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Discussion of a mixed binomial model inspired by the Merton model.

Derive the large-portfolio approximation formula in this framework.

Discussion of a mixed binomial model where the factor has discrete distribution.
Consider a credit portfolio model, not necessary homogeneous, with \( m \) obligors, and where each obligor can default up to fixed time point, say \( T \).

Assume that each obligor \( i \) (think of a firm named \( i \)) follows the Merton model, in the sense that obligor \( i \)-s assets \( V_{t,i} \) follows the dynamics

\[
dV_{t,i} = rV_{t,i}dt + \sigma_i V_{t,i}dB_{t,i} \tag{1}
\]

where \( B_{t,i} \) is a stochastic process defined as

\[
B_{t,i} = \sqrt{\rho} W_{t,0} + \sqrt{1-\rho} W_{t,i}. \tag{2}
\]

Here \( W_{t,0}, W_{t,i}, \ldots, W_{t,m} \) are independent standard Brownian motions

It is then possible to show that \( B_{t,i} \) is also a standard Brownian motion. Hence, due to (1) we then know that \( V_{t,i} \) is a GBM so by using Ito´s lemma, we get

\[
V_{t,i} = V_{0,i}e^{(r-\frac{1}{2}\sigma_i^2)t+\sigma_i B_{t,i}}
\]
The mixed binomial model inspired by the Merton Model

- The intuition behind (1) and (2) is that the asset for each obligor \( i \) is driven by a **common** process \( W_{t,0} \) representing the **economic environment**, and an **individual** process \( W_{t,i} \) unique for obligor \( i \), where \( i = 1, 2, \ldots, m \).

- This means that the asset for each obligor \( i \), depend both on a macroeconomic random process (common for all obligors) and an idiosyncratic random process (i.e. unique for each obligor). This will create a **dependence** among these obligors. To see this, recall that

\[
\text{Cov}(X_i, X_j) = \mathbb{E}[X_i X_j] - \mathbb{E}[X_i] \mathbb{E}[X_j]
\]

so due to (2)

\[
\text{Cov} (B_{t,i}, B_{t,j}) = \mathbb{E}[B_{t,i} B_{t,j}] - \mathbb{E}[B_{t,i}] \mathbb{E}[B_{t,j}]
\]

\[
= \mathbb{E} \left[ \left( \sqrt{\rho} W_{t,0} + \sqrt{1-\rho} W_{t,i} \right) \left( \sqrt{\rho} W_{t,0} + \sqrt{1-\rho} W_{t,j} \right) \right]
\]

\[
= \mathbb{E} [\rho W_{t,0}^2] + \sqrt{\rho} \sqrt{1-\rho} \left( \mathbb{E} [W_{t,0} W_{t,i}] + \mathbb{E} [W_{t,0} W_{t,j}] \right)
\]

\[
+ (1 - \rho) \mathbb{E} [W_{t,j} W_{t,i}]
\]

\[
= \rho \mathbb{E} [W_{t,0}^2] = \rho t
\]

where the third equality is due to \( \mathbb{E} [W_{t,j} W_{t,i}] = 0 \) when \( i \neq j \).
The mixed binomial model inspired by the Merton Model

- Hence, \( \text{Cov} (B_{t,i}, B_{t,j}) = \rho t \) which implies that there is a dependence of the processes that drives the asset values \( V_{t,i} \). To be more specific,

\[
\text{Corr} (B_{t,i}, B_{t,j}) = \frac{\text{Cov} (B_{t,i}, B_{t,j})}{\sqrt{\text{Var} (B_{t,i}) \text{Var} (B_{t,i})}} = \frac{\rho t}{\sqrt{t} \sqrt{t}} = \rho
\]  

so \( \text{Corr} (B_{t,i}, B_{t,j}) = \rho \) which is the mutual dependence among the obligors created by the macroeconomic latent variable \( W_{t,0} \)

- Note that if \( \rho = 0 \), we have \( \text{Corr} (B_{t,i}, B_{t,j}) = 0 \) which makes the asset values \( V_{t,1}, V_{t,2}, \ldots, V_{t,m} \) independent (so the obligors are independent).

- Next, let \( D_i \) be the debt level for each obligor \( i \) and recall from the Merton model that obligor \( i \) defaults if \( V_{T,i} \leq D_i \), that is if

\[
V_{0,i} e^{(r - \frac{1}{2} \sigma_i^2) T + \sigma_i B_{T,i}} < D_i
\]  

which, by using the definition of \( B_{t,i} \) is equivalent with the event

\[
\ln V_{0,i} - \ln D_i + (r - \frac{1}{2} \sigma_i^2) T + \sigma_i \left( \sqrt{\rho} W_{T,0} + \sqrt{1 - \rho} W_{T,i} \right) < 0
\]
The mixed binomial model inspired by the Merton Model

Next, recall that for each $i$, $W_{i,T} \sim N(0, T)$, i.e $W_{i,T}$ is normally distributed with zero mean and variance $T$. Hence, if $Y_i \sim N(0, 1)$, $W_{i,T}$ has the same distribution as $\sqrt{T}Y_i$ for $i = 0, 1, \ldots, m$ where $Y_0, Y_1, \ldots, Y_M$ also are independent. Furthermore, define $Z$ as $Y_0$, i.e $Z = Y_0$. This in (5) yields

$$\ln V_{0,i} - \ln D_i + (r - \frac{1}{2}\sigma_i^2)T + \sigma_i \left(\sqrt{\rho}\sqrt{T}Z + \sqrt{1 - \rho}\sqrt{T}Y_i\right) < 0 \quad (6)$$

and dividing with $\sigma_i\sqrt{T}$ renders

$$\frac{\ln V_{0,i} - \ln D_i + (r - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}} + \sqrt{\rho}Z + \sqrt{1 - \rho}Y_i < 0. \quad (7)$$

We can rewrite the inequality (7) as

$$Y_i < -\left(\frac{C_i + \sqrt{\rho}Z}{\sqrt{1 - \rho}}\right) \quad (8)$$

where $C_i$ is a constant given by

$$C_i = \frac{\ln (V_{0,i}/D_i) + (r - \frac{1}{2}\sigma_i^2)T}{\sigma_i\sqrt{T}} \quad (9)$$
The mixed binomial model inspired by the Merton Model

- Hence, from the previous slides we conclude that

\[ V_{T,i} < D_i \quad \text{is equivalent with} \quad Y_i < \frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}} \tag{10} \]

where \( C_i \) is a constant given by (9).

- Next define \( X_i \) as

\[ X_i = \begin{cases} 1 & \text{if } V_{T,i} < D_i \\ 0 & \text{if } V_{T,i} > D_i \end{cases} \tag{11} \]

- Then (10) implies that

\[ \mathbb{P} [ X_i = 1 | Z ] = \mathbb{P} [ V_{T,i} < D_i | Z ] = \mathbb{P} \left[ Y_i < \frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}} \right] \bigg| Z \]

\[ = N \left( \frac{-(C_i + \sqrt{\rho}Z)}{\sqrt{1-\rho}} \right) \tag{12} \]

where \( N(x) \) is the distribution function of a standard normal distribution.

- The last equality in (12) follows from the fact that \( Y_i \sim N(0,1) \) and that \( Y_i \) is independent of \( Z \) in (10).
The mixed binomial model inspired by the Merton Model

- Next, assume that all obligors in the model are identical, so that $V_{0,i} = V_0$, $D_i = D$ and thus $C_i = C$ for $i = 1, 2, \ldots, m$.

- Then we have a homogeneous static credit portfolio, where we consider the time period up to $T$.

- Furthermore, Equation (12) implies that

$$
P [ X_i = 1 | Z ] = N \left( \frac{-(C + \sqrt{\rho}Z)}{\sqrt{1 - \rho}} \right)$$

where $C$ is a constant given by (9) with $V_{0,i} = V_0$, $D_i = D$, $\sigma_i = \sigma$ and thus $C_i = C$ for all obligors $i$.

- Let $Z$ be the "economic background variable" in our homogeneous portfolio and define $p(Z)$ as

$$
p(Z) = N \left( \frac{-(C + \sqrt{\rho}Z)}{\sqrt{1 - \rho}} \right)$$

where $N(x)$ is the distribution function of a standard normal distribution.
The mixed binomial model inspired by the Merton Model

Since, \( p(Z) \in [0, 1] \), we would like to use \( p(Z) \) in a mixed binomial model.

To be more specific, let \( X_1, X_2, \ldots, X_m \) be identically distributed random variables such that \( X_i = 1 \) if obligor \( i \) defaults before time \( T \) and \( X_i = 0 \) otherwise.

Furthermore, conditional on \( Z \), the random variables \( X_1, X_2, \ldots, X_m \) are independent and each \( X_i \) have default probability \( p(Z) \), that is

\[
P[X_i = 1 | Z] = p(Z) = N \left( \frac{- \left( C + \sqrt{\rho Z} \right)}{\sqrt{1 - \rho}} \right).
\] (15)

We call this the mixed binomial model inspired by the Merton model or sometimes simply a mixed binomial Merton model.
The mixed binomial Merton model

Let \( \tilde{L}_m = \sum_{i=1}^{m} \ell X_i \) denote the total credit loss in our portfolio at time \( T \).

We now want to study \( \mathbb{P}\left[ \tilde{L}_m \leq x \right] \) in our portfolio where \( X_i \), conditional on \( Z \), have default probabilities \( p(Z) \) given by (15).

Since the portfolio is homogeneous, all losses are the same and constant given by, say \( \ell \), so

\[
\tilde{L}_m = \sum_{i=1}^{m} \ell X_i = \ell \sum_{i=1}^{m} X_i = \ell N_m \quad \text{where} \quad N_m = \sum_{i=1}^{m} X_i
\]

thus, \( N_m \) is the number of defaults in the portfolio up to time \( T \). Hence, since \( \mathbb{P}\left[ \tilde{L}_m = k\ell \right] = \mathbb{P}\left[ N_m = k \right] \), it is enough to study \( \mathbb{P}\left[ N_m \leq n \right] \) where \( n = 0, 1, 2 \ldots, m \) instead of \( \mathbb{P}\left[ \tilde{L}_m \leq x \right] \).

Next, note that \( \mathbb{P}\left[ N_m \leq n \right] = \sum_{k=0}^{n} \mathbb{P}\left[ N_m = k \right] \) and

\[
\mathbb{P}\left[ N_m = k \right] = \int_{-\infty}^{\infty} \binom{m}{k} p(z)^{k}(1 - p(z))^{m-k} f_Z(z)dz \quad (16)
\]

where \( f_Z(z) \) is the density of \( Z \).
The mixed binomial Merton model, cont.

In our case $Z$ is a standard normal random variable so

$$\mathbb{P}[N_m = k] = \int_{-\infty}^{\infty} \left( \begin{pmatrix} m \\ k \end{pmatrix} p(u)^k (1 - p(u))^{m-k} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \right) du.$$  \quad (17)

Furthermore, $p(u)$ is given by

$$p(u) = N\left( \frac{- (C + \sqrt{\rho} u)}{\sqrt{1-\rho}} \right)$$

where $N(x)$ is the distribution function of a standard normal distribution.

Hence, $\mathbb{P}[N_m \leq n]$ is given by

$$\mathbb{P}[N_m \leq n] = \sum_{k=0}^{n} \left( \begin{pmatrix} m \\ k \end{pmatrix} \int_{-\infty}^{\infty} N\left( \frac{- (C + \sqrt{\rho} u)}{\sqrt{1-\rho}} \right)^k \right.$$

$$\left. \cdot \left( 1 - N\left( \frac{- (C + \sqrt{\rho} u)}{\sqrt{1-\rho}} \right) \right)^{m-k} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \right) du.$$  \quad (18)
So if we know \( C \) (later we show how to find \( C \)) we can therefore find \( \mathbb{P}[N_m \leq n] \) by numerically evaluate the expression in the RHS in (18).

However, there is another way to find a very convenient approximation of \( \mathbb{P}[N_m \leq n] \).

To see this, recall from the last lecture that in any mixed binomial distribution we have that

\[
\mathbb{P} \left[ \frac{N_m}{m} \leq \theta \right] \rightarrow F(\theta) \quad \text{as } m \rightarrow \infty
\]  

where \( F(x) \) is the distribution function of \( p(Z) \), i.e. \( F(x) = \mathbb{P}[p(Z) \leq x] \)

But for any \( x \) we then have

\[
\mathbb{P}[N_m \leq x] = \mathbb{P} \left[ \frac{N_m}{m} \leq \frac{x}{m} \right] \approx F \left( \frac{x}{m} \right) \quad \text{if } m \text{ is ”large”}.
\]

Hence, we can approximate \( \mathbb{P}[N_m \leq n] \) with \( F \left( \frac{n}{m} \right) \) instead of numerically compute the quite involved expression in the RHS in (18).
We therefore next want to find an explicit expression of $F(\theta)$ where $F(\theta) = \mathbb{P}[p(Z) \leq \theta]$. From (15) we know that $p(Z) = \mathcal{N}\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right)$ where $Z$ is a standard normal random variable, i.e. $Z \sim \mathcal{N}(0,1)$.

Hence, $F(\theta) = \mathbb{P}[p(Z) \leq \theta] = \mathbb{P}\left[\mathcal{N}\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \leq \theta\right]$ so

$$
\mathbb{P}\left[\mathcal{N}\left(\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}}\right) \leq \theta\right] = \mathbb{P}\left[\frac{-\left(C + \sqrt{\rho}Z\right)}{\sqrt{1-\rho}} \leq \mathcal{N}^{-1}(\theta)\right]
= \mathbb{P}\left[-Z \leq \frac{1}{\sqrt{\rho}} \left(\sqrt{1-\rho}\mathcal{N}^{-1}(\theta) + C\right)\right]
= \mathcal{N}\left(\frac{1}{\sqrt{\rho}} \left(\sqrt{1-\rho}\mathcal{N}^{-1}(\theta) + C\right)\right)
$$

where the last equality is due to

$\mathbb{P}[-Z \leq x] = \mathbb{P}[Z \geq -x] = 1 - \mathbb{P}[Z \leq -x]$ and $1 - \mathcal{N}(-x) = \mathcal{N}(x)$ for any $x$, due to the symmetry of a standard normal random variable.
The mixed binomial Merton model and LPA, cont.

- Hence, \( F(\theta) = N\left( \frac{1}{\sqrt{\rho}} \left( \sqrt{1 - \rho} N^{-1}(\theta) + C \right) \right) \) so what is left is to find \( C \).

- Since our model is inspired by the Merton model, we have that

\[
X_i = \begin{cases} 
1 & \text{if } V_T < D \\
0 & \text{if } V_T > D
\end{cases} \quad (20)
\]

so \( \mathbb{P}[X_i = 1] = \mathbb{P}[V_T < D] \). However, from (7) and (10) we conclude that

\[
V_T < D \iff \sqrt{\rho} Z + \sqrt{1 - \rho} Y_i \leq -C \quad (21)
\]

where \( C \) is given by Equation (9) in the homogeneous case where \( V_{0,i} = V_0, D_i = D, \sigma_i = \sigma \) and consequently \( C_i = C \) for \( i = 1, 2, \ldots, m \).

Furthermore, since \( Z \) and \( Y_i \) are standard normals then \( \sqrt{\rho} Z + \sqrt{1 - \rho} Y_i \) will also be standard normal. Hence, \( \mathbb{P}[\sqrt{\rho} Z + \sqrt{1 - \rho} Y_i \leq -C] = N(-C) \) and this observation together with (21) implies that

\[
\mathbb{P}[X_i = 1] = \mathbb{P}[V_T < D] = N(-C). \quad (22)
\]
Recall that $\bar{p} = \mathbb{E}[p(Z)] = \int_0^1 p(z)f_Z(z)dz$ so $\bar{p} = \mathbb{P}[X_i = 1]$ since $\mathbb{P}[X_i = 1 | Z] = p(Z)$ and thus

$$\mathbb{P}[X_i = 1] = \mathbb{E}[\mathbb{P}[X_i = 1 | Z]] = \mathbb{E}[p(Z)] = \bar{p}$$

Hence, from (22) we have $\bar{p} = N(-C)$ so

$$C = -N^{-1}(\bar{p}) \quad (23)$$

which means that we can ignore $C$ (and thus also ignore $V_0, D, \sigma$ and $r$, see (9)) and instead directly work with the default probability $\bar{p} = \mathbb{P}[X_i = 1]$. Hence, we estimate $\bar{p}$ to 5%, say, which then implicitly defines the quantizes $V_0, D, \sigma$ and $r$ via (9) and (23).

Finally, going back to $F(\theta) = N\left(\frac{1}{\sqrt{\rho}} \left(\sqrt{1-\rho}N^{-1}(\theta) + C\right)\right)$ and using (23) we conclude that

$$F(\theta) = N\left(\frac{1}{\sqrt{\rho}} \left(\sqrt{1-\rho}N^{-1}(\theta) - N^{-1}(\bar{p})\right)\right) \quad (24)$$

where $F(\theta) = \mathbb{P}[p(Z) \leq \theta]$. 
Hence, if $m$ is large enough, we can in a mixed binomial model inspired by the Merton model, do the following approximation of the portfolio loss probability $\mathbb{P}[N_m \leq n] = \mathbb{P}\left[\frac{N_m}{m} \leq \frac{n}{m}\right] \approx F\left(\frac{n}{m}\right)$, that is

$$
\mathbb{P}[N_m \leq n] \approx N\left(\frac{1}{\sqrt{\rho}}\left(\sqrt{1-\rho}N^{-1}\left(\frac{n}{m}\right) - N^{-1}(\bar{p})\right)\right).
$$

(25)

where $\bar{p} = \mathbb{P}[X_i = 1]$ is the individual default probability for each obligor.

The approximation (24) or equivalently (25), is sometimes denoted the LPA in a static Merton framework, and was first introduced by Vasicek 1991, at KMV, in the paper "Limiting loan loss probability distribution".

The LPA in a Merton framework and its offsprings (i.e. variants) is today widely used in the industry (Moody's-KMV, CreditMetrics etc. etc.) for risk management of large credit/loan portfolios, especially for computing regulatory capital in Basel II and Basel III (Basel III is to be implemented before end of 2013).
Recall from (3), that $\rho$ was the correlation parameter describing the dependence between the Brownian motions $B_{t,i}$ that drives each obligor $i$’s asset price, i.e. $\text{Cov}(B_{t,i}, B_{t,j}) = \rho t$ so that $\text{Corr}(B_{t,i}, B_{t,j}) = \rho$.

Since $X_i = 1\{V_{T,i} \leq D\}$ we know that $X_i$ and $X_j$ are dependent because $\text{Cov}(B_{t,i}, B_{t,j}) = \rho t$ where $\rho \neq 0$. Furthermore, if $\rho \neq 0$ it generally holds that $\text{Cov}(X_i, X_j) \neq 0$ since

$$\text{Cov}(X_i, X_j) = \mathbb{E}\left[1\{V_{T,i} \leq D\}1\{V_{T,j} \leq D\}\right] - \mathbb{E}\left[1\{V_{T,i} \leq D\}\right] \mathbb{E}\left[1\{V_{T,j} \leq D\}\right]$$

$$= \mathbb{P}[V_{T,i} \leq D, V_{T,j} \leq D] - \mathbb{P}[V_{T,i} \leq D] \mathbb{P}[V_{T,j} \leq D]$$

$$= \mathbb{P}[V_{T,i} \leq D, V_{T,j} \leq D] - \bar{p}^2$$

and $\mathbb{P}[V_{T,i} \leq D, V_{T,j} \leq D] \neq \bar{p}^2$ since $\text{Cov}(B_{t,i}, B_{t,j}) = \rho t$ with $\rho \neq 0$ implies (see also Equation (21) and (22))

$$\mathbb{P}[V_{T,i} \leq D, V_{T,j} \leq D] = \mathbb{P}\left[B_{T,i} < -\sqrt{T}C, B_{T,j} < -\sqrt{T}C\right]$$

$$\neq \mathbb{P}\left[B_{T,i} < -\sqrt{T}C\right] \mathbb{P}\left[B_{T,j} < -\sqrt{T}C\right] = \bar{p}^2.$$

Hence, $\text{Cov}(X_i, X_j) \neq 0$ when $\rho \neq 0$. 
Next, assume that $\rho = 0$ so that $\text{Cov}(B_t,i, B_t,j) = 0$. Furthermore, by (2) we have that $B_t,i = W_t,i$ when $\rho = 0$ since

$$B_t,i = \sqrt{0}W_{t,0} + \sqrt{1-0}W_{t,i} = W_{t,i}$$  \hspace{1cm} (27)$$

where $W_{t,0}, W_{t,i}, \ldots, W_{t,m}$ are independent standard Brownian motions.

Equation (27) and the independence among $W_{t,0}, W_{t,i}, \ldots, W_{t,m}$ then imply

$$\mathbb{P}[V_{T,i} \leq D, V_{T,j} \leq D] = \mathbb{P}[B_{T,i} < -\sqrt{T}C, B_{T,j} < -\sqrt{T}C]$$

$$= \mathbb{P}[W_{T,i} < -\sqrt{T}C, W_{T,j} < -\sqrt{T}C]$$

$$= \mathbb{P}[W_{T,i} < -\sqrt{T}C] \mathbb{P}[W_{T,j} < -\sqrt{T}C]$$

$$= \mathbb{P}[V_{T,i} \leq D] \mathbb{P}[V_{T,j} \leq D] = \bar{p}^2$$

and plugging this into (26) yields that $\text{Cov}(X_i, X_j) = 0$. 

From the above studies we conclude that

\[ \text{Cov}(X_i, X_j) = 0 \quad \text{if} \quad \rho = 0 \quad (28) \]

and

\[ \text{Cov}(X_i, X_j) \neq 0 \quad \text{if} \quad \rho \neq 0. \quad (29) \]

We therefore conclude that \( \rho \) is a measure of default dependence among the zero-one variables \( X_1, X_2, \ldots, X_m \) in the mixed binomial Merton model.
Large portfolio approximation for different correlations. Individual default probability, p=5%
Given the limiting distribution \( F(\theta) \)

\[
F(\theta) = N \left( \frac{1}{\sqrt{\rho}} \left( \sqrt{1 - \rho} N^{-1}(\theta) - N^{-1}(\bar{\rho}) \right) \right)
\]  

we can also find the density \( f_{LPA}(\theta) \) of \( F(\theta) \), that is \( f_{LPA}(\theta) = \frac{dF(\theta)}{d\theta} \).

It is possible to show that

\[
f_{LPA}(\theta) = \sqrt{\frac{1 - \rho}{\rho}} \exp \left( \frac{1}{2} (N^{-1}(\theta))^2 - \frac{1}{2\rho} \left( N^{-1}(\bar{\rho}) - \sqrt{1 - \rho} N^{-1}(\theta) \right)^2 \right)
\]

This density is just an approximation, and fails for small number of the loss fraction.
Density of large portfolio approximation for different correlations. Individual default probability, $p=1\%$.
Density of large portfolio approximation for different correlations. Individual default probability, $p=5\%$

- $\rho=0.1\%$
- $\rho=10\%$
- $\rho=20\%$
- $\rho=30\%$
- $\rho=95\%$
Density of large portfolio approximation for different correlations. Individual default probability, p=5%
VaR in the mixed binomial Merton model

Consider a static credit portfolio with $m$ obligors in a mixed binomial model inspired by the Merton framework where

- the individual one-year default probability is $\bar{p}$
- the individual loss is $\ell$
- the default correlation is $\rho$

By assuming the LPA setting we can now state the following result for the one-year credit Value-at-Risk $\text{VaR}_\alpha(L)$ with confidence level $1 - \alpha$.

**VaR in the mixed binomial Merton model using the LPA setting**

With notation and assumptions as above, the one-year $\text{VaR}_\alpha(L)$ is given by

$$\text{VaR}_\alpha(L) = \ell \cdot m \cdot N \left( \frac{\sqrt{\rho}N^{-1}(\alpha) + N^{-1}(\bar{p})}{\sqrt{1 - \rho}} \right).$$

(32)

Useful exercise: Derive the formula (32).

Note that variants of the formula (32) is extensively used for computing regulatory capital in **Basel II** and **Basel III**
In all our previous examples the random variable $Z$ (modelling the common background factor) have been continuous and the mixing function $p(x) \in [0, 1]$ were chosen to be continuous too.

However, we can also model $Z$ to be a discrete random variable as follows. Let $Z$ be a random variable such that

$$Z \in \{z_1, z_2, \ldots, z_N\} \text{ where } \mathbb{P}[Z = z_n] = q_n \text{ and } \sum_{n=1}^{N} q_n = 1. \quad (33)$$

where it obviously must hold that $q_n \in [0, 1]$ for each $n = 1, 2, \ldots, N$.

Furthermore, we model the mixing function $p(x) \in [0, 1]$ as

$$p(Z) \in \{p_1, p_2, \ldots, p_N\} \text{ where } p(z_n) = p_n \in [0, 1] \text{ for each } n \quad (34)$$

where we without loss of generality may assume that $p_1 < p_2 < \ldots < p_N$. 
Furthermore, note that

$$P[Z = z_n] = P[p(Z) = p_n] = q_n \quad \text{for} \quad n = 1, 2, \ldots, N.$$  \hfill \text{(35)}

Recall that $\bar{p} = P[X_i = 1] = E[p(Z)]$ so in the model described by (33) and (35) we have

$$\bar{p} = \sum_{n=1}^{N} p_n q_n.$$  \hfill \text{(36)}

Given (33) and (35) the distribution function $F(x) = P[p(Z) \leq x]$ is then for any $x \in [0, 1]$ expressed as

$$F(x) = \sum_{n: p_n \leq x} q_n.$$  \hfill \text{(37)}

Due to the LPA approach we then know that for any $x \in [0, 1]$ it holds that

$$\mathbb{P} \left[ \frac{N}{m} \leq x \right] \rightarrow \sum_{n: p_n \leq x} q_n \quad \text{as} \quad m \rightarrow \infty.$$  \hfill \text{(38)}
Thank you for your attention!