Optimal discrete hedging and approximation of stochastic integrals

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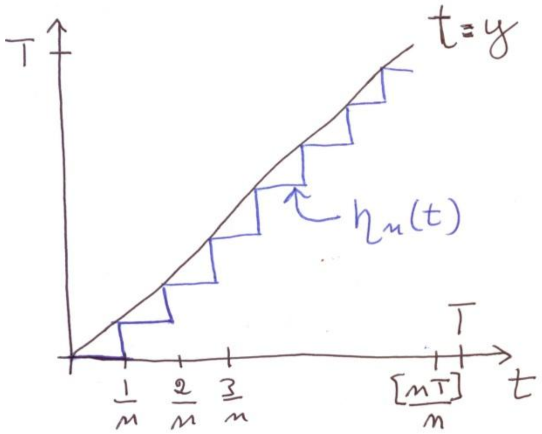
The Euler method



$$\int_0^t f\,dB = \int_0^t f(B_s, s)\,dB_s$$

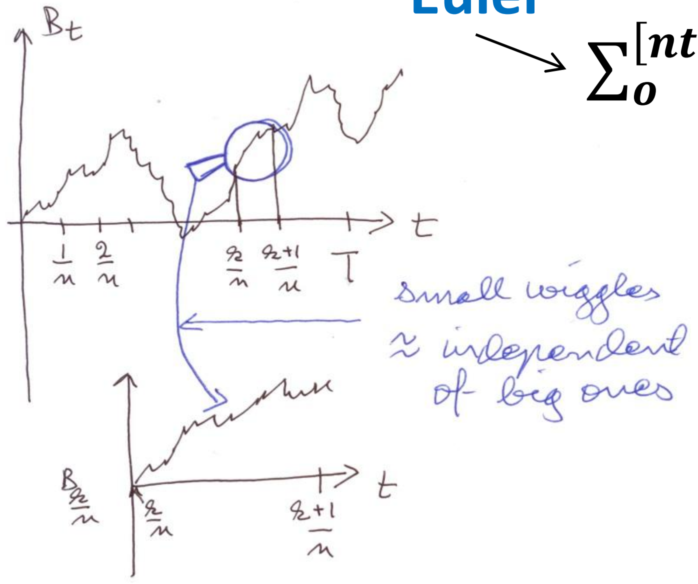
$$\tau_0 = \frac{0}{n}, \tau_1 = \frac{1}{n}, \ldots \tau_{[nT\}} = \frac{[nT]}{n}$$

$$\eta_n(t) = \tau_k; \tau_k \le t < \tau_{k+1}$$

Euler

$$\sum_o^{[nt]} f(\tau_i)(B_{\tau_{i+1}} - B_{\tau_i}) + f(\frac{[nt]}{n})(B_t - B_{\frac{[nt]}{n}})$$

$$= \int_0^t f \circ \eta_n\,dB$$



$$\boxed{\sqrt{n}\int_0^t (f - f \circ \eta_n)\,dB = \text{error} \Rightarrow \frac{1}{\sqrt{2}}\int_0^t f'\,dW}$$

Intuition: $\int_{k/n}^{k+1/n}(f - f\circ\eta_n)dB$

$$\approx f'(B_{\frac{k}{n}}) \int_{k/n}^{k+1/n} \left(B_s - B_{\frac{k}{n}}\right) dB$$

Thm (-80)

$$\left\{ \begin{array}{l} \lim\limits_{n\to\infty} \sup \left| \int_0^t \psi_n(s)ds \right| \to 0 \\ \lim\limits_{n\to\infty} \int_0^t \psi_n(s)^2\, ds \xrightarrow{P} \int_0^t \psi(s)^2\, ds \end{array} \right.$$

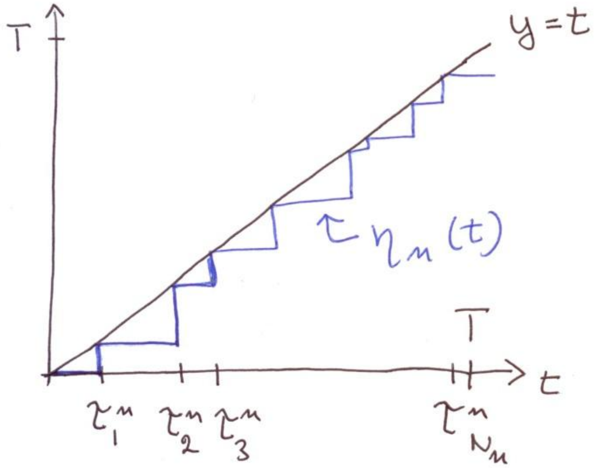
$$\Rightarrow$$

Independent!

$$\int_0^t \psi_n(s)\, dB_s \Rightarrow \int_0^t \psi(s)\, dW_s$$

Renyi-mixing
Renyi-stability

Variable meshes – variable interval hedging



$$\boxed{\tau_0 = 0; \tau_{k+1} = \tau_k + \frac{1}{n\theta(\tau_k)}}$$

$$N_n \approx n \int_0^T \theta(t)\,dt$$

= # of interventions

= # of mesh points

$$\boxed{\text{Error} = \sqrt{n}\int_0^t (f - f \circ \eta_n)\,dB \Rightarrow \int_0^t \frac{f'}{\sqrt{2\theta}}\,dW}$$

Independent!

$$\boxed{\text{Error} = \sqrt{n}\int_0^t (f - f\circ\eta_n)\,dB \Rightarrow \int_0^t \frac{f'}{\sqrt{2\theta}}\,dW}$$

No bad days

$$\theta(t) = f'^2 \qquad \Rightarrow \qquad \boxed{\begin{array}{l} \text{error} \approx \dfrac{1}{\sqrt{2n}}\,W(t) \sim \mathrm{N}\left(0, \dfrac{t}{2n}\right) \\[2em] E(N_n) \approx n\displaystyle\int_0^T E(f'^2)\,dt \end{array}}$$

$E(\text{Error}^2) \approx \frac{1}{n}\int_0^T E\frac{f'^2}{2\theta}\,\text{dt},$ $\text{E}N_n \approx n\int_0^T E\theta(t)dt$

Minimum variance:

Minimize $E(\text{Error}^2)$ under the restriction $\text{EN} \leq \text{C}$

Optimal: $E(\text{Error}^2) = \frac{1}{2C}\left(\int_0^T E|f'|\text{dt}\right)^2$

$\theta = |f'|,$ $EN = C = n\int_0^T E|f'|(t)dt$

(Proof: use Cauchy-Schwarz several times)

$$\boldsymbol{E}(|\mathbf{error}|) \approx \frac{\mathbf{1}}{\sqrt{\mathbf{2}\boldsymbol{n}}}\boldsymbol{E}\sqrt{\int_{\mathbf{0}}^{\boldsymbol{T}} \frac{\boldsymbol{f'}^{\mathbf{2}}}{\mathbf{2}\boldsymbol{\theta}}\,\mathbf{dt}}, \qquad \mathbf{E}\boldsymbol{N}_{\boldsymbol{n}} \approx \boldsymbol{n}\int_{\mathbf{0}}^{\boldsymbol{T}} \boldsymbol{E\theta}(\boldsymbol{t})\boldsymbol{dt}$$

Minimum standard deviation – not solved (but ”min var” better than ”no bad day”)

An extension

$B = (B_i; 1 \le i \le d)$ Brownian motion, $\{H^n_{i,j}\}$ adapted

$$\{H^n_{i,j} \bullet B^i\} = \{\int_0^t H^n_{i,j}\, dB^i,\ \ 1 \le i \le d,\ \ 1 \le j \le k_i,\ \ 0 \le t \le T\}$$

$$sup_{0 \le t \le T} \left| \int_0^t H^n_{i,j}\, \mathrm{ds} \right| \to_P 0, \qquad \int_0^t (H^n_{i,j})^2\, \mathrm{ds} \to_P \int_0^t (H_{i,j})^2\, \mathrm{ds}$$

$$\Rightarrow$$

$$\boxed{\{H^n_{i,j} \bullet B^i\} \Rightarrow_S \{H_{i,j} \bullet W^i\}}$$

$W = (W_i; 1 \le i \le d)$
Brownian motion
independent of B

Extension to ct. Brownian semimartingale immediate

SDE, variable meshes

$\alpha, \beta, \boldsymbol{f}$: $\mathrm{R}^{d} \to \mathrm{R}^{d}$, β: $\mathrm{R}^{d} \to \mathrm{R}^{dxd}$ $\mathrm{B(t)} : \mathrm{R} \to \mathrm{R}^{d}$

$$dY(t) = \alpha(Y(t))dt + \beta(Y(t))dB(t)$$

$$\boxed{\begin{aligned} &\sqrt{n}\int_0^t (f(Y(s)) - f(Y \circ \eta_n(s))\mathrm{dY(s)} \\ &\qquad \Rightarrow \sum_{k,i=1}^{d} \int_0^t d_{k,i}(s)\,dW_i(s) \end{aligned}}$$

(:R → R)

$W = (W_i; 1 \leq i \leq d)$
Brownian motion independent of B

$$d_{k,i}(s) = \sum_{m,j=1}^{d} \frac{\frac{\partial f_m}{\partial y_j}(Y(t)\beta_{m,k}(Y(t)\beta_{j,i}(Y(t)}{\sqrt{2\theta(t)}}$$

Black-Scholes, European call option, strike price K

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$$
$$dR(t) = rR(t)dt$$

No bad days:

$$\theta(t) = \varphi\left(\frac{\frac{\log S(t)}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)^2 \sigma^2 S(t)^4/(K^2\,(T-t))$$

Minimum variance:

$$\theta(t) = \varphi\left(\frac{\frac{\log S(t)}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \sigma\, S(t)^2/(K\sqrt{T-t})$$

- Portfolio hedging – compare simultaneous hedging with individual hedging

- Restrict intervals between hedging

- Simulation to investigate properties straightforward
  -- simulation error can be evaluated

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