PRICING k^{th}-TO-DEFAULT SWAPS UNDER DEFAULT CONTAGION: THE MATRIX-ANALYTIC APPROACH

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ABSTRACT. We study a model for default contagion in intensity-based credit risk and its consequences for pricing portfolio credit derivatives. The model is specified through default intensities which are assumed to be constant between defaults, but which can jump at the times of defaults. The model is translated into a Markov jump process which represents the default status in the credit portfolio. This makes it possible to use matrix-analytic methods to derive computationally tractable closed-form expressions for single-name credit default swap spreads and k^{th}-to-default swap spreads. We "semi-calibrate" the model for portfolios (of up to 15 obligors) against market CDS spreads and compute the corresponding k^{th}-to-default spreads. In a numerical study based on a synthetic portfolio of 15 telecom bonds we study a number of questions: how spreads depend on the amount of default interaction; how the values of the underlying market CDS-prices used for calibration influence k^{th}-th-to default spreads; how a portfolio with inhomogeneous recovery rates compares with a portfolio which satisfies the standard assumption of identical recovery rates; and, finally, how well k^{th}-th-to default spreads in a nonsymmetric portfolio can be approximated by spreads in a symmetric portfolio.

1. INTRODUCTION

In this paper we study dynamic dependence modelling in intensity-based credit risk. We focus on the concept of default contagion and its consequences for pricing k^{th}-to-default swaps. The paper is an extension of Chapter 6 of the licentiate thesis [28].

Default dependency has attracted a much interest during the last few years. A main reason is the growing financial market of products whose payoffs are contingent on the default behavior of a whole credit portfolio consisting of, for example, corporate bonds or single-name credit default swaps (CDS-s). Example of such instruments that have gained popularity are k^{th}-to-default swaps and (synthetic) CDO-s. These products are designed to manage and trade the risk of default dependencies. We refer to [6], [8], [16], [18], [28], [41], [45] or [54] for more detailed descriptions of the instruments. Models which capture

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default dependencies in a realistic way is at the core of pricing, hedging and managing such instruments.

As the name suggest, default contagion, treats the phenomenon of how defaults can "propagate" like a disease in a financial market (see e.g. [13]). There may be many reasons for this kind of domino effect. For a very interesting discussion of sources of default contagion, see pp. 1765-1768 in [39].

It is, of course, important for credit portfolio managers to have a quantitative grasp of default contagion. This paper describes a new numerical approach to handle default interactions. The underlying idea is the same as in [5], [7], [19], [21], which is to model default contagion via a Markov jump process that represents the joint default status in the credit portfolio. The main difference is that [19], [21] use time-varying parameters in their practical examples and solve the Chapman-Kolmogorov equation by using numerical methods for ODE-systems. In [7], the authors implement results from [5] by using Monte Carlo simulations to calibrate and price credit derivatives.

In this article, we focus on intensities which are constant between defaults, but which may jump at the default times. This makes it possible to obtain compact and computationally tractable closed-form expressions for many quantities of interest, including k^{th}-to-default spreads. For this we use the so-called matrix-analytic approach, see e.g. [1]. From a portfolio credit risk modeling point of view, it also turns out that this method posses useful intuitive and practical features, both analytical and computationally. We believe that these features in many senses are at least as attractive as the copula approach which is current a standard for practitioners. (For a critical study of the copula approach in financial mathematics, see [46]).

The number of articles on dynamic models for portfolio credit risk has grown exponentially during the last years. The subtopic of default contagion in intensity based models is not an exception and has been studied in for example [3], [6], [9], [10], [12], [14], [16], [24], [25], [32] [34], [39], [40], [41], [45], [51], [52], [53], [57], [58].

The paper [3] considers a chain where states record if obligors have defaulted or not, and implemented this model for a basket of two bonds. The intensities in the model were calibrated to market data using linear regression. In [14] the authors model default contagion in symmetric portfolio by using a piecewise-deterministic Markov process and find the default distribution. The book [41], pp. 126-128, studies a Markov chain model for two firms that undergo default contagion. Further, [58] treats default contagion using the total hazard construction of [50], [55], as first suggested in [15]. This method allows for general time dependent and stochastic intensities and that the intensities are functionals of the default times. The latter seems difficult to handle in a Markov jump process framework. Given the parameters of the model, the total hazard method gives a way to simulate default events. The total hazard construction seems rather complicated to implement even in simple cases such as piece-wise deterministic intensities considered in this paper.

The paper [39] assumes a so called *primary-secondary structure*, were obligors are divided into two groups called *primary obligors* and *secondary obligors*. The idea is that the default-intensities of primary obligors only depend on macroeconomic market variables while the default intensity for secondary obligors can depend on both the macroeconomic variables

and on the default status of the primary firms, but not on the default status of the other secondary firms. Assuming this structure, [39] derives closed formulas for defaultable bonds, default swaps, etc, also for stochastic intensities. In the article [10] the authors propose a method where one can value defaultable claims without having to use the so called "no-jump condition". This technique is then applied to find survival distributions for a portfolio of two obligors that undergo default contagion. In [57] the author studies counterparty risk in CDS valuation by using a four state Markov process that includes contagion effects. [57] considers time dependent intensities and then uses perturbation techniques to approximately solve the Chapman-Kolmogorov equation. The framework in [57] is similar to [12], where the author treats the same problem in a setup where the intensities are constant.

The rest of this paper is organized as follows. In Section 2 we give a short introduction to pricing of credit k^{th}-to-default swaps. Section 3 contains the formal definition of default contagion used in this paper, given in terms of default intensities. It is then used to construct such default times as hitting times of a Markov jump process.

In Section 5 we use the results of Section 4, for numerical investigation of a number of properties of k^{th}-to-default spreads. Specifically, we semi-calibrate portfolios with up to 15 obligors against market CDS spreads and then compute the corresponding k^{th}-to-default spreads. The results are used to illustrate how k^{th}-to-default spreads depend on the strength of default interaction, on the underlying market CDS-prices used for calibration, and on the amount of inhomogeneity in the portfolios.

Section 6 discusses numerical issues and some possible extensions, and the final section, Section 7 summarizes and discusses the results.

2. Pricing k^{th}-to-default swap spreads

In this section and in the sequel all computations are assumed to be made under a risk-neutral martingale measure $\mathbb{P}$. Typically such a $\mathbb{P}$ exists if we rule out arbitrage opportunities.

Consider a k^{th}-to-default swap with maturity T where the reference entity is a basket of m bonds, or obligors, with default times $\tau_1, \tau_2, \ldots, \tau_m$ and recovery rates $\phi_1, \phi_2, \ldots, \phi_m$. Further, let $T_1 \leq \ldots \leq T_k$ be the ordering of $\tau_1, \tau_2, \ldots, \tau_m$. For k^{th}-to-default swaps, it is standard to let the notional amount on each bond in the portfolio have the same value, say, N, so we assume this is the case.

The protection buyer $\boldsymbol{A}$ pays a periodic fee $R_k N \Delta_n$ to the protection seller $\boldsymbol{B}$, up to the time of the k-th default T_k, or to the time T, whichever comes first. The payments are made at times $0 < t_1 < t_2 < \ldots < t_n = T$. Further let $\Delta_n = t_n - t_{n-1}$ denote the times between payments (measured in fractions of a year). Furthermore, if default happens for some $T_k \in [t_n, t_{n+1}]$, $\boldsymbol{A}$ will also pay $\boldsymbol{B}$ the accrued default premium up to T_k. On the other hand, if $T_k < T$, $\boldsymbol{B}$ pays $\boldsymbol{A}$ the loss occurred at T_k, that is $N(1-\phi_i)$ if it was obligor i which defaulted at time T_k.

The constant R_k, often called k-th-to default spread, is expressed in bp per annum and determined so that the expected discounted cash-flows between $\boldsymbol{A}$ and $\boldsymbol{B}$ coincide at $t = 0$.

This implies that R_k is given by

$$R_k = \frac{\sum_{i=1}^m \mathbb{E}\left[1_{\{T_k \le T\}} D(T_k)(1-\phi_i) 1_{\{T_k=\tau_i\}}\right]}{\sum_{i=1}^n \mathbb{E}\left[D(t_i)\Delta_i 1_{\{T_k > t_i\}} + D(T_k)\left(T_k - t_{i-1}\right) 1_{\{t_{i-1}<\tau\le t_i\}}\right]}, \tag{2.1}$$

where $D(T) = \exp\left(-\int_0^T r_s ds\right)$, and r_t is the so called short term risk-free interest rate at time t. Note that N does not enter into this expression. Thus, we will from now on without loss of generality let $N = 1$ when discussing spreads on credit swaps.

In the credit derivative literature today, it is standard to assume that the default times and the short time riskfree interest rate are mutually independent, and that the recovery rates are deterministic. Under these assumptions Equation (2.1) can be simplified to

$$R_k = \frac{\sum_{i=1}^m (1-\phi_i) \int_0^T B(s) dF_{k,i}(s)}{\sum_{i=1}^n \left(B(t_i)\Delta_i(1-F_k(t_i)) + \int_{t_{i-1}}^{t_i} B(s)\left(s - t_{n-1}\right) dF_k(s)\right)} \tag{2.2}$$

where $B(t) = \mathbb{E}\left[D(t)\right]$ is the expected value of the discount factor, and $F_k\left(t\right) = \mathbb{P}\left[T_k \le t\right]$ and $F_{k,i}\left(t\right) = \mathbb{P}\left[T_k \le t, T_k = \tau_i\right]$ are the distribution functions of the ordered default times, and the probability that the k-th default is by obligor i and that it occurs before t, respectively. It may be noted that in the special case when all recovery rates are the same, say $\phi_i = \phi$ the denominator in (2.2) can be simplified to $(1-\phi)\int_0^T B(s) dF_k(s)$, and hence the $F_{k,i}$ are not needed in this case.

The latter of course in particular holds if there is only one bond (or obligor) so that $m = 1$. This case gives the most liquidly traded instrument, called a single-name Credit Default Swap (CDS), which has special importance in this paper as our main calibration tool.

3. Intensity based models reinterpreted as Markov jump processes

In this section we define the intensity-based model for default contagion which is used throughout the paper. The model is then reinterpreted in terms of a Markov jump process. This interpretation makes it possible to use a matrix-analytic approach to derive computationally tractable closed-form expressions for single-name CDS spreads and k-th-to default spreads. These matrix analytic methods has largely been developed for queueing theory and reliability applications, and in these context are often called phase-type distributions, or multivariate phase-type distributions in the case of several components (see e.g. [2]).

With $\tau_1, \tau_2 \ldots, \tau_m$ default times as above, define the point process $N_{t,i} = 1_{\{\tau_i \le t\}}$ and introduce the filtrations

$$\mathcal{F}_{t,i} = \sigma\left(N_s^i; s \le t\right), \quad \mathcal{F}_t = \bigvee_{i=1}^m \mathcal{F}_t^i.$$

Let $\lambda_{t,i}$ be the $\mathcal{F}_t$-intensity of the point processes $N_{t,i}$. Below, we will for convenience often omit the filtration and just write intensity or "default intensity". With a further extension of language we will sometimes also write that the default times $\{\tau_i\}$ have intensities $\{\lambda_{t,i}\}$.

The model studied in this paper is specified by requiring that the default intensities have the following form,

$$\lambda_{t,i} = a_i + \sum_{j \neq i} b_{i,j} 1_{\{\tau_j \leq t\}}, \qquad t \leq \tau_i, \tag{3.1}$$

and $\lambda_{t,i} = 0$ for $t > \tau_i$. Further, $a_i \geq 0$ and $b_{i,j}$ are constants such that $\lambda_{t,i}$ is non-negative.

The financial interpretation of (3.1) is that the default intensities are constant, except at the times when defaults occur: then the default intensity for obligor i jumps by an amount $b_{i,j}$ if it is obligor j which has defaulted. Thus a positive $b_{i,j}$ means that obligor i is put at higher risk by the default of obligor j, while a negative $b_{i,j}$ means that obligor i in fact benefits from the default of j, and finally $b_{i,j} = 0$ if obligor i is unaffected by the default of j.

The intensities in Equation (3.1) only depend on which obligors that have defaulted, and not by the order in which the defaults have occurred. Thus it is a model for *Unordered Default Contagion*. A more general case is when the intensities also are affected by the order in which defaults have happened. The approach outlined below works equally well for such *Ordered Default Contagion*. We make some further comments on this at the end of the present section.

Equation (3.1) determines the default times through their intensities. However, the expressions (2.1) and (2.2) for the k^{th}-to-default spreads are in terms of their joint distributions. It is by no means obvious how to go from one to the other. Here we will use the following observation.

Proposition 3.1. *There exists a Markov jump process $(Y_t)_{t\geq 0}$ on a finite state space $\boldsymbol{E}$ and a family of sets $\{\Delta_i\}_{i=1}^m$ such that the stopping times*

$$\tau_i = \inf\{t > 0 : Y_t = \Delta_i\}, \qquad i = 1, 2, \ldots, m, \tag{3.2}$$

have default intensities (3.1). Hence, any distribution derived from the multivariate stochastic vector $(\tau_1, \tau_2 \ldots, \tau_m)$ can be obtained from $\{Y_t\}_{t\geq 0}$.

In this paper, Proposition 3.1 is throughout used for computing distributions. However, we still use Equation (3.1) to describe the dependencies in a credit portfolio since it is more compact and intuitive. Proposition 3.1 is rather obvious, and perhaps most easily understood by examples, see below. However, we still give one possible formal construction, since it provides notation which anyhow is needed later on.

Proof of Proposition 3.1. We construct the state space as a union,

$$\boldsymbol{E} = \bigcup_{k=0}^{m} \boldsymbol{E}_k, \tag{3.3}$$

where $\boldsymbol{E}_k$ is set of states consisting of precisely k elements of $\{1, \ldots m\}$,

$$\boldsymbol{E}_k = \{\boldsymbol{j} = \{j_1, \ldots j_k\} : \quad 1 \leq j_i \leq m, \quad i = 1, \ldots k\}. \tag{3.4}$$

for $k = 1, \ldots m$, and where $\boldsymbol{E}_0 = \{0\}$. The interpretation is that on the set $\boldsymbol{E}_0$ no obligors have defaulted, on $\{j_1, \ldots j_k\}$ the obligors in the set have defaulted, and on $\boldsymbol{E}_m$ all obligors have defaulted.

The Markov jump process $(Y_t)_{t\geq 0}$ on $\boldsymbol{E}$ is specified by making $\{1,\dots m\}$ absorbing and starting Y in $\{0\}$, and by specifying its intensity matrix $\boldsymbol{Q}$. The latter specification is that transitions are only possible from $\boldsymbol{E}_k$ to $\boldsymbol{E}_{k+1}$, and that for a state $\boldsymbol{j}=\{j_1,j_2,\dots,j_k\}\in \boldsymbol{E}_k$ a transition can only occur to a state $\boldsymbol{j}'=(\boldsymbol{j},j_{k+1})\in \boldsymbol{E}_{k+1}$ where $j_{k+1}\neq j_i$ for $i=1,2,\dots,k$. Further, the intensity for transitions from $\boldsymbol{j}=\{j_1,j_2,\dots,j_k\}\in \boldsymbol{E}^k$ to such a $\boldsymbol{j}'$ is

$$\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}'}=a_{j_{k+1}}+\sum_{i=1}^{k} b_{j_{k+1},j_i}. \tag{3.5}$$

The diagonal elements of $\boldsymbol{Q}$ is determined by the requirement that the row sums of an intensity matrix is zero.

Next, set

$$\Delta_i=\{\boldsymbol{j}\in\boldsymbol{E}: j_n=i \text{ for some } j_n\in\boldsymbol{j}\}$$

and define the hitting times $\tau_1,\dots,\tau_m$ by

$$\tau_i=\inf\{t>0: Y_t=\Delta_i\}. \tag{3.6}$$

This construction is illustrated in Figure 1 for the case $m=3$. It is clear from the construction that $\tau_1,\dots,\tau_m$ have the intensities (3.1), see e.g. [35], Chapter 4. As a final

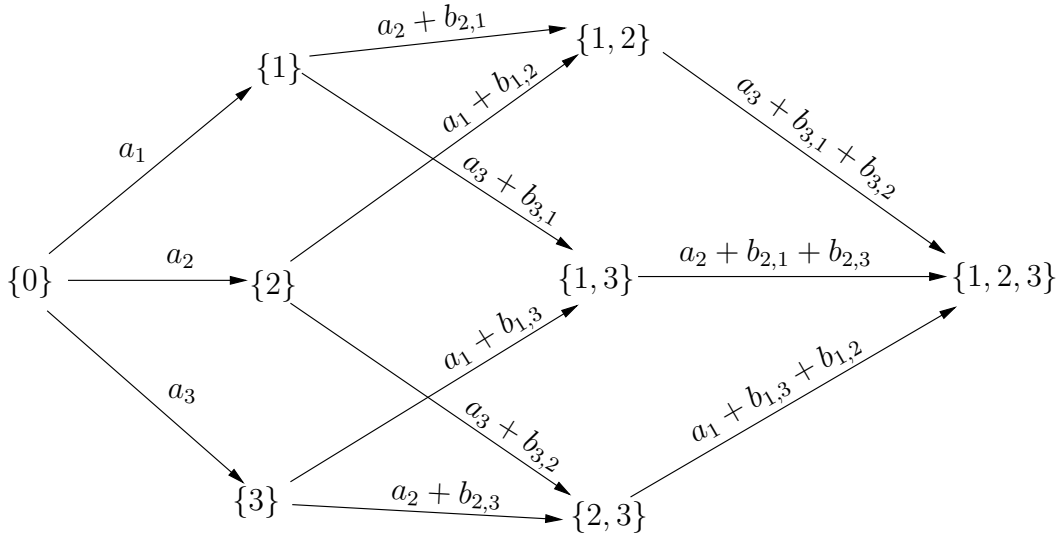

Figure 1: Illustration of the construction for $m=3$. Arrows indicate possible transitions, and the transition intensities are given on top of the arrows.

aside, when we write down $\boldsymbol{Q}$ as a matrix it is computationally convenient to order the states in $\boldsymbol{E}$ so that $\boldsymbol{Q}$ is upper triangular. This can be done by letting $\{0\}$ be first, then taking the states in $\boldsymbol{E}_1$ in some arbitrary order, followed by the states in $\boldsymbol{E}_2$ in some arbitrary order, and so on. □

Table 1: The number of states in the unordered and ordered case for different number of obligors, m.

m	unordered	ordered
5	32	326
6	64	1957
7	128	13700
8	256	109601
9	512	986410
10	1024	9864101

So far we have considered Unordered Default Contagion. In Ordered Default Contagion, also the order in which the defaults occur influence default intensities. In our setup, this corresponds to changing the form (3.1) of the intensities to

$$\lambda_{t,i} = a_i + \sum_{\boldsymbol{j}\in\mathcal{P}} b_{i,\boldsymbol{j}} 1_{\left\{\tau_{j_1}<\ldots<\tau_{j_{|\boldsymbol{j}|}}\leq t\right\}}, \quad \tau_i \geq t, \tag{3.7}$$

and $\lambda_{t,i} = 0$ when $t > \tau_i$. Here $\mathcal{P}$ contains all the ordered subsets $\boldsymbol{j} = \left(j_1, \ldots, j_{|\boldsymbol{j}|}\right)$ of the set $\{1, 2, \ldots, m\}$. Furthermore, a_i and $b_{i,\boldsymbol{j}}$ are constants such that $\lambda_{t,i} \geq 0$.

It is easy to see that the construction for Proposition 3.1 can be extended to the case (3.7). The basic change which has to be made is to change $\boldsymbol{E}_k$ from the set of all subsets of size k of $\{1, \ldots m\}$ to the set of all ordered subsets of size k.

Changing from unordered to ordered default contagion however increases the number of states in $\boldsymbol{E}$ violently. For unordered default contagion

$$|\boldsymbol{E}| = 2^m$$

while for ordered default contagion

$$|\boldsymbol{E}| = \sum_{n=0}^{m} n! \binom{m}{n}.$$

Table 1 shows the number of states in the unordered respectively ordered case for different sizes of the number m of obligors.

It is of course up to the modeler to decide if it is appropriate to use ordered or unordered default contagion. However, from the table we see that in practice it is mainly convenient to work with unordered default contagion. Further, if possible one should for large m try to reduce the number of states in $\boldsymbol{E}$ further, for example by using symmetries.

4. The matrix-analytic method

We now use the matrix analytic method, see e.g. [1] to find expressions for $F_k(t) = \mathbb{P}[T_k \leq t]$ and $F_{k,i}(t) = \mathbb{P}[T_k \leq t, T_k = \tau_i]$, the distribution functions of the ordered default times, and the probability that the k-th default is by obligor i and that it occurs before t.

The first one is more or less standard, while the second one is less so. These expressions in turn give possibilities to compute the quantities which are at the center of interest in this paper, the k^{th}-to-default spreads. Our development is closely related to so-called multivariate phase type distributions, see e.g. [2].

Define the probability vector $\boldsymbol{p}(t) = (\mathbb{P}[Y_t = \boldsymbol{j}])_{\boldsymbol{j}\in\boldsymbol{E}}$ and let $\boldsymbol{\alpha} = (1, 0, \ldots, 0) \in \mathbb{R}^{|\boldsymbol{E}|}$ be the initial distribution of the Markov jump process and let its generator be $\boldsymbol{Q}$. From Markov theory we know that

$$\boldsymbol{p}(t) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}, \quad \text{and} \quad \mathbb{P}[Y_t = \boldsymbol{j}] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{e}_{\boldsymbol{j}}, \tag{4.1}$$

where $\boldsymbol{e}_{\boldsymbol{j}} \in \mathbb{R}^{|\boldsymbol{E}|}$ is a column vector where the entry at position $\boldsymbol{j}$ is 1 and the other entries are zero. Furthermore, $e^{\boldsymbol{Q}t}$ is the matrix exponential which has a closed form expression in terms of the eigenvalue decomposition of $\boldsymbol{Q}$.

Next, define vectors $\boldsymbol{m}^{(k)}$ of length $|\boldsymbol{E}|$ by requiring that

$$\boldsymbol{m}^{(k)}_{\boldsymbol{j}} = \begin{cases} 1 \text{ if } & \boldsymbol{j} \in \cup_{i=0}^{k-1} \boldsymbol{E}_i \\ 0 \text{ otherwise .} & \end{cases} \tag{4.2}$$

Then,

$$\mathbb{P}[T_k > t] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \boldsymbol{m}^{(k)}, \tag{4.3}$$

since $\boldsymbol{m}^{(k)}$ sums the probabilities of states where there has been less than k defaults.

Hence, what is left to compute is $\mathbb{P}[T_k > t, T_k = \tau_i]$. For this we use the imbedded Markov chain $(Y_{T_n})_{n=0}^m$. By definition, the transition probability matrix $\boldsymbol{P}$ for $(Y_{T_n})_{n=0}^m$ is given by

$$\boldsymbol{P}_{\boldsymbol{j},\boldsymbol{j}'} = \mathbb{P}\left[Y_{T_n} = \boldsymbol{j}' \,|\, Y_{T_{n-1}} = \boldsymbol{j}\right] = \frac{\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}'}}{\sum_{\boldsymbol{k}\neq\boldsymbol{j}} \boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{k}}}, \quad \boldsymbol{j}, \boldsymbol{j}' \in \boldsymbol{E},$$

with the ordering of the states in $\boldsymbol{P}$ the same as for $\boldsymbol{Q}$.

Further, let $\boldsymbol{h}^{i,k}$ be vectors of length $|\boldsymbol{E}|$ and let $\boldsymbol{G}^{i,k}$ be $|\boldsymbol{E}| \times |\boldsymbol{E}|$ diagonal matrices, defined by

$$\boldsymbol{h}^{i,k}_{\boldsymbol{j}} = \begin{cases} 1 \text{ if } & \boldsymbol{j} \in \Delta_i \cap \boldsymbol{E}_k \\ 0 \text{ otherwise ,} & \end{cases}$$

and

$$\boldsymbol{G}^{i,k}_{\boldsymbol{j},\boldsymbol{j}} = \begin{cases} 1 \text{ if } & \boldsymbol{j} \in (\Delta_i)^C \cap \boldsymbol{E}_k \\ 0 \text{ otherwise .} & \end{cases}$$

We now establish the following result.

Proposition 4.1. *With notation as above,*

$$\mathbb{P}[T_k > t, T_k = \tau_i] = \boldsymbol{\alpha} e^{\boldsymbol{Q}t} \sum_{\ell=0}^{k-1} \left(\prod_{p=\ell}^{k-1} \boldsymbol{G}^{i,p} \boldsymbol{P} \right) \boldsymbol{h}^{i,k}, \tag{4.4}$$

for $k = 1, \ldots m$.

Proof of Proposition 3.1. We will use the following fact, which is straightforward to establish, and standard in Markov chain theory:

If $\{X_n\}$ is a stationary, discrete time, finite state space, Markov chain with initial distribution $\boldsymbol{p}$ and transition matrix $\boldsymbol{P}$, and $\boldsymbol{E}_0, \dots \boldsymbol{E}_k$ are subsets of the state space, then

$$\mathbb{P}\left[X_0 \in \boldsymbol{E}_0, \dots X_k \in \boldsymbol{E}_k\right] = \boldsymbol{p}\boldsymbol{G}_0\boldsymbol{P}\boldsymbol{G}_1\boldsymbol{P}\dots\boldsymbol{G}_{k-1}\boldsymbol{P}\boldsymbol{h}_k,$$

where the $\boldsymbol{G}_\ell$-s are diagonal matrices with diagonal elements equal to one for state j if $j \in \boldsymbol{E}_\ell$ and zero otherwise, for $\ell = 0, \dots k-1$, and $\boldsymbol{h}_k$ is a column vector with a 1 in position j if $j \in \boldsymbol{E}_k$.

Let Δ_i^C be the complement of Δ_i in $\boldsymbol{E}$, i.e. the set of all states where i has not defaulted. By an appropriate translation to the situation and notation above, in particular replacing $\boldsymbol{p}$ by $\boldsymbol{\alpha}e^{\boldsymbol{Q}t}$, and if $T_\ell \le t < T_{\ell+1} < T_k$, we obtain that

$$\begin{aligned}\mathbb{P}\left[Y_t \in \Delta_i^C \cap \boldsymbol{E}_\ell, Y_{T_{\ell+1}} \in \Delta_i^C \cap \boldsymbol{E}_{\ell+1}, \dots Y_{T_{k-1}} \in \Delta_i^C \cap \boldsymbol{E}_{k-1}, Y_{T_k} \in \Delta_i \cap E_k\right] \\ = \boldsymbol{\alpha}e^{\boldsymbol{Q}t}\boldsymbol{G}^{i,\ell}\boldsymbol{P}\boldsymbol{G}^{i,\ell+1}\dots\boldsymbol{G}^{i,k-1}\boldsymbol{P}\boldsymbol{h}^{i,k}.\end{aligned}$$

Since

$$\begin{aligned}&\mathbb{P}\left[T_k > t, T_k = \tau_i\right] \\ &\quad = \sum_{\ell=0}^{k-1} \quad \mathbb{P}\left[Y_t \in \Delta_i^C \cap \boldsymbol{E}_\ell, Y_{T_{\ell+1}} \in \Delta_i^C \cap \boldsymbol{E}_{\ell+1}, \dots Y_{T_{k-1}} \in \Delta_i^C \cap \boldsymbol{E}_{k-1}, Y_{T_k} \in \Delta_i \cap E_k\right],\end{aligned}$$

this proves (4.4). □

5. Numerical studies

In this section we will use the theory developed in previous sections to study, in a realistic numerical example, how different factors affect the size of k-th to default spreads. For this it is convenient to reparameterize the basic description (3.1) of the default intensities to the form

$$\lambda_{t,i} = a_i\left(1 + c\sum_{j=1, j\neq i}^{m}\theta_{i,j}1_{\{\tau_j \le t\}}\right), \tag{5.1}$$

which was suggested in [20]. In this parametrization, the a_i are the base default intensities, c measures the general "interaction level" and the $\theta_{i,j}$ measure the "relative dependence structure".

First, in Subsection 5.1 we introduce a portfolio consisting of 15 telecom companies which is used as a basis for the numerical studies. We further "semi-calibrate" our model to this portfolio, using CDS spreads taken from Reuters.

We then study the influence of portfolio size on k^{th}-to-default spreads, (Subsection 5.2), of changing the interaction level (Subsection 5.3), the impact of using inhomogeneous recovery rates (Subsection 5.4), the sensitivity to the underlying CDS spreads (Subsection 5.5), and finally compare a model with non-symmetric dependence to a corresponding symmetric model (Subsection 5.6).

For the rest of this paper we will assume that the $\theta_{i,j}$ are given - hence the term "semi-calibrations", cf. Subsection 5.1. It is a topic for future research to find out how to estimate the $\theta_{i,j}$. For example, using liquid market data on CDO's will give us more information which can be used for some cases. The rapidly increasing market of credit portfolio products may also help. In Section 7 we discuss this topic in more detail.

Numerical studies always carry the risk of programming errors and numerical instability. However, fortunately we have been able to benchmark our numerical methods to an example from [19], pp. 19-20 and [20], which as far as we know, are the only available results on default contagion for nonsymmetric portfolios with more than three bonds.

The paper [19] studies a portfolio with five bonds and time-dependent default intensities, and uses numerical solution of differential equations to compute spreads for a number of cases. Our model for intensities doesn't directly allow for time-dependence, but it was still possible to approximate the portfolio in [19] with our model, calibrate it as discussed in Section 6 below, and compare the spreads thus obtained with those in [20]. The results agreed to at least four significant digits in all cases, which lends some confidence to our numerical implementation.

5.1. **A telecom portfolio.** Table 2 describes the portfolio which is used in our numerical studies. The data was obtained from `Reuters` at August 23, 2005. We have assumed a fictive recovery rate structure and also a fictive relative dependence structure $\theta_{i,j}$ which is given in Table 8 in Appendix, and we used the interaction level $c = 0.5$. The interest rate was assumed to be constant and set to 3%, and the protection fees were assumed to be paid quarterly. The maturity was 5 years. The a_i-s are obtained by individual calibration to the CDS spreads in Table 2. From Table 8 we see that the intensities can jump up to 284% of their "base values" a_i, when $c = 0.5$. In case both bid and ask prices for the CDS-s were given, we used their average. The calibration is described in more detail in Section 6. We refer to the entire procedure - using the fictive recovery rates, the fictive dependence structure and the calibrated base intensities - as *semi-calibration*.

5.2. **Dependence on portfolio size.** To study the dependence on portfolio size, we considered 6 different sub-portfolios. The first portfolio consisted of the 10 first bonds from Table 2, the second of the 11 first bonds, and so on, until the last portfolio which contained all the 15 bonds in the table. Each subportfolio with m obligors had a dependence structure given by upper left $m \times m$ submatrix of the matrix given in Table 8. When we calibrated the subportfolios against the market CDS spreads, the corresponding sum of the absolute calibration error never exceeded two hundreds of a bp. For each portfolio the k^{th}-to-default spreads were computed from Equation (2.2). The results are shown in Table 3. The spreads are only shown for $k \leq 5$. The remaining spreads were all less than six hundreds of a basis point.

In the table the spreads increase as the size of the portfolio increases, as they should. Quantitatively, the increase from a portfolio of size 10 to one of size 15 is 47% for a 1^{st}-to-default swap, 92% for a 2^{nd}-to-default swap, 168% for a 3^{rd}-to-default swap, and for a 5^{th}-to-default swap the increase is 700%. Further, for a portfolio of size 10 the price

Table 2: The Telecom companies and their 5 year CDS spreads.

Company	bid	ask	time	recovery %
British Telecom	40	44	23 Aug, 09:33	32%
Deutsche Telecom	34		23 Aug, 19:18	48%
Ericsson	54	54	23 Aug, 18:27	45%
France Telecom	38	42	23 Aug, 17:13	34%
Nokia	21	23	23 Aug, 12:25	42%
Hellenic Telecom		43	23 Aug, 19:18	41%
Telefonica	34	38	23 Aug, 09:34	29%
Telenor	26		23 Aug, 12:25	39%
Telecom Italia	47		23 Aug, 19:34	51%
Telia		35	23 Aug, 12:25	41%
Port Telecom Int	34	38	23 Aug, 12:10	47%
MM02		47	23 Aug, 16:29	33%
Vodafone	24	28	23 Aug, 12:59	35%
KPN	38	42	23 Aug, 09:33	43%
Telekom Aus		35	04 Aug, 19:59	50%

Table 3: The k-th-to default swap premiums in basis points (bp). The first column is for the 10 first obligors in Table 2, the second is for the 11 first obligors, and so on.

k	$m = 10$	$m = 11$	$m = 12$	$m = 13$	$m = 14$	$m = 15$
1	357.7	389.8	432.3	456.6	493.3	526.1
2	55.38	65.27	77.48	84.34	95.96	106.8
3	7.649	9.963	12.84	14.49	17.47	20.40
4	0.8698	1.281	1.814	2.132	2.744	3.366
5	0.08026	0.1373	0.2167	0.2678	0.3701	0.4795

of a 1^{st}-to-default swap is about 4500 times higher than for a 5^{th}-to-default swap. The corresponding ratio for a portfolio of size 15 is about 1100.

5.3. **Dependence on the interaction level.** In this subsection we use a portfolio consisting of the 9 first obligors in Table 2 to study how spreads are affected by the interaction parameter c which was taken to be 0.5 in the previous section. As above, we let the dependence parameters be given by the upper left part of Table 8. We first note that by (5.1) the value of c enters into the calibration of the base intensities a_i: a higher value of c will lead to smaller a_i-s, and hence c affects spreads both directly, and indirectly through its influence on the base intensities.

The dependence of the spreads on the interaction level is illustrated in Figure 2. The 1^{st}-to-default spread decreases with increasing interaction level, and for k larger than 2

the spreads increase. However, it looks as if the 2^{nd}-to-default spread may have a local maxima. To confirm that this is indeed possible, we experimented with different dependence structures. One result was Figure 3, which depicts the same graphs as Figure 2 but for different $\theta_{i,j}$, given by Table 7 where some of elements $\theta_{i,j}$ are much bigger than the corresponding numbers in Table 8. In this figure, the 2^{nd}-to-default spread has a clear local maximum.

It might be worth noting that the graph of the 1^{st}-to-default spread as a function of the interaction level c, roughly had the same structure as the corresponding graph for the intensity for T_1, see Figure 6. However, the same was not true for the 2^{nd}-to-default spread. This can be seen from Figure 7 which shows the intensity $\sum_{i=1,i\neq j}^{9} a_i(1+c\theta_{i,j})$ for the second default T_2, when the first default was by obligor j, for $j = 1, 2, \ldots 9$.

The case $c = 0$ is of special interest since it means that the defaults are independent of one-another. In particular, Figures 2, 3 and 4 quantifies the errors made in computing spreads as if obligors where independent in cases where there in fact is default contagion. Further, in Figure 3 we note that for very large interaction levels, the spreads for $1 \leq k \leq 5$ tend to converge into a narrow interval, compared with the case with very small interactions. The intuitive explanation for this may be that once one obligor default, several other will quickly follow. Finally, note that as the interaction level increases, the spreads for $6 \leq k \leq 9$ drastically increases and can thus no longer be neglected, as for example in the Table 3 where $c = 0.5$.

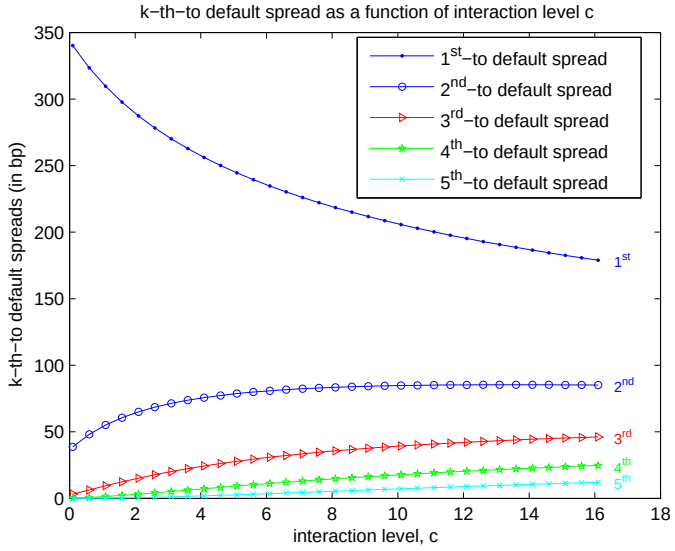

Figure 2: The k^{th}-to-default spreads as a function of the interaction level c, for a portfolio consisting of the first 9 obligors in Table 2.

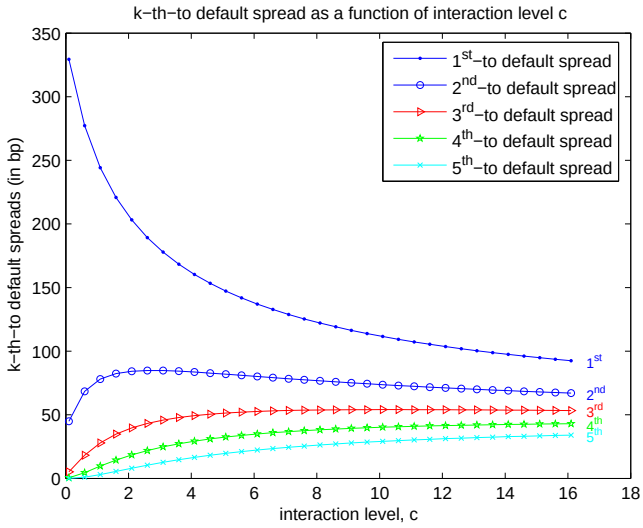

Figure 3: The different k^{th}-to-default spreads for $k \leq 5$ as a function of the interaction level c, for a portfolio consisting of the first 9 obligors in Table 2 with dependence structure given by Table 7.

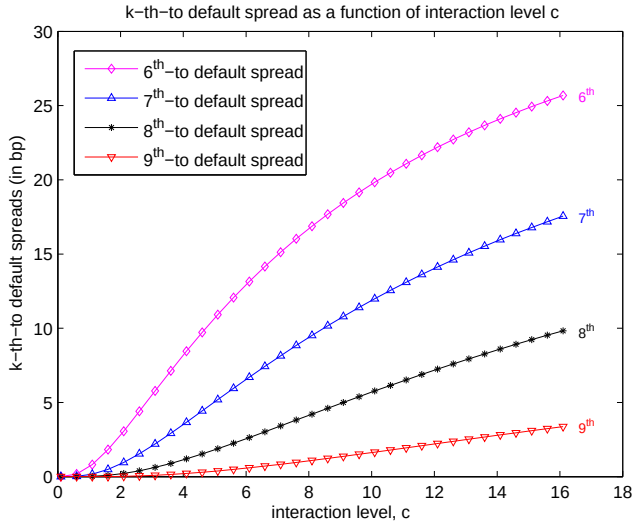

Figure 4: The different k^{th}-to-default spreads for $6 \leq k \leq 9$ as a function of the interaction level c, for a portfolio consisting of the first 9 obligors in Table 2 with dependence structure given by Table 7.

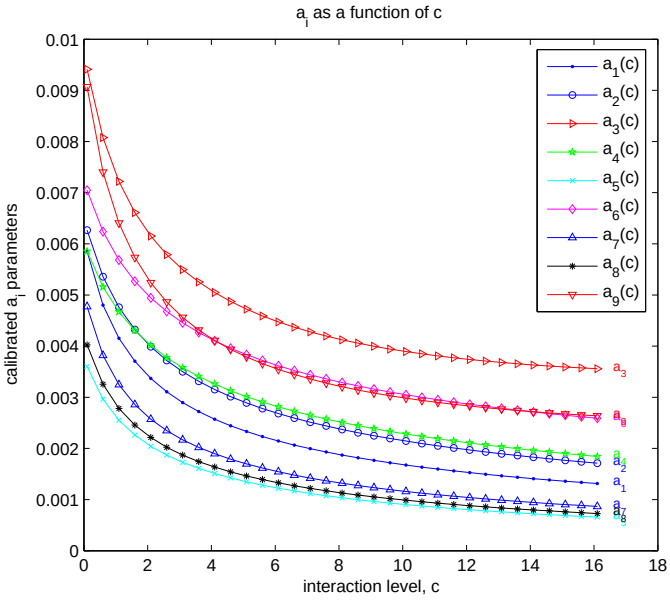

Figure 5: The base intensities a_i as functions of the interaction level c for a portfolio consisting of the first 9 obligors in Table 2 with dependence structure given by Table 7.

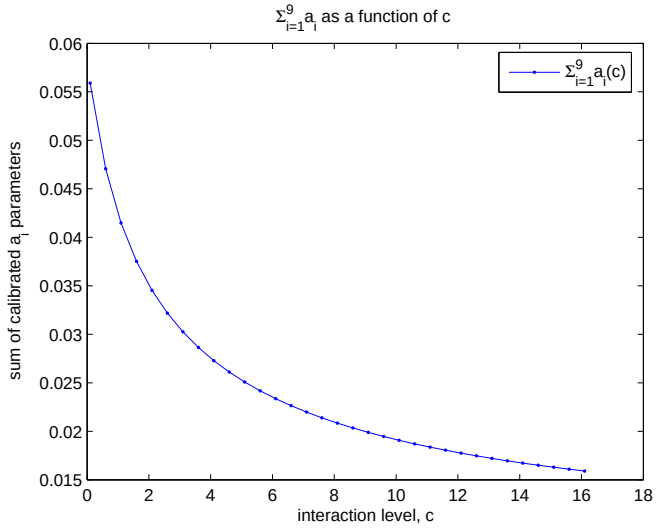

Figure 6: The intensity for T_1, as a function of the interaction level c, for a portfolio consisting of the 9 first obligors in Table 2 with dependence structure given by Table 7.

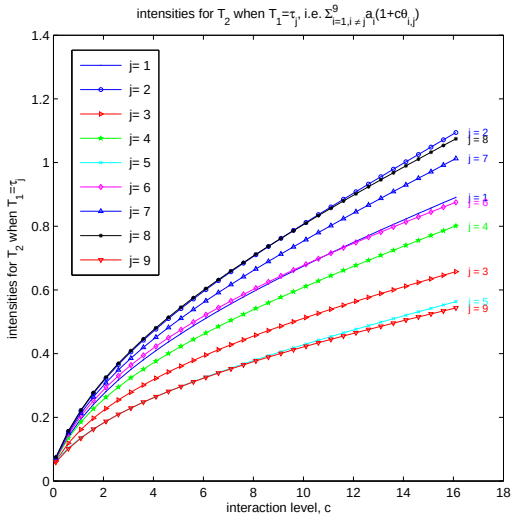

Figure 7: Intensities for T_2 when the first default was by obligor j, for a portfolio consisting of the 9 first obligors in Table 2 with dependence structure given by Table 7.

Table 4: The relative difference in percent between k^{th}-to-default swap spreads priced with homogeneous recovery rates and nonhomogeneous recovery rates. The different recovery rates are displayed in Table 6 in Appendix 8

k	std = 3.24	std = 4.90	std = 6.92	std = 11.10
1	0.13	0.16	0.20	0.23
2	1.27	1.35	2.05	2.74
3	2.81	3.25	5.13	6.71
4	4.27	5.17	8.45	11.10
5	5.60	6.99	11.69	15.51

5.4. **Dependence on the recovery rates.** In this subsection the numerical experiment aimed at investigating to what extent k^{th}-to-default swaps spreads for portfolios with nonhomogeneous recovery rates differ from the spreads in a corresponding portfolio where all recovery rates are the same and equal to the average of the nonhomogeneous rates.

The experiment was performed on a portfolio consisting of the first 11 obligors in Table 2 with dependence structure given by the upper left 11×11 submatrix of the matrix given in Table 8. We studied five different cases. In the first one all recovery rates were set to 40%. In the other four cases the recovery rates were varied "randomly" around approximately the mean 40%, but with the different standard deviations $3.24, 4.90, 6.92$ and 11.10, respectively. The results are displayed in Table 4. For the 1^{st}- and 2^{nd}-to-default spreads, the inhomogeneous cases differed from the homogenous one by at most

3%, and even the largest difference, for $k = 5$ and for the standard deviation equal to 11.10%, was only 15%. The different recoveries are displayed in Table 6 in Appendix 8

5.5. **Dependence on the market spreads.** In this subsection we investigate in numerical experiments how the k^{th}-to-default swap prices change when the market prices of the underlying single-name CDS prices change.

The first experiment used a portfolio consisting of the 5 first obligors in Table 2. The CDS spreads for obligors $1, 3, 4$ were held at their market values, and while the CDS spreads for obligors 2 and 5 were varied from 10 to 225 in steps of 10 bp. The resulting k^{th}-to-default spreads increased smoothly as the CDS spreads increased, and this increase was more dramatic for larger k-s, see Figure 8.

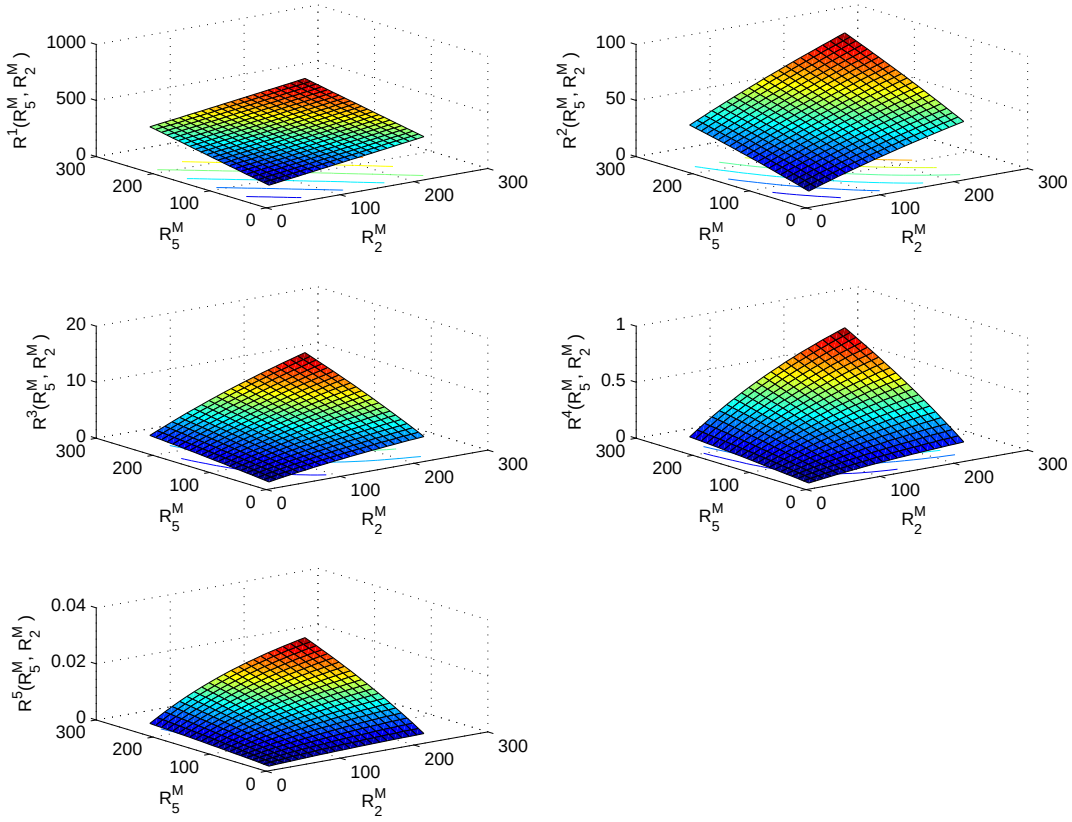

Figure 8: k^{th}-to-default spreads as function of market CDS spreads. The portfolio consisted of the first 5 obligors in Table 2. The CDS spreads for obligors 1 and 5 were varied, while the others were held constant.

5.6. **Approximation by a symmetric portfolio.** A portfolio is symmetric if the obligors are completely interchangeable. In the intensity formulation (5.1) for unordered default contagion, this means that the parameters a_i all are equal, and similarly the $\theta_{i,j}$ are the same for each obligor and the ϕ_i are equal. To compute spreads it then is sufficient to keep track of how many obligors have defaulted, but there is no need to know which ones it was. Here we consider the special case of (3.1) where all obligors have the same default intensities $\lambda_{t,i} = \lambda_t$ specified by $\lambda_t = a + \sum_{k=1}^{m-1} b_k 1_{\{T_k \le t\}}$ where $\{T_k\}$ is the ordering of the default times $\{\tau_i\}$. For this symmetric case, the Markov jump process constructed in Proposition 3.1 can be collapsed into a chain with the $m+1$ states $\{0\}, \{1\}, \ldots \{m\}$. The interpretation is that the chain is in the state $\{k\}$ if precisely k obligors have defaulted. In the literature such a process is called a death process. This new state space is very much smaller state than the one in Proposition 3.1, which means that it is possible to do numerical computation for much larger portfolios, with hundreds or thousands of obligors, see e.g. [31] where CDO-tranche spreads are computed on portfolios with 125 obligors in such a symmetric model.

It is hence of interest to understand how well non-symmetric portfolios can be approximated by symmetric ones. To explore this we constructed and semi-calibrated three different portfolios, 1) the telecom portfolio from Table 2 with all recovery rates set to 40%, 2) the telecom portfolio with the recovery rates given in Table 2 (the standard deviation of these rates are 6.88%), and 3) the telecom portfolio with the first 11 recovery rates given by the last row in Table 6 and the remaining rates for obligors 12 to 15 set to $30, 32, 40$ and 60% (the rates then have standard deviation 11.71%). In all cases the dependence structure was given by Table 8 and the interaction level was $c = 0.5$.

Each of these three portfolios was then approximated by a symmetric portfolio. In the symmetric approximating portfolio all recovery rates were set equal to the average of the recovery rates in the original portfolio. The jump parameters where assumed to be the same so $b_k = b$ at each default time T_k. Further, the parameters a and b were chosen so that the model CDS spread in the symmetric portfolio coincided with the average market CDS spread in the nonsymmetric counterpart, and so that the first-to-default spreads also agreed.

From Table 5 we see that the relative differences are small. For $k = 2$ and 3 they increase as the standard deviation of the recovery rates increase. However, for $k = 4$ and 5 the relative differences for the second portfolio (std = 6.88) somewhat surprisingly are smaller than for the other two cases. For the nonhomogeneous recovery rate cases, the relative differences are not monotone in k. Finally, for $6 \le k \le 15$ the differences can range between 9% up to 130% where the error on average increases with k. However, for each case the sum of these spreads are smaller than one tenth of a bp.

6. Calibration and numerical implementation

In this section we discuss the computational aspects in more detail. Subsection 6.1 gives a short description how to calibrate (or *semi-calibrate*) the model (5.1) against market data and formulas for the CDS and k^{th}-to-default spreads, and Subsection 6.2 discusses how to

Table 5: The difference in k^{th}-to-default swap spreads between an approximating symmetric portfolio and a nonsymmetric portfolio, in percent of the spreads for the nonsymmetric portfolio; for three different cases. For $k = 1$ the difference is by construction (almost) zero.

k	std = 0	std = 6.88	std = 11.71
2	0.153	0.587	1.98
3	0.811	0.850	3.37
4	2.24	0.339	3.82
5	4.65	1.26	3.13

compute the matrix exponential. Subsections 6.3 and 6.4 consider more general models and computation by simulation, respectively.

6.1. **Calibration.** As discussed above, we assume that the relative dependence structure $\theta_{i,j}$, the interaction level c and the recovery rate structure $\phi_1, \dots, \phi_m$ all are exogenously given - hence the term "semi-calibration". The base intensities a_i are then obtained by individual calibration to the market CDS spreads. The calibration uses a nonlinear least squares method where the model CDS spreads $R^{(i)}$ are matched against the corresponding market CDS spreads $R^{i,M}$. Thus the base intensities are computed as

$$(a_1, \dots, a_m) = \underset{\tilde{\boldsymbol{a}}=(\tilde{a}_1,\dots,\hat{a}_m)}{\operatorname{argmin}} \sum_{i=1}^{m} \left(R^{i,M} - R^{(i)}(\hat{\boldsymbol{a}})\right)^2$$

where we have emphasized that the $R^{(i)}$ are functions of the parameters $\boldsymbol{a} = (a_1, \dots, a_m)$. We then use the calibrated base intensities a_i with $\{\theta_{i,j}, \phi_i, c\}$ to compute the k-th-to default spreads R_k.

Reverting to the notation of (3.1), closed-form expressions for R_k and $R^{(i)}$ may be obtained by inserting (4.3) and (4.4) into (2.2). For ease of reference we exhibit the resulting formulas (detailed proofs can be found in [28] or [30]).

Proposition 6.1. *Consider m obligors with default intensities* (3.1) *and assume that the interest rate r is constant. Then,*

$$R_k = \frac{\boldsymbol{\alpha}\left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)\boldsymbol{\phi}^{(k)}}{\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n}e^{-rt_n} + \boldsymbol{C}(t_{n-1}, t_n)\right)\right)\boldsymbol{m}^{(k)}}$$

and

$$R^{(i)} = \frac{(1-\phi_i)\boldsymbol{\alpha}\left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)\boldsymbol{g}^{(i)}}{\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n}e^{-rt_n} + \boldsymbol{C}(t_{n-1}, t_n)\right)\right)\boldsymbol{g}^{(i)}}.$$

Here

$$\boldsymbol{\phi}^{(k)} = \sum_{i=1}^{m}(1-\phi_i)\,\boldsymbol{R}^{i,k}\boldsymbol{h}^{i,k} \quad \text{and} \quad \boldsymbol{R}^{i,k} = \sum_{\ell=0}^{k-1}\left(\prod_{p=\ell}^{k-1}\boldsymbol{G}^{i,p}\boldsymbol{P}\right)$$

and

$$\boldsymbol{C}(s,t) = s\left(\boldsymbol{A}(t) - \boldsymbol{A}(s)\right) - \boldsymbol{B}(t) + \boldsymbol{B}(s),$$

where

$$\boldsymbol{A}(t) = e^{\boldsymbol{Q}t}\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}$$
$$\boldsymbol{B}(t) = e^{\boldsymbol{Q}t}\left(t\boldsymbol{I} + \left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\right)\left(\boldsymbol{Q} - r\boldsymbol{I}\right)^{-1}\boldsymbol{Q}e^{-rt}$$

Finally, $\boldsymbol{g}^{(i)}$ *is an* $|\boldsymbol{E}|$ *column vector such that*

$$\boldsymbol{g}^{(i)}_{\boldsymbol{j}} = \begin{cases} 1 & \text{if} \quad \boldsymbol{j} \in (\Delta_i)^C \\ 0 & \text{otherwise .} \end{cases}$$

There are several possible computational shortcuts. The quantities $\boldsymbol{g}^{(i)}$, $\boldsymbol{h}^{i,k}$, $\boldsymbol{g}^{i,k}$ and $\boldsymbol{m}^{(k)}$ do not depend on the parametrization, and hence only have to be computed once. The row vectors $\boldsymbol{\alpha}\left(\boldsymbol{A}(0) - \boldsymbol{A}(T)\right)$ and $\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n}e^{-rt_n} + \boldsymbol{C}(t_{n-1},t_n)\right)\right)$ are the same for all CDS spreads and all k^{th}-to-default swap spreads and hence only have to be computed once for each parametrization $\{a_i, \theta_{i,j}, \phi_i, c\}$. The same holds for $\boldsymbol{\phi}^{(k)}$ for each k. Furthermore, if Δ_n is constant then $\boldsymbol{\alpha}\left(\sum_{n=1}^{n_T}\left(\Delta_n e^{\boldsymbol{Q}t_n}e^{-rt_n} + \boldsymbol{C}(t_{n-1},t_n)\right)\right)$ can be simplified in terms of $e^{\boldsymbol{Q}t_n}$. If the ϕ_i are the same for all obligors, $\boldsymbol{\phi}^{(k)}$ can be replaced by $(1-\phi)\boldsymbol{m}^{(k)}$. Next, if we rewrite the sum $\boldsymbol{R}^{i,k}$ as $\boldsymbol{R}^{i,k} = \sum_{\ell=0}^{k-1}\boldsymbol{M}^{i,\ell,k}$ where $\boldsymbol{M}^{i,\ell,k} = \prod_{p=\ell}^{k-1}\boldsymbol{G}^{i,p}\boldsymbol{P}$ then $\boldsymbol{M}^{i,\ell-1,k} = \boldsymbol{G}^{i,\ell-1}\boldsymbol{P}\boldsymbol{M}^{i,\ell,k}$ which is useful for computation. Also note that $(\boldsymbol{Q} - r\boldsymbol{I})$ is invertible since it is upper diagonal with strictly negative diagonal elements. The conditioning number of $(\boldsymbol{Q} - r\boldsymbol{I})$ is often large, but still we have not encountered numerical problems in computing the inverse.

6.2. **Computation of the matrix exponential.** The main challenge in Proposition 6.1 is to compute the matrix exponential $e^{\boldsymbol{Q}t}$. We have mainly experimented with two different numerical methods; direct series expansion of the matrix exponential and the uniformization method (which sometimes also is called the randomization method). Both were fast and robust for our problems. However, while the series expansion method lacks lower bounds on possible worst case scenarios (see [47]), the uniformization method provides analytical expressions for the residual error. Furthermore, previous studies indicate that it can handle large sparse matrices with remarkable robustness, see e.g. [56], [26] or Appendix C.2.2 in [41]. We have therefore chosen the uniformization method. A probabilistic interpretation of the method can be found in [26] and pure matrix arguments which motivates the method are given in [56] and Appendix C.2.2 in [41].

There are many different methods to compute the matrix exponential ([47] [48]). However, most of the other standard methods are not adapted to very large, but sparse, matrices and don't seem possible when the state space is larger than a few hundred (see [56],[26]). As one example, it is tempting to try eigenvalue decomposition since the eigenvalues are given by the diagonal of $\boldsymbol{Q}$. However, in our examples this method failed already for $m = 9$ since the eigenvalue matrices turned out to be ill-conditioned, which introduced large numerical errors in the inversions. Compared to e.g. Krylov-based methods or ODE methods the uniformization method is relatively simple to implement.

The uniformization method works as follows. Let $\Lambda = \max\left\{|\boldsymbol{Q}_{\boldsymbol{j},\boldsymbol{j}}| : \boldsymbol{j} \in \boldsymbol{E}\right\}$ and set $\widetilde{\boldsymbol{P}} = \boldsymbol{Q}/\Lambda + \boldsymbol{I}$. Then

$$e^{\boldsymbol{Q}t} = \sum_{n=0}^{\infty} \widetilde{\boldsymbol{P}}^n e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}. \tag{6.2.1}$$

Recall that $\boldsymbol{p}(t) = \boldsymbol{\alpha} e^{\boldsymbol{Q}t}$ and define $\widetilde{\boldsymbol{p}}(t, N) = \boldsymbol{\alpha} \sum_{n=0}^{N} \widetilde{\boldsymbol{P}}^n e^{-\Lambda t} \frac{(\Lambda t)^n}{n!}$. Let $\varepsilon > 0$ be arbitrary and pick $N(\varepsilon)$ so that $1 - \sum_{n=0}^{N(\varepsilon)} e^{-\Lambda t} \frac{(\Lambda t)^n}{n!} < \varepsilon$. Then the L_1 error $\|\boldsymbol{p}(t) - \widetilde{\boldsymbol{p}}(t, N(\varepsilon))\|_1$ is also less than ε, i.e., $\widetilde{\boldsymbol{p}}(t, N(\varepsilon))$ approximates $\boldsymbol{p}(t)$ with an accumulated absolute error which is less than ε. Furthermore, since all entries in $\widetilde{\boldsymbol{p}}(t, N(\varepsilon))$ are positive there are no cancelation effects and the approximation error decreases monotonically with increasing N. Furthermore, for fixed N, the error $\|\boldsymbol{p}(t) - \widetilde{\boldsymbol{p}}(t, N)\|_1$ is also decreasing in t.

Further useful properties of (6.2.1) are that it separates the computation of $\widetilde{\boldsymbol{P}}^n$ from the time dependent components and that

$$\int_0^T e^{(\boldsymbol{Q} - r\boldsymbol{I})t} dt = \sum_{n=0}^{\infty} \widetilde{\boldsymbol{P}}^n \frac{\Lambda^n}{n!} I_k^{(\Lambda + r)}(T), \tag{6.2.2}$$

where

$$\begin{aligned} I_k^{(\beta)}(t) &= \int_0^t s^k e^{-\beta s} ds \\ &= \frac{e^{-\beta t}}{(-\beta)^{k+1}} \left[(-\beta t)^k - k(-\beta t)^{k-1} + k(k-1)(-\beta t)^{k-2} - \ldots + (-1)^{k-1}(k-1)!(-\beta t)\right]. \end{aligned}$$

Since $I_k^{(\beta)}(t) > 0$, there are no cancelation effects in the approximation of the RHS in (6.2.2). Truncating the sum in the RHS in (6.2.2) gives an approximation to the integral in the LHS. In this case, the error control requires a little more work.

For $m \leq 13$, we used a standard laptop with 1025 MB RAM. For $m = 14$ and $m = 15$ the memory requirements were too big for the laptop so the computations were done on a Sun Solaris, 2x900 MHz UltraSPARC-III with 5GB RAM. As an example if $m = 15$, $c = 0.5$, $T = 5$ and $\theta_{i,j}$ were as in Table 8 and if we put $\varepsilon = 3.33 \cdot 10^{-16}$ (the floating-point relative accuracy for the number 1 in Matlab is $2.22 \cdot 10^{-16}$) then our calibration against Table 2 implied that $\Lambda = 0.2500$ and that $N(\varepsilon) = 19$ terms were needed in the computation of $\widetilde{\boldsymbol{p}}(T, N)$. Furthermore, the calibration errors were neglige: the sum of the individual absolute calibration errors were less than two tenths of a bp.

A further point is that our matrices in general are very large, for example if $m = 15$ then the generator has $2^{15} = 32768$ rows and thus contain $2^{30} \approx 1.$ billion entries. However, at the same time it is extremely sparse and the sparseness is increasing with m. E.g., for $m = 15$ there are only 0.025% nonzero entries in $\boldsymbol{Q}$, and hence only about 280.000 elements have to be stored.

A final point is that we are not interested in finding the matrix exponential itself, but only the probability vector $\boldsymbol{p}(t)$. This is important, since computing $e^{\boldsymbol{Q}t}$ is very time and memory consuming compared with computing $\boldsymbol{\alpha} e^{\boldsymbol{Q}t}$. For example, using the uniformization method with $t = 5$, $m = 14$ and $\varepsilon = 3.331 \cdot 10^{-16}$, which implies that $N(\varepsilon) = 19$ the time

to find $\boldsymbol{p}(t)$ in Matlab, by first computing $e^{\boldsymbol{Q}t}$ with the uniformization method and then multipling this matrix with $\boldsymbol{\alpha}$, was 42.3 seconds. On the other hand, computing $\boldsymbol{p}(t)$ via the vectors $\boldsymbol{\alpha}\widetilde{\boldsymbol{P}}^n$ only took 0.14 seconds, and hence was about 300 times faster.

To get the same accuracy with direct Taylor method required 32 terms in the truncated sum. For this case, the corresponding computational times where 108 seconds and 0.25 seconds. The quick method was about 450 times faster than the slow one. Further, the uniformization method was about 2.5 and 1.8 times faster compared to the corresponding slow and quick Taylor methods. The reason was that the Taylor method required 32 terms in the sums, compared to the 19 needed for the uniformization method.

6.3. **Extensions.** It is natural to believe that the default intensity of a obligor in addition to dependence on defaults of other obligors also depends on exogenous macroeconomic and market factors. It is possible generalize Proposition 3.1 in Section 3 to the case when the parameters a_t^i and $b_t^{i,j}$ in the formula (3.1) for the intensity depend on some background random process. The idea is to first condition on the whole realization of the background process and then treat $(Y_t)_{t\geq 0}$ as an inhomogeneous Markov jump process with time-dependent generator. It is usually impossible to find tractable closed-form expressions for distributions derived from $(\tau_1, \ldots, \tau_m)$ for this case. Instead, to use our method one has to rely on a fine discretisation of time.

We also note that Proposition 3.1 does not seem easily applicable if the parameters in the intensities depend of the default times. For example, so called *self-exciting* point processes (or *Hawkes processes*), see e.g. [27] with intensities given by

$$\lambda_{t,i} = a_i + \sum_{j\neq i} b_{i,j} e^{-(t-\tau_j)} 1_{\{\tau_j \leq t\}} \tag{6.3.1}$$

are therefore not suited to our setup. Applications of Hawkes processes in credit risk are discussed in e.g. [23], [22] and [17].

6.4. **Simulation.** An alternative to numerical computation is to use simulation to produce realizations of the random vector $(\tau_1, \tau_2 \ldots, \tau_m)$. If the intensities are given by (3.1) this is straightforward, see e.g. [35].

The so called *total hazard construction* ([50], [55]) can be used in more general circumstances where the intensities may be functionals of the default times as well as of time, see e.g. (6.3.1). The paper, [15], we believe, was the first to point out that the total hazard construction can be used in credit risk. In Chapter 5 of [58], the author used the total-hazard construction to study four cases of default contagion. Three of these cases can be handled without invoking the total-hazard construction, for example using our approach, cf also [50]. The forth case, which considers stochastic parameters, can actually also be treated without the total hazard construction, see e.g. the forthcoming paper [29]. The case with a stochastic processes $\boldsymbol{X}_t$ in the parameters as in [58], is handled by doing a straightforward extension of the results in [50], [55].

To simulate one needs to know the values of the parameters. It is far from trivial how to estimate the parameters in general. Although one can repeat the simulations with different parameters until the model is calibrated, this is often very time consuming.

7. Discussion and conclusions

In this paper we considered the intensity based default contagion model (3.1), where the default intensity of one firm is allowed to change when other firms default. The model was reinterpreted in terms of a Markov jump process, and this reinterpretation made it possible to derive closed form expressions for k^{th}-to-default spreads. With the computational resources available to us these expressions were tractable for general portfolios with up to 15 obligors. These are much larger than the general examples treated by other authors.

We used a synthetic telecom portfolio with 15 companies taken from `Reuters` at August 23, 2005 in a numerical study of how default contagion influences k^{th}-to-default spreads. For this we performed a "semi-calibration" of the portfolios where interaction parameters, interest rates and recovery rates were assumed obtained from prior knowledge but where the baseline default intensities were calibrated to market CDS spreads. The questions we tried to illustrate were:

How did the size of the portfolio influence k^{th}-to-default spreads? In our example, the 1^{th}-to-default spread increased by about 50% when the portfolio size increased from 10 to 15 and by about 700% for a 5^{th}-to-default spread. For a portfolio of size 10 the prize of a 1^{st}-to-default spread was about 4500 times higher than for a 5^{th}-to-default spread and for a portfolio of size 15 the spread was about 1100 times higher. Qualitatively this is completely as expected. However it would seem rather impossible to guess the sizes of the effects without computation.

How were k^{th}-to-default spreads influenced by the strength of interaction in the portfolio? We considered two examples. In those the 1^{th}-to-default spread decreased when the interaction became stronger, and the higher order than 2^{nd}-to-default spreads decreased. In the first example there was some indication that 2^{nd}-to-default spreads first increased and then decreased, and this was clear in the second example. This last finding may be somewhat counterintuitive. A possible intuitive explanation may be that as the interaction increases, the 2^{nd}-to-default will have a behavior more like the 1^{th}-to-default spread. This view is also supported by nothing that when the interaction level is extremely big, all swap spreads tend to converge into a narrow interval, compared with the case with very small interaction, see e.g. Figure 3. The results also illustrate the error made if one assumes independence in cases where there in fact is default contagion.

How did k^{th}-to-default spreads depend on the underlying market CDS spreads? The k^{th}-to-default spreads increased smoothly with increasing market spreads. The increase was greater for larger k-s. This first result was complectly as expected, and the second one perhaps slightly less obvious.

How were spreads affected by non-homogeneities in the recovery rates? The spreads were virtually unchanged by moderate inhomogeneities in the recovery rates. This agrees with earlier findings for single-name CDS-s, see e.g. [33].

Does approximation of a non-homogeneous portfolio with a homogenous one work well? In our example the approximation worked quite well. The interest of this comparison is that the computational burden for a homogenous portfolios is very much smaller than for non-homogeneous ones.

As a general comment, qualitatively most of the results summarized above were as was expected beforehand. However, it seems difficult to guess the sizes of the effects without actually doing the computations.

How can we estimate the dependence structure ? There are several possible ways to estimate, or calibrate the dependence matrix θ. One approach is to use historical time series data on the traded CDS spreads and consider the quadratic covariance process between the obligors i and j to get an indication of $\theta_{i,j}$. A similar procedure can also be done on the corresponding bonds. Another approach is to estimate equity correlation and use this as a proxy for default correlation. To be more specific, using Proposition 3.1 it is straightforward to find computational tractable closed-form expression for pairwise default correlation (i.e. $\mathrm{Corr}(1_{\{\tau_i \leq t\}}, 1_{\{\tau_j \leq t\}})$) as function of time, the matrix θ, c and the baseline intensities a_i. These analytical expressions for the default correlations can be used to extract θ from the numerical values on the corresponding equity correlations. However, further assumptions have to be done since the equity correlations are only half as many as the entities in the matrix θ. Using equity correlation as a proxy for default correlation has previously been very common when modelling default dependencies, see e.g. [4], [36], [42], [43], [44] and [49]. For example, in [42], the authors value k^{th}-to-default swaps by using two different copulas for a portfolio with 6 obligors. First, a one-factor Gaussian copula is used where the six correlations are estimated from equity returns. The one-factor Gaussian copula lacks correlations in the tails and [42] redoes the computations with a Clayton copula where the lower tail dependence is estimated from equity returns by using Kendall's tau on this data. Similar techniques are also pursued in e.g. [4] and [43].

A third approach to extract θ is first to estimate default correlation in an intensity based model from historical corporate data under the statistical probability measure. This is done in e.g. [11], [37] and [38]. Then, if we know the relationship between the statistical probability measure and the risk-neutral martingale measure, this can be used to extract θ. Determining the relationship between these two measures is equivalent to finding the connection between the intensities when changing the measures. Such a procedure is far from trivial and a discussion of how this can be done is given in e.g. [5], [6] and [59].

Finally, as discussed above the approach of this paper can be generalized to time-dependent and stochastic intensities. Further, it also gives a possibility to price other products, such as CDO-s. This will be presented in further papers by the first author.

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8. Appendix

In this appendix we display the recovery rate cases and the dependence structures used in Section 5.

Table 6: Different recovery rate cases used in Table 4 and Table 5. Recoveries expressed in %.

ϕ_1	ϕ_2	ϕ_3	ϕ_4	ϕ_5	ϕ_6	ϕ_7	ϕ_8	ϕ_9	ϕ_{10}	ϕ_{11}	mean	std
36	43	45	37	41	44	38	37	43	39	43	40.55	3.24
33	47	46	34	44	41	36	39	42	39	46	40.64	4.90
32	48	45	34	42	41	29	39	51	41	47	40.82	6.93
27	52	41	31	45	31	29	35	52	41	61	40.45	11.10

Table 7: The θ matrix describing the alternative dependence structure, rounded to two decimal places.

i	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$
1	0	3.03	5.13	2.5	4.69	6.36	5.69	6.7	1.32
2	3.84	0	2.96	3.49	1.41	4.18	4.27	1.03	3.44
3	1.84	5.62	0	3.04	1.92	2.31	4.91	6.09	2.87
4	4.18	0.59	0.51	0	4.39	3.35	0.65	5.39	3.25
5	0.35	6.62	3.88	4.32	0	4.18	2.98	3.11	4.28
6	4	6.41	2.05	0.8	0.42	0	2.63	4.35	0.5
7	4.91	4.22	6.01	6.29	0.63	5.81	0	6.66	2.2
8	6.74	1.78	2.35	5.28	1.9	6.69	5.83	0	4.26
9	5.25	6.11	4.76	5.54	2.87	4.17	5.87	1.74	0

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Table 8: The θ matrix describing the dependence structure, rounded to two decimal places. This matrix is used in all examples except in Figure 3 to Figure 7.

i	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$	$j=8$	$j=9$	$j=10$	$j=11$	$j=12$	$j=13$	$j=14$	$j=15$
1	0	5.68	0.11	0.12	0.13	0.16	0.11	0.16	0.13	0.1	0.12	0.12	0.13	0.14	0.15
2	0.15	0	2.91	2.22	0.14	0.2	0.16	0.11	0.2	0.1	0.15	0.11	0.14	0.19	0.14
3	0.13	0.16	0	0.11	4.61	5.21	0.18	0.16	0.11	0.13	0.13	0.15	0.19	0.14	0.16
4	2.5	4.79	0.17	0	0.14	0.13	2.11	1.83	0.14	0.16	0.17	0.12	0.17	0.11	0.19
5	0.1	0.13	0.18	0.18	0	0.2	0.16	0.16	3.87	1.29	0.13	0.17	0.2	0.11	0.19
6	0.1	0.13	0.16	0.2	0.19	0	0.14	0.16	0.11	0.17	1.87	3.28	0.17	0.17	0.12
7	0.19	0.19	0.12	0.13	0.2	0.17	0	0.14	0.1	0.17	0.1	0.18	2.42	2.8	0.15
8	1.57	0.17	2.8	0.15	2.94	0.17	3.36	0	2.04	0.16	2.6	0.18	1.19	0.18	1.38
9	0.18	2.46	0.11	2.76	0.13	2.5	0.14	0.9	0	2.3	0.13	2.22	0.16	1.92	0.15
10	1.62	2.45	1.29	0.12	1.76	2.47	1.87	0.18	0.13	0	1.72	2.6	1.44	0.14	0.18
11	1.59	1.52	0.12	1.28	1.05	0.87	0.13	0.71	0.11	2.15	0	0.15	0.13	0.58	1.4
12	0.1	0.11	0.84	0.12	1.08	0.14	0.8	0.18	1.51	0.14	0.16	0	0.2	0.15	1.14
13	0.19	0.69	0.12	0.41	0.12	0.19	0.11	0.64	0.17	1.32	0.12	0.17	0	0.76	0.11
14	0.5	0.15	0.94	0.16	0.27	0.2	0.74	0.12	0.18	0.12	0.39	0.13	0.7	0	0.16
15	0.2	0.33	0.16	0.16	0.1	0.4	0.15	0.19	0.14	0.14	0.15	0.57	0.14	0.17	0