) \triangleq f(t + \tau) \) and \( \theta_\tau N(B) = N(B + \tau) \)
The Palm distribution $P_f, N$ is defined by its expectation,

$$E_{f,N}(F(f,N)) \triangleq \frac{E\left(\int_B F(\theta_{\tau}(f,N))N(d\tau)\right)}{E(N(B))}$$

for $B \in \mathbb{R}^N$ with $0 < \mu(B) < \infty$ and $F$ a realvalued function of sample paths of $f$ and positive integervalued measures on $\mathbb{R}^N$. It doesn’t depend on the choice of $B$.

Let $t = (t_1, ..., t_k) \in \mathbb{R}^N$. The the Palm distribution describes a new field $\tilde{f}_t$ which has finite dimensional distributions

$$F_t(x) \triangleq P(\tilde{f}(t_1) \leq x_1, ..., \tilde{f}(t_k) \leq x_k)$$

$$= \frac{E(\#\{s_j \in B; f(s_j + t_i) \leq x_i, i = 1, ... k\})}{E(\#\{s_j \in B\})},$$

where $\{s_j\}$ are the points of $N$. 
\[ B^N_\lambda \triangleq \{ x \in \mathbb{R}^N; \|x\| \leq \lambda \} \]

• If \( \{f_t\} \) is ergodic, then

\[
F_t(x) = \lim_{\lambda \to \infty} \frac{\#\{s_j \in B^N_\lambda; f(s_j + t_i) \leq x_i, i = 1, \ldots, k\}}{\#\{s_j \in B^N_\lambda\}},
\]

"\( F_t(x) \) is the ergodic conditional distribution of \( f_t \) given there is a point at 0",

• e.g., if the \( s_j \) are the times when \( f_t \) has a local maximum then \( F_t(x) \) describes the typical behaviour around a local maximum

• \( \tilde{f}_t \) is the Slepian model process
\[ P(N(B^N_\lambda) = 1; f(t_i) \leq x_i, i = 1, ... k) \]
\[ P(N(B^N_\lambda) = 1) \sim \frac{E(\#\{s_j \in B^N_\lambda; f(s_j + t_i) \leq x_i, i = 1, ... k\})}{E(\#\{s_j \in B^N_\lambda\})} = F_t(x), \text{ as } \lambda \to 0 \]

- Both these relations motivate writing
  \[ F_t(x) = P(f(t_i) \leq x_i, i = 1, ... k \mid N \text{ has point at 0}) \]

- This is not the same as the "usual" conditional probability
Slepian model processes for behavior around local maxima: (ATW p. 196-203)

\( f_u(t) \) the Slepian model process for a standard Gaussian \( f_t \) conditional on a local maximum above \( u \) at 0.

Long, complicated and not quite correct computations give that

\[
f_u(t) = uC(t) - W_u \beta^T(t) + g(t),
\]

for

- \( W_u \) an \( N(N+1)/2 \)-dimensional random vector which converges to a Gaussian vector \( W \) as \( u \to \infty \)
- \( \beta(t) \) is a nonrandom \( N(N+1)/2 \)-dimensional function
- \( g(t) \) is a non-homogeneous Gaussian field, independent of \( W_u \)
\[ f_u(t) = uC(t) - W_u\beta^T(t) + g(t) \]

implies that

\[ f_u(t) = u - \frac{u}{2} t\Lambda t^T + O_P(1), \text{ where} \]

- \( \Lambda = (\lambda_{i,j}) \) is the matrix of second order spectral moments
- \( O_P(1) \) is a random variable which depends on \( u \) but is such that for any \( \epsilon > 0 \), there is a \( K = K(\epsilon) \) such that \( P(|O_P(1)| > K) \leq \epsilon \) for all \( u \) (in weak convergence language this means that \( O_P(1) \) is tight).