Error distributions for random grid approximations of multidimensional stochastic integrals

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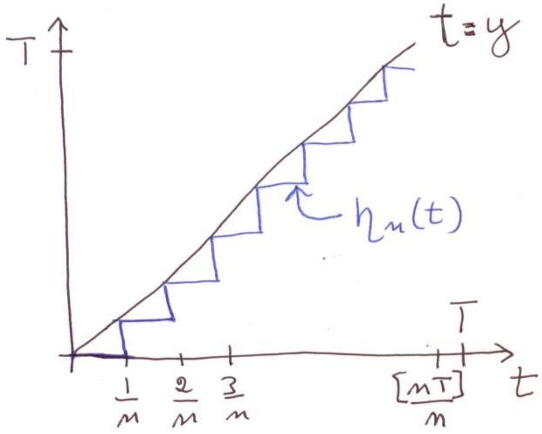
The Euler method for $\int_0^t f\,dB \; = \int_0^t f(B_s)\,dB_s$



the grid:

$$\tau_0 = \frac{0}{n}, \tau_1 = \frac{1}{n}, \ldots \tau_{[nT\}} = \frac{[nT]}{n}$$

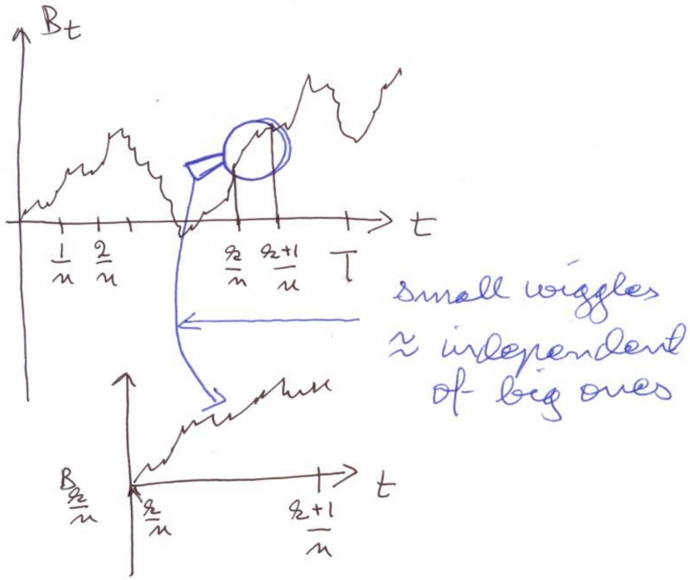
$$\eta_n(t) = \tau_k ; \tau_k \leq t < \tau_{k+1}$$

Euler: $\sum_{o}^{[nt]} f(B_{\tau_k})(B_{\tau_{k+1}} - B_{\tau_k}) + f(B_{[nt]/n})(B_t - B_{[nt]/n})$

$$= \int_0^t f \circ \eta_n \, dB$$

Independent!

error = $\sqrt{n} \int_0^t (f - f \circ \eta_n)\,dB \Rightarrow \frac{1}{\sqrt{2}} \int_0^t f'(B_s)\,dW_s$



Intuition: $\int_{k/n}^{k+1/n} (f - f \circ \eta_n) dB$

$$\approx f'(B_{\frac{k}{n}}) \int_{k/n}^{k+1/n} \left(B_s - B_{\frac{k}{n}}\right) dB_s$$

Thm

$$\begin{cases} \lim\limits_{n\to\infty} \sup_t \left| \int_0^t \psi_n(s)ds \right| \to 0 \\ \int_0^t \psi_n(s)^2 \, ds \xrightarrow{P} \int_0^t \psi(s)^2 \, ds \end{cases}$$

$$\Rightarrow$$

independent!

$$\int_0^t \psi_n(s) \, dB_s \Rightarrow \int_0^t \psi(s) \, dW_s$$

Renyi-stability:

$\Rightarrow$ denotes weak convergence, usually in $C([0,T],R^K)$

Renyi-stable convergence of X_n to X if one of following:

- $X_n \Rightarrow X$, with respect to $P(\cdot\,|A)$ for any A with $P(A) > 0$
- $E(U\,g(X_n)) \to E(Ug(X))$ for any bounded continuous g and $U > 0$ with $E(U) = 1$
- $(U_n, X_n) \Rightarrow (U, X)$ for any sequence $U_n \underset{P}{\to} U$

Renyi-mixing convergence if U and X independent. Holds if limit in second characterization is $E(g(X))$

PF: the assumption was

$$\lim_{n\to\infty} \sup_t \left| \int_0^t \psi_n(s)ds \right| \to 0$$

$$\tau_n(t) = \int_0^t \psi_n(s)^2\, ds \xrightarrow{P} \int_0^t \psi(s)^2\, ds$$

Define probability Q by $dQ/dP = U$. By Girsanov theorem and Lévy time-change

$$E\left(Uf\left(\int_0^{\tau_n^{-1}(\cdot)} \psi_n\, dB\right)\right) = E_Q\left(f\left(\int_0^{\tau_n^{-1}(\cdot)} \psi_n\, d\bar{B} + \int_0^{\tau_n^{-1}(\cdot)} \psi_n c\, ds\right)\right)$$

$$\to E(f(W))$$

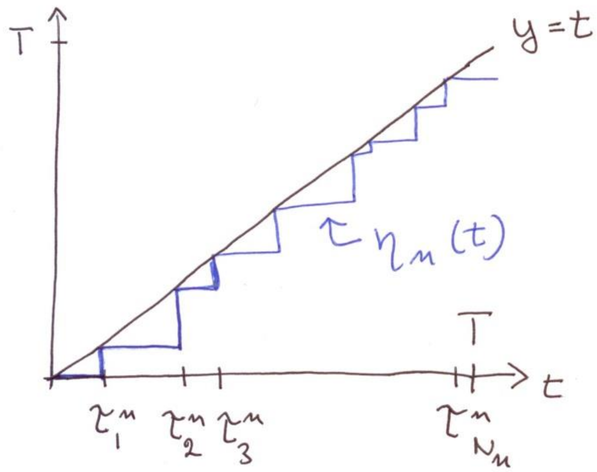
Hence $\int_0^{\tau_n^{-1}(\cdot)} \psi_n\, d\bar{B} \Rightarrow W$ mixing. Reversing the time change shows that $\int_0^t \psi_n(s)\, dB_s \Rightarrow W(\tau(t)) =_d \int_0^t \psi(s)\, dW_s$

Kurz, Protter (1991a, 1991b, 1996), Jacod, Protter (1998)
necessary and sufficient conditions for convergence of error in Euler method for d-dimensional SDE-s

Main effort in proving tightness, stable convergence important tool, approximation error obtained as solution to a SDE

Gobet, Tenam (2001), Hayashi, Mykland (2005), Fukusawa (2011), Lindberg, R (2013)
Random grids, multidimensional, inspired by heding errors in mathematical finance.

Variable meshes – variable interval hedging



$$\tau_0 = 0; \tau_{k+1} = \tau_k + \frac{1}{n\theta(\tau_k)}$$

$$N_n \approx n \int_0^T \theta(t)dt$$

= # of grid points

= # of interventions

$$\text{error} = \sqrt{n}\int_0^t (f - f \circ \eta_n)dB \Rightarrow \int_0^t \frac{f'}{\sqrt{2\theta}}dW$$

Independent!

$$\text{error} = \sqrt{n}\int_0^t (f - f\circ\eta_n)\,dB \Rightarrow \int_0^t \frac{f'}{\sqrt{2\theta}}\,dW$$

$$\text{cost} = CN_n \approx Cn\int_0^T \theta(t)\,dt$$

No bad days strategy:

$$\theta(t) = \frac{f'^2}{c^2} \quad \Rightarrow$$

$$\text{error} \approx \frac{c}{\sqrt{2n}}\,W(t) \sim \mathrm{N}\!\left(0, \frac{c^2 t}{2n}\right)$$

$$\text{cost} \approx C\frac{n}{c^2}\int_0^T f'^2\,dt$$

$$E(\text{error}^2) \approx \frac{1}{n}\int_0^T E\frac{f'^2}{2\theta}\,\mathrm{dt}, \qquad E(\text{cost}) = EN_n \approx n\int_0^T E\theta(t)\,dt$$

Minimum variance:

Minimize $E(\text{error}^2)$ under the restriction EN$\leq$ C

Optimal: $$E(\text{error}^2) = \frac{1}{2C}\left(\int_0^T E|f'|\,\mathrm{dt}\right)^2$$

$\theta = C|f'|$, $$E(\text{cost}) = C = n\int_0^T E|f'|(t)\,dt$$

(Proof: use Cauchy-Schwarz several times)

Multivariate result

$B = (B_k; 1 \le k \le d)$ Brownian motion, $\{H^n_{i,k}\}$ adapted

$\{H^n_{i,k} \bullet B_k\} = \{\int_0^t H^n_{i,k}\, dB_k,\ \ 1 \le i \le d,\ \ 1 \le k \le d,\ \ 0 \le t \le T\}$

$\sup_{0 \le t \le T} | \int_0^t H^n_{i,k}\, \mathrm{ds}| \to_P 0,$ **Covariation converges**

$\Rightarrow$

$$\{H^n_{i,k} \bullet B_k\} \Rightarrow_S \{\sum_{j=1}^d H_{i,k}\ \sigma^k_{i,j} \bullet W_{j,k}\}$$

square root of "limit correlation matrix"

W **Brownian motion independent of B**

Extension to continuous Brownian semimartingales

SDE, variable meshes

$\alpha, f: \mathrm{R}^d \to \mathrm{R}^d,\ \beta: \mathrm{R}^d \to \mathrm{R}^{dxd}\ \ B(t): \mathrm{R} \to \mathrm{R}^d$

$$dY(t) = \alpha(Y(t))dt + \beta(Y(t))dB(t)$$

$$\sqrt{n}\int_0^t (f(Y(s)) - f(Y \circ \eta_n(s))\mathrm{d}Y(\mathrm{s})$$

$$\Rightarrow \sum_{r,k=1}^{d} \int_0^t d_{r,k}(s)dW_{r,k}(s)$$

$(: \mathrm{R} \to \mathrm{R})$

$W = (W_{r,k}; 1 \le r, k \le d)$
Brownian motion independent of B

$$d_{r,k}(s) = \sum_{i,j=1}^{d} \frac{\frac{\partial f_j}{\partial y_j}(Y(t)\beta_{i,r}(Y(t)\beta_{j,k}(Y(t)}{\sqrt{2\theta(t)}}$$

Black-Scholes, European call option, strike price K

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t)$$
$$dR(t) = rR(t)dt$$

No bad days:

$$\theta(t) = \frac{1}{c^2}\varphi\left(\frac{\frac{\log S(t)}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)^2 \sigma^2 S(t)^2/(2(T-t))$$

Minimum variance:

$$\theta(t) = C\,\varphi\left(\frac{\frac{\log S(t)}{K}+\left(r+\frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\sigma\, S(t)/(2\sqrt{T-t}))$$

(Technical difficulty at $t = T$)

New problems

SPDE-s

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