Adv. Appl. Prob. **20**, 371–390 (1988)
Printed in N. Ireland

MAXIMA AND EXCEEDANCES OF STATIONARY MARKOV CHAINS

HOLGER ROOTZÉN,* *University of Copenhagen*

Abstract

Recent work by Athreya and Ney and by Nummelin on the limit theory for Markov chains shows that the close connection with regeneration theory holds also for chains on a general state space. Here this is used to study extremal behaviour of stationary (or asymptotically stationary) Markov chains. Many of the results center on the 'clustering' of extremes of adjacent values of the chains. In addition one criterion for convergence of extremes of general stationary sequences is derived. The results are applied to waiting times in the $GI/G/1$ queue and to autoregressive processes.

REGENERATIVE PROCESSES; CLUSTERING OF EXTREME VALUES; POINT PROCESS OF EXCEEDANCES; STATIONARY SEQUENCES; GI/G/1 QUEUE; AUTOREGRESSIVE PROCESSES

1. Introduction

Although stationary Markov chains are important both from the applied and theoretical points of view, their extremal behaviour has been comparatively little studied. A basic (albeit elementary) observation was, however, made early; implicitly in [3], [4], and explicitly in [1], and was further developed in [20]. This is that if the Markov chain is regenerative then parts of the extreme value theory for independent identically distributed sequences carry over in a straightforward way. Recent advances in the limit theory for Markov chains (briefly reviewed in Section 2 below) have given this observation much wider applicability, and we will use it as a starting point here. Some further scattered results connected with extremes of stationary Markov chains are given in [6]–[10].

Specifically, a sequence $\{Z_t; t \geqq 0\}$ with values in a measurable space $(E, \mathscr{E})$ is regenerative if there exist integer-valued random variables $0 < S_0 < S_1 < \cdots$ which split the sequence up into independent 'cycles', $C_0, C_1, \cdots$, i.e. if

$$C_0 = \{Z_t; 0 \leqq t < S_0\}, \quad C_1 = \{Z_{t+S_0}; 0 \leqq t < S_1 - S_0\},$$
$$C_2 = \{Z_{t+S_1}; 0 \leqq t < S_2 - S_1\}, \cdots \tag{1.1}$$

are independent and if in addition $C_1, C_2, \cdots$ have the same distribution. Clearly $\{S_k\}_{k=0}^{\infty}$ then is a renewal process, i.e. $Y_0 = S_0$, $Y_1 = S_1 - S_0$, $Y_2 = S_2 - S_1, \cdots$ are independent and $Y_1, Y_2, \cdots$ are identically distributed. We also need the more

Received 20 March 1985; revision received 7 April 1987.

* Present address: Department of Mathematical Statistics, University of Lund and Lund Institute of Technology, Box 118, S-221 00 Lund, Sweden.

general concept that $\{Z_t\}$ is 1-dependent regenerative if there exists a renewal process $\{S_k\}_{k=0}^{\infty}$ as above, which splits $\{Z_t\}$ up into 1-dependent cycles $C_0, C_1, \cdots$, with $C_1, C_2, \cdots$ forming a stationary sequence. Thus adjacent cycles might be dependent, while cycles separated by at least one cycle are independent. Now, suppose $\{Z_t\}$ is real-valued and regenerative, let $M_n = \max\{Z_t; 0 < t \leqq n\}$, write $\zeta_0 = \sup\{Z_t; 0 \leqq t < S_0\}$ and $\zeta_1 = \sup\{Z_t; S_0 \leqq t < S_1\}$, $\zeta_2 = \sup\{Z_t; S_1 \leqq t < S_2\}, \cdots$ for the 'submaxima' over cycles, and let $v_t = \inf\{k; S_k > t\}$. If $\mu = EY_1 < \infty$ then by the law of large numbers $v_t/t \to 1/\mu$ a.s., and M_n is easily approximated by $\max\{\zeta_0, \cdots, \zeta_{v_n}\}$, which in turn can be approximated by $\max\{\zeta_1, \cdots, \zeta_{[n/\mu]}\}$, cf. Theorem 3.1 below. Thus asymptotically $P(M_n \leqq x)$ equals $G(x)^n$, where $G(x) = P(\zeta_1 \leqq x)^{1/\mu}$. Since G is a distribution function (d.f.), it follows at once that the extremal types theorem applies so that the only possible limit laws of $a_n(M_n - b_n)$ are the three extreme value distributions. Further, the i.i.d. criteria for convergence to each of the possible limit distributions (see [15], pp. 10, 16, 17) can be applied to G.

It is of course easy to find, say, the limiting distribution of the kth largest of the ζ_t's or of the point process of large ζ_t-values. However, it is entirely possible that there is a strong dependence between large values within a cycle, so that a large ζ_t-value might correspond to a small cluster of large Z_t's in the same cycle and then these results for the ζ_t's do not translate directly to the Z_t's. A further problem is that often the tail of $G(x) = P(\zeta_1 \leqq x)^{1/\mu}$ is hard to find, while sometimes the marginal d.f. of the Z_t's themselves is more accessible. It turns out that this problem also is solved if one knows the 'degree of clustering', since then knowledge of the tails of either one of ζ_1 or Z_1 is enough to determine the other.

As mentioned above, Section 2 of this paper contains a brief account of recent developments on regeneration and Markov chain limit theory. Motivated by this, a detailed study of extremes of regenerative processes, with emphasis on clustering behaviour, is given in Section 3 together with a brief discussion of the 1-dependent case. Clustering has already been studied for moving average processes in [18], [19] and for general stationary processes in [11], [14]. Parts of the results in Section 3 are closely related to results in [14]. Section 4 contains some criteria for convergence of extremes of general stationary processes, which are of a form which seems particularly appropriate for Markov chains. The case where no clustering occurs is of special interest, and is singled out for study in Section 5, leading in particular to a criterion of a 'martingale-like' flavour. In Section 6 the results are applied to a queueing problem and to autoregressive processes.

2. Limit theory for Markov chains on a general state space

In this section we introduce some notation and for later reference, briefly discuss recent developments on the connection between asymptotic stationarity of Markov chains and regeneration, closely following [2]. Let $(E, \mathscr{E})$ be a measurable space. The E-valued sequence $\{X_t; t=0, 1, \cdots\}$ is a Markov chain with (stationary) transition probabilities $P(\cdot, \cdot)$ if $P(\cdot, \cdot)$ is a Markov kernel such that

$$P(X_{t+r} \in A \mid X_t, \cdots, X_0) = P_r(X_t, A) \quad \text{a.s.},$$

for $r \geqq 1$, $t=0, 1, \cdots$, and $A \in \mathscr{E}$, where the $P_r(\cdot, \cdot)$ are given recursively by $P_r(x, A) = \int P_{r-1}(y, A)P(x, dy)$, for $P_1(\cdot, \cdot) = P(\cdot, \cdot)$. The distribution of $\{X_t\}$ is determined by the transition probabilities and the initial distribution, i.e. the distribution of X_0. We often write P_λ for the distribution of the chain with initial probability distribution λ, and P_x if λ gives probability 1 to the set $\{x\}$, and E_λ or E_x for the corresponding expectations. A probability π is stationary for $P(\cdot, \cdot)$ if $\pi(A) = \int P(x, A)\pi(dx)$, for all $A \in \mathscr{E}$ and then $\{X_t\}$ is strictly stationary under P_π. Throughout this paper, P_π denotes such distributions. Further, writing $\tau(R) = \inf\{t \geqq 1; X_t \in R\}$ for the first time X_t enters R, the set $R \in \mathscr{E}$ is recurrent if $P_x(\tau(R) < \infty) = 1$ for all $x \in E$.

A set $R \in \mathscr{E}$ is a *regeneration set* if it is recurrent and if there exist $r > 0$, $\varepsilon \in (0, 1)$ and a probability λ on $\mathscr{E}$ such that

$$(2.1) \qquad P_r(x, A) \geqq \varepsilon\lambda(A), \qquad \forall x \in R, A \in \mathscr{E}.$$

It can be shown that $\{X_t\}$ has a regeneration set if and only if it is Harris recurrent ([2], Section VI.3). There are two main situations when a regeneration set exists:

(i) When there is a recurrent one-point set $\{x_0\}$ (one can then take $r=1$, $\varepsilon=1$, $R=\{x_0\}$, and $\lambda(A) = P(x_0, A)$).

(ii) When, for some $r > 0$, a transition density $f_r(\cdot, \cdot)$ exists (i.e. when $P_r(x, dy) = f_r(x, y)\mu(dy)$ for some measure μ) together with a recurrent set R and a set S with $0 < \mu(S) < \infty$ such that $f_r(x, y) \geqq \varepsilon > 0$ for any $x \in R$, $y \in S$.

If $\{X_t\}$ has a regeneration set, it can be constructed simultaneously with a renewal process $\{S_k\}$ which makes $\{X_t\}$ regenerative if $r=1$ and 1-dependent regenerative if $r \neq 1$. Loosely the idea is to let a renewal occur with probability ε with r time units delay after a visit to R and then to restart the process with initial distribution λ. The remaining part of the construction ([2], p. 151) is to patch together the rest of the X_t-process to make it have the right distribution—of course (2.1) is essential for this. Below, when we discuss a Markov chain $\{X_t\}$ with a regeneration set, $Y_0, Y_1, \cdots$ will always be the intervals between the renewals $\{S_k\}$ obtained by this construction. Clearly, under P_λ the entire sequence $Y_0, Y_1, \cdots$ has the same distribution, so that in particular $E_\lambda(Y_0^\alpha) = E_\lambda(Y_k^\alpha)$, for $k=1, 2, \cdots$. Further we will say that $\{X_t\}$ is aperiodic if $P_\lambda(Y_0 \in \{d, 2d, \cdots\}) = 1$ only for $d=1$. The main result of [2], Section VI is that the following three conditions are equivalent for a Markov chain $\{X_t; t=0, 1, \cdots\}$ which has a regeneration set and is aperiodic:

(i) $E_\lambda(Y_0) = \mu < \infty$,

(ii) there exists a (necessarily unique) stationary initial probability distribution π,

(iii) the P_x-distribution of $(X_n, X_{n+1}, \cdots)$ converges in total variation to the P_π-distribution of $(X_1, X_2, \cdots)$.

If (i), (ii), or (iii) holds, then, for any real measurable function f,

$$\begin{aligned} E_\pi(f(X_0, X_1, \cdots)) &= E_\lambda\left\{\sum_{k=0}^{Y_0-1} f(X_k, X_{k+1}, \cdots)\right\}\Big/E_\lambda(Y_0) \\ &= E_\lambda\left\{\sum_{k=1}^{Y_0} f(X_k, X_{k+1}, \cdots)\right\}\Big/E_\lambda(Y_0). \end{aligned} \tag{2.2}$$

For later use, we note that this equivalence and (2.2) holds also for non-Markovian (1-dependent) aperiodic regenerative processes, if P_λ and E_λ are replaced by P_0 and E_0, the probability and expectation in the 'zero-delayed' case, when the first cycle, C_0, has the same distribution as all the other cycles. In this, P_π is the unique distribution which makes $\{X_t\}$ stationary, without changing the distribution of $C_1, C_2, \cdots$.

Returning to the Markov case, there is a sizeable literature on moments of $\tau(R) = \inf\{t \geqq 1; X_t \in R\}$, see e.g. [21], but unfortunately that a moment of $\tau(R)$ is finite does not in general ensure that this moment of Y_0 is finite, since Y_0 also includes the 'probability ε randomization'. Here we will list three criteria for moments of Y_0. The first one includes the case when R is a one-point set and is completely obvious, since $\varepsilon = 1$ means that $Y_0 = \tau(R)$. The proofs of the other two criteria are relegated to the appendix.

(2.3) If $\varepsilon = 1$ then $E_\lambda(\tau(R)^\alpha) < \infty$ if and only if $E_\lambda(Y_0^\alpha) < \infty$.

(2.4) If $E_\lambda(\tau(R)^\alpha) < \infty$ and $E_x(\tau(R)^\alpha)$ is uniformly bounded for $x \in R$, then $E_x(Y_0^\alpha) < \infty$.

(2.5) If the set R is a regeneration set for $\lambda = \pi_R = \pi((\cdot) \cap R)/\pi(R)$ then $E_{\pi_R}(\tau(R)^\alpha) < \infty$ if and only if $E_{\pi_R}(Y_0^\alpha) < \infty$.

The theory described above applies, with sums replaced by integrals, also to continuous-parameter Markov chains. For this we throughout assume that in the continuous-parameter case $(E, \mathscr{E})$ is Polish, that the sample paths are in $D([0, \infty))$, and that the cycle length distribution satisfies Stone's condition on the existence of an absolutely continuous component.

3. Extremes of regenerative processes

In this section we first develop the extreme value theory for regenerative processes in some detail, and then briefly discuss the changes needed in the 1-dependent case. Since a function of a (1-dependent) regenerative process also is regenerative, this in particular applies to instantaneous functions $Z_t = f(X_t)$ of Markov chains $\{X_t\}$ with regeneration sets.

Since the distribution of the first cycle, C_0, in general is arbitrary, we need the condition

$$P(\zeta_0 > \max(\zeta_1, \cdots, \zeta_k)) \to 0, \quad \text{as} \quad k \to \infty, \tag{3.1}$$

to ensure that its effect on extremes is asymptotically unimportant. It is trivial to see that (3.1) holds if $\{Z_t\}$ is zero-delayed, since $\zeta_0, \zeta_1, \cdots$ then are i.i.d. and in the appendix we prove that (3.1) also holds if $\{Z_t\}$ is stationary. We now for completeness state a fairly general version of the results of [1], [3] and [4] which were discussed in the introduction. In this we also consider l-dimensional processes $\{Z_t\} = \{(Z_t^{(1)}, \cdots, Z_t^{(l)})\}$ with the coordinatewise ordering and maxima, so that e.g. $M_n = \max\{Z_1, \cdots, Z_n\} \leqq x$ if $\max\{Z_1^{(i)}, \cdots, Z_n^{(i)}\} \leqq x^{(i)}$ for $i = 1, \cdots, l$. Further, recall the notation $v_t = \inf\{k \geqq 0; S_k > t\}$ where $\{S_k\}$ is the renewal sequence associated with $\{Z_t\}$.

Theorem 3.1. Let $\{Z_t; t = 0, 1, \cdots\}$ be an l-dimensional regenerative process with $\mu = EY_1 < \infty$ and put $G(x) = P(\zeta_1 \leqq x)^{1/\mu}$. Then, for $\delta < 0$, with $\delta' = \delta + 1/n$,

$$(3.2) \qquad |P(M_n \leqq x) - G(x)^n| \leqq \mu\delta' + P(|v_n/n - 1/\mu| \geqq \delta) + P(\zeta_0 > \max\{\zeta_1, \cdots, \zeta_{[n(1/\mu+\delta)]}\}).$$

Hence if (3.1) holds then $\sup_x |P(M_n \leqq x) - G(x)^n| \to 0$ as $n \to \infty$.

The same result holds for a continuous-parameter regenerative process $\{Z_t; t \in [0, \infty)\}$, with M_n replaced by $M_T = \sup_{0 \leqq t \leqq T} Z_t$ and n replaced by T.

Proof. For $k_n = [n(1/\mu + \delta)]$ we have that

$$(3.3) \qquad \begin{aligned} P(M_n \leqq x) &\geqq P\Big(\max_{0 \leqq k \leqq v_n} \zeta_k \leqq x\Big) \\ &\geqq P\Big(\max_{1 \leqq k \leqq k_n} \zeta_k \leqq x\Big) - P(|v_n/n - 1/\mu| > \delta) \\ &\quad - P(\zeta_0 > \max\{\zeta_1, \cdots, \zeta_{k_n}\}) \\ &\geqq G(x)^{n+n\mu\delta} - P(|v_n/n - 1/\mu| > \delta) \\ &\quad - P(\zeta_0 > \max\{\zeta_1, \cdots, \zeta_{k_n}\}), \end{aligned}$$

since $\zeta_1, \zeta_2, \cdots$ are i.i.d. with d.f. $G(x)^\mu$. From the elementary inequality $|z(z^\gamma - 1)| \leqq \gamma/(1+\gamma) \leqq \gamma$, for $0 \leqq z \leqq 1$ and $0 \leqq \gamma$, applied with $z = G(x)^n$, $\gamma = \mu\delta$, it follows that $|G(x)^{n+n\mu\delta} - G(x)^n| \leqq \mu\delta$, and half of (3.2) thus follows from (3.3). The proof of the other half is similar, and partly simpler since ζ_0 does not have to be included in the right-hand side of (3.3), but instead the algebra is slightly more involved since one gets the exponent $n - n\mu\delta - \mu$ on $G(x)$ in (3.3). Next, $P(|v_n/n - 1/\mu| > \delta) \to 0$ by the law of large numbers, and using this and (3.1) in (3.2) shows that $\sup_x |P(M_n \leqq x) - G(x)^n| \to 0$, since $\delta > 0$ is arbitrary. The proof is the same in the continuous case.

As mentioned in the introduction, one immediate consequence is the extremal types theorem; if the conditions of Theorem 3.1, including (3.1), hold and $a_n(M_n - b_n)$ converges in distribution, for some constants $a_n > 0$, b_n, then the limit is an extreme value distribution. For cases when the tail of the distribution of ζ_1 can be found, the theorem gives detailed information on M_n. However, often the distribution of ζ_1 is inaccessible, and in addition one is often also interested in, say, the location of the maximum or in the distribution of the kth largest value. Below we study these problems by means of more general point process results, for the rest of the paper specializing to the one-dimensional case.

Perhaps the most basic object is the point process, N_n, of time-normalized exceedances of u_n by $\{Z_t\}$, defined by

$$N_n(A) = \#\{k/n \in A; Z_n > u_n\} \tag{3.4}$$

if $\{Z_t\}$ has discrete parameter, and the corresponding quantity if $\{Z_t\}$ has continuous parameter, viz. $N_T(A) = |\{t; t/T \in A \text{ and } Z_t > u_T\}|$, where $|\ |$ denotes the length (i.e. Lebesgue measure) of the set, for Borel sets $A \subseteqq [0, \infty)$. However, we first study a related point process, N'_n, which sometimes is easier to handle. Let r_n be a sequence of integers with

$$r_n \to \infty, \quad r_n = o(n), \quad \text{and} \quad nP(Y_1 > r_n) \to 0, \quad \text{as} \quad n \to \infty. \tag{3.5}$$

An easy argument shows that such sequences always exist if $\mu = E(Y_1) < \infty$. Now let $t_1 = \inf\{t \geqq 0; Z_t > u_n\}$ and define t_j for $j \geqq 2$ recursively by $t_{j+1} = \inf\{t > t_j + r_n; Z_t > u_n\}$, and then let N'_n be given by

$$N'_n(A) = \#\{i \geqq 1; t_i/n \in A\},$$

for Borel sets $A \subset [0, \infty)$. The idea behind this is (cf. the introduction) that exceedances of u_n by Z_t may come in small clusters, where each cluster belongs to the same cycle, but that (3.5) ensures that asymptotically each such cycle is at most of length r_n, so that the t_j's are the locations of the clusters. For the next result, we refer to the appendix of [15] (cf. also [13]) for definition and properties of convergence in distribution of point processes, and use the notation $\xrightarrow{d}$ for it.

Theorem 3.2. Let $\{Z_t; t = 0, 1, \cdots\}$ be an aperiodic regenerative sequence with $\mu = EY_1 < \infty$ which satisfies (3.1). Then

$$P(M_n \leqq u_n) \to \exp(-\eta), \quad \text{as} \quad n \to \infty \tag{3.6}$$

if and only if

$$nP(\zeta_1 > u_n)/\mu \to \eta, \quad \text{as} \quad n \to \infty \tag{3.7}$$

and then also $N'_n \xrightarrow{d} N'$ as $n \to \infty$, in $[0, \infty)$, where N' is a Poisson process with intensity η, provided r_n satisfies (3.5). The same result holds for a continuous-parameter process $\{Z_t; t \in [0, \infty)\}$ if M_n is replaced by $M_T = \sup_{0 \leqq t \leqq T} Z_t$ and n by T.

Proof. With the notation of Theorem 3.1, it is readily seen that $G(u_n) \to \eta$ if and only if (3.7) holds, and hence the equivalence of (3.7) and (3.6) follows from this theorem.

The remaining result, that $N'_n \xrightarrow{d} N'$ is closely related to [16], Theorem 1 and to [20], Theorem 3. However, we give a direct proof, since this seems less complicated than to translate to the present setting. We will use the main idea of [16] to approximate with a suitably time-scaled point process, $\tilde{N}_n$, of exceedances of u_n by $\{\zeta_t\}$, defined as

$$\tilde{N}_n(A) = \#\{t; t\mu/n \in A \quad \text{and} \quad \zeta_t > u_n\},$$

for $A \subseteqq [0, \infty)$. Since $\zeta_0, \zeta_1, \cdots$ are i.i.d. under P_0, and $E_0(\tilde{N}_n((0, 1])) \sim n\mu^{-1}P_0(\zeta_0 > u_n) = n\mu^{-1}P(\zeta_1 > u_n) \to \eta$ it follows at once that $\tilde{N}_n \xrightarrow{d} N'$ under P_0. Further, for any k,

$$\begin{aligned} P(\zeta_0 > u_n) &\leqq P(\zeta_0 > \max\{\zeta_1, \cdots, \zeta_k\}) + P(\tilde{N}_n((0, k\mu/n]) \geqq 1) \\ &= P(\zeta_0 > \max\{\zeta_1, \cdots, \zeta_k\}) + P_0(\tilde{N}_n((0, k\mu/n]) \geqq 1). \end{aligned}$$

By the Poisson convergence of $\tilde{N}_n$, the last term tends to 0, and since k is arbitrary it then follows from (3.1) that $P(\zeta_0 > u_n) \to 0$, as $n \to \infty$. Since $\zeta_1, \zeta_2, \cdots$ have the same distribution under P and P_0 this shows that $\tilde{N}_n \xrightarrow{d} N'$, as $n \to \infty$ in $[0, \infty)$, also under P.

We conclude the proof by showing that each event of $\tilde{N}_n$ in a fixed bounded interval—for simplicity the interval $[0, 1)$, the general case being similar—asymptotically corresponds to precisely one event in N'_n at the same location. Since $\tilde{N}_n \xrightarrow{d} N'$ this will prove $N'_n \xrightarrow{d} N'$, cf. e.g. Lemma 3.3 of [19]. First, by (3.5), $P(\max\{Y_0, \cdots, Y_n\} > r_n) \leqq P(Y_0 > r_n) + nP(Y_1 > r_n) \to 0$ as $n \to \infty$, and hence asymptotically an event of $\tilde{N}_n$ in $[0, 1)$ corresponds to at most one event in N'_n. Next

$$\limsup_{n\to\infty} P(\tilde{N}_n([(j-2)/k, j/k]) \geqq 2, \quad \text{for some} \quad j \in \{2, \cdots, k\} \to 0, \quad \text{as} \quad k \to \infty,$$

since $\tilde{N}_n$ converges to a Poisson process, and this by a straightforward argument shows that asymptotically each event of $\tilde{N}_n$ in $[0, 1)$ corresponds to at least one event in N'_n. Further, if $\tilde{N}_n$ has an event at a point $s \in [0, 1)$ then N'_n has its corresponding event at some t satisfying $nt \in [S_{[ns/\mu]-1}, S_{[ns/\mu]})$. Hence, if $\max\{Y_0, \cdots, Y_n\} \leqq r_n$, then

$$|s - t| \leqq \frac{r_n}{n} + \frac{1}{n}|S_{ns/\mu} - ns| \leqq \frac{r_n}{n} + \sup_{0<k\leqq n/\mu} \frac{k}{n}\left|\frac{S_k}{k} - \mu\right|. \tag{3.8}$$

Here $r_n = o(n)$ and $S_k/k \to \mu$ a.s. so that the bound tends to 0 almost surely. Since it in addition does not depend on the event considered, the locations of the corresponding events in $\tilde{N}_n$ and N'_n asymptotically coincide, as required to complete the proof. (Note that this proof applies to continuous-parameter processes too.)

From Theorem 3.2 the asymptotic distribution of, say, the location of the maximum or of the height of the kth highest cluster can readily be obtained in the manner of [15], Chapter 5. We do not state this explicitly, but instead turn to the process N_n of exceedances of u_n by the Z_t-process itself. Rather than, say, the kth highest cluster this concerns the kth largest individual Z_t-value. The limit of $\tilde{N}_n$ is a compound Poisson process N which, informally, can be constructed as follows: the locations of events in N are determined by a Poisson process with intensity η and the multiplicities of the events are independent, with distribution given by the 'compounding d.f.' G. In the continuous-parameter case, $\tilde{N}_n$ measures the time spent over u_n, and the 'multiplicities' typically have a non-discrete d.f., so that N is a random measure, rather than a point process, and then convergence of N_n to N is in the sense discussed in [13], Chapter 4. Further, let $\xi_k = \xi_k(u_n)$ be defined for $k \geqq 0$ by

$$\xi_k = \#\{t; S_{k-1} \leqq t < S_k \quad \text{and} \quad Z_t > u_n\}$$

if $\{Z_t\}$ has discrete parameter, and by $\xi_k = |\{t; S_{k-1} \leqq t < S_k \text{ and } Z_t > u_n\}|$ if $\{Z_t\}$ has continuous parameter (with $S_{k-1} = 0$ for $k = 0$).

Theorem 3.3. Suppose that the assumptions of Theorem 3.2 are satisfied for some $\eta > 0$ and that

$$P(\xi_1 \leqq x \mid \xi_1 > 0) \to G(x), \quad \text{as} \quad n \to \infty, \tag{3.9}$$

for continuous points x of the d.f. G. Then $N_n \xrightarrow{d} N$ as $n \to \infty$, ($N_T \xrightarrow{d} N$, as $T \to \infty$, in the continuous-parameter case) in $[0, \infty)$ where N is the compound Poisson process described just before the theorem. Conversely, if N_n (or N_T) converges in distribution to a non-zero point process (random measure) then the limit necessarily is a compound Poisson process, and (3.7) and (3.9) are satisfied.

Proof. This time we define the approximating process $\tilde{N}_n$ by $\tilde{N}_n(A) = \sum_{t; t/n \in A} \xi_t$ (with n replaced by T in the continuous-parameter case). Clearly, N_n and $\tilde{N}_n$ differ only in the location of points. A slight variation of the last argument in the proof of Theorem 3.2 shows that these differences asymptotically vanish, so that $\tilde{N}_n([t_1, t_2)) - N_n([t_1, t_2))$ tends to 0 in probability as $n \to \infty$ for any $t_2 > t_1 > 0$. Thus, again using [19], Lemma 3.3, $\tilde{N}_n \xrightarrow{d}$ N for some process N if and only if $\tilde{N}_n \xrightarrow{d} N$. However, since $P(\xi_i > 0) = P(\zeta_i > u_n)$, since $P(\xi_0 > 0) \to 0$ by (3.8), and since $\xi_1, \xi_2, \cdots$ are i.i.d. it follows at once that $\tilde{N}_n \xrightarrow{d} N$, where N is not identically 0, if and only if $nP(\zeta_1 > u_n)/\mu \to \eta$ and (3.9) holds, for some $\eta > 0$ and G. (Alternatively the result follows from [16], Theorem 1 or from [20], Theorem 2.3.)

We conclude this section with a brief discussion of 1-dependent regenerative processes. It is straightforward to extend Theorem 3.1 also to this case—in (3.2) the term $G(x)^n$ is replaced by $P(\max\{\zeta_1, \cdots, \zeta_{[n/\mu]}\} \leqq x)$ and the term $\mu\delta$ has to be increased. Since the extremal types theorem holds for 1-dependent sequences it then applies to $\{Z_t\}$ and thus the only possible limit distributions of $a_n(M_n - b_n)$ for 1-dependent regenerative sequences with $\mu = EY_1 < \infty$ are the extreme value distributions. Also Theorems 3.2 and 3.3 have counterparts in this case.

Theorem 3.4. Let $\{Z_t; t=0, 1, \cdots\}$ be an aperiodic 1-dependent regenerative sequence with $\mu = EY_1 < \infty$, which satisfies (3.1). If

$$P(\zeta_1 \leqq u_n, \zeta_2 > u_n)/\mu \to \eta, \quad \text{as} \quad n \to \infty$$

then

$$P(M_n \leqq u_n) \to \exp(-\eta), \quad \text{as} \quad n \to \infty,$$

and $N_n' \xrightarrow{d} N'$ in $[0, \infty)$, where N' is a Poisson process with intensity η, provided r_n satisfies (3.5). If in addition

$$P(\xi_2 + \xi_3 \leqq x \mid \xi_1 = 0, \xi_2 > 0) \to G(x), \quad \text{as} \quad n \to \infty, \tag{3.10}$$

(note that $\{\xi_1 = 0, \xi_2 > 0\} = \{\zeta_1 \leqq u_n, \zeta_2 > u_n\}$) for continuity points x of the d.f. G, then $N_n \xrightarrow{d} N$ in $[0, \infty)$, where N is the compound Poisson process described just before Theorem 3.3.

We give only a brief comment on the proof. Since $\zeta_0, \zeta_1, \cdots$ are 1-dependent there may be 'clusters' consisting of two adjacent large ζ-values, and then the process $\tilde{N}_n$ used in Theorem 3.2 may not converge to a Poisson process. However, if $\tilde{N}_n$ is defined instead in the same way as N_n' but with Z_s replaced by ζ_s and jr_n/n replaced by $jr_n\mu/n$, then if $nP(\zeta_1 \leqq u_n, \zeta_2 > u_n) \to \eta$ it follows that $\tilde{N}_n$ converges in distribution to a Poisson process with intensity η for any $r_n = o(n)$, with $r_n \to \infty$. This can be seen for instance from [14], Corollary 3.2 or from Theorem 4.1 below. The approximation of N_n' by $\tilde{N}_n$ can be made along similar lines as in Theorem 3.2. Next, $N_n \xrightarrow{d} N$ follows as in the proof of Theorem 3.3. The only difference is that the variables ξ_t in the appropriate process $\tilde{N}_n$ now are 1-dependent instead of independent, and that hence the conditions for convergence of $\tilde{N}_n$ has to be changed to those approximate for 1-dependent sequences, as stated in the theorem. It may be noted that the result holds also if the order of the variables is reversed, so that (ζ_1, ζ_2) is replaced by (ζ_2, ζ_1) and (ξ_1, ξ_2, ξ_3) by (ξ_3, ξ_2, ξ_1) in these conditions.

4. Extremal index for general stationary sequences

Let $\{Z_t\}$ be a general, not necessarily Markovian, strictly stationary sequence, with marginal d.f. $F(x) = P(Z_t \leqq x)$. The sequence satisfies Leadbetter's 'distributional mixing' condition $D(u_n)$ if there are constants $\{\alpha_{n,l}\}$ with $\alpha_{n,[n\lambda]} \to 0$ as $n \to \infty$, for all $\lambda > 0$, such that

$$|P(AB) - P(A)P(B)| \leqq \alpha_{n,l}, \tag{4.1}$$

for all sets A of the form $\{Z_{i_1} \leqq u_n, \cdots, Z_{i_p} \leqq u_n\}$ and sets B of the form $\{Z_{j_1} \leqq u_n, \cdots, Z_{j_{p'}} \leqq u_n\}$, with $1 \leqq i_1 < \cdots < i_p < j_1 < \cdots < j_{p'} \leqq n$ and $j_1 - i_p \geqq l$ ([15], p. 53). In [11] a slightly stronger mixing condition, $\Delta(u_n)$ is used, where it is required that (4.1) holds for all sets A, B such that $A \in \sigma\{\{Z_1 \leqq u_n\}, \cdots,$

$\{Z_k \leqq u_n\}\}$ and $B \in \sigma\{\{Z_{k+l} \leqq u_n\}, \cdots, \{Z_n \leqq u_n\}\}$, for some $k \in [1, n-l]$. In particular it is known (and easy to prove) that if $\{Z_t\}$ is regenerative and stationary and satisfies the conditions of Theorem 3.2 then it is strong mixing and hence satisfies $D(u_n)$ and $\Delta(u_n)$ for any sequence $\{u_n\}$. If $D(u_n)$ holds, then (see [15], Section 3.7, and [14]) the only possible limit laws of $a_n(M_n - b_n)$ are the extreme value distributions, and under weak further restrictions there exists an 'extremal index', i.e. a constant θ such that if $n(1-F(u_n)) \to \tau$ then $P(M_n \leqq u_n) \to \exp(-\theta\tau)$—of course here $n(1-F(u_n)) \to \tau$ is equivalent to the assumption $P(\tilde{M}_n \leqq u_n) \to \exp(-\tau)$, where $\tilde{M}_n$ is the maximum of n i.i.d. variables with common d.f. F.

From Theorem 3.2 above it follows immediately that a stationary regenerative sequence $\{Z_t\}$ has extremal index θ if and only if

$$\frac{P(\zeta_1 > u_n)/\mu}{P(Z_1 > u_n)} \to \theta \tag{4.2}$$

for some sequence u_n with $n(1-F(u_n)) \to \tau > 0$. By (2.2) the condition (4.2) in turn is equivalent to

$$\lim_{n\to\infty} \frac{P_0(\zeta_0 > u_n)}{E_0\left(\sum_{k=0}^{Y_0-1} 1_{\{Z_r > u_n\}}\right)} = \theta,$$

where the subscript 0 in P_0, E_0 refers to the zero-delayed case.

According to Theorem 3.4 the same applies to 1-dependent regenerative sequences provided $P(\zeta_1 > u_n)$ and $P_0(\zeta_0 > u_n)$ is replaced by $P(\zeta_1 \leqq u_n, \zeta_2 > u_n)$ and $P_0(\zeta_0 \leqq u_n, \zeta_1 > u_n)$, respectively.

Some criteria for finding θ and proving convergence of N'_n and N_n, (defined in Section 3) for general stationary sequences are given in [14] and [11]. (Actually [14] uses a slightly different definition of N'_n. However, it is immediate that convergence of this process entails convergence of N'_n.) Here we will use these criteria to find conditions which seem particularly useful for Markov chains. The result is somewhat related to methods used by Berman in a continuous-parameter context. It is supposed to hold for all $r_n = o(n)$ with $r_n/n \to 0$ 'slowly enough', i.e. there is some sequence r'_n for which the result holds, and it then holds for any $r_n \geqq r'_n$ with $r_n = o(n)$.

Theorem 4.1. Let $\{Z_t; t = 0, 1, \cdots\}$ be a stationary sequence such that for each $\tau > 0$ there are constants $\{u_n = u_n(\tau)\}$ with $n(1-F(u_n)) \to \tau$.

(i) Suppose $D(u_n(\tau))$ holds for each $\tau > 0$. Then $\{Z_t\}$ has extremal index $\theta > 0$ if and only if

$$\limsup_{n\to\infty} |P(M_{[n\varepsilon]} \leqq u_n \mid Z_0 > u_n) - \theta| \to 0, \quad \text{as} \quad \varepsilon \downarrow 0, \tag{4.3}$$

for $u_n = u(\tau_0)$ for some $\tau_0 > 0$. Further then $N'_n \xrightarrow{\text{d}} N'$ in $[0, \infty)$, for any sequence $r_n = o(n)$ with $r_n/n \to 0$ slowly enough, where N' is a Poisson process with intensity $\theta\tau$, for $u_n = u_n(\tau)$, for any $\tau > 0$.

(ii) Suppose $\Delta(u_n(\tau))$ holds for each $\tau>0$ and that $\{Z_t\}$ has extremal index $\theta>0$. Then $N_n \xrightarrow{d} N$, as $n\to\infty$, for some point process N if and only if there are constants $\theta_2 \geqq \theta_3 \geqq \cdots$ such that

$$\limsup_{n\to\infty} |P(N_n((0,\varepsilon])=k-1 \,|\, Z_0>u_n)-\theta_k| \to 0, \quad \text{as} \quad \varepsilon \downarrow 0, \tag{4.4}$$

for $k=2,3,\cdots$ and N then is a compound Poisson process, with intensity $\theta\tau$ for the locations of points, and probability $(\theta_k-\theta_{k+1})/\theta$, (with $\theta_1=\theta$), that an event has multiplicity k.

Proof. Theorem 4.1 of [14] implies that $N'_n \xrightarrow{d} N'$ if $\{Z_t\}$ has extremal index θ, and hence it is sufficient to show that this is equivalent to (4.3). However, by combining (4.3) with Theorem 3.1 of [14] it is seen that part (i) follows if we prove that for $n'=[n\varepsilon]$,

$$\limsup_{n\to\infty} |P(M_{n'}>u_n)/(\varepsilon\tau_0)-P(M_{n'}\leqq u_n \mid Z_0>u_n)| \to 0, \quad \text{as} \quad \varepsilon\downarrow 0. \tag{4.5}$$

Let $v=\max\{t\leqq n';\, Z_t>u_n\}$ be the time of the last exceedance of u_n before time n'. Then, splitting up according to the value of v, and using in turn stationarity and $nP(Z_1>u_n)\to\tau_0$ we have that

$$\begin{aligned} P(M_{n'}>u_n) &\geqq \sum_{t=1}^{n'} P(v=t,\, Z_{t+1}\leqq u_n, \cdots, Z_{t+n'}\leqq u_n) \\ &= n'P(Z_0>u_n,\, M_{n'}\leqq u_n) \\ &= n'P(Z_0>u_n)P(M_{n'}\leqq u_n \mid Z_0>u_n) \\ &\sim \varepsilon\tau_0 P(M_{n'}\leqq u_n \mid Z_0>u_n), \quad \text{as} \quad n\to\infty. \end{aligned} \tag{4.6}$$

To prove the opposite inequality we first note that by [14], Equation (2.1),

$$\begin{aligned} P(M_{n'}>u_n, \max\{Z_{n'+1},\cdots,Z_{2n'}\}>u_n) &\sim P(M_{n'}>u_n)^2 \\ &\leqq (n'P(Z_1>u_n))^2 \sim (\varepsilon_0)^2, \quad \text{as} \quad n\to\infty. \end{aligned} \tag{4.7}$$

Further,

$$\begin{aligned} P(M_{n'}>u_n, \max\{Z_{n'+1},\cdots,Z_{2n'}\}\leqq u_n) &\leqq \sum_{t=1}^{n'} P(v=t,\, Z_{t+1}\leqq u_n,\cdots,Z_{t+n'}\leqq u_n) \\ &= n'P(Z_0>u_n,\, M_{n'}\leqq u_n) \sim \varepsilon\tau_0 P(M_{n'}\leqq u_n \mid Z_0>u_n), \quad \text{as} \quad n\to\infty \end{aligned} \tag{4.8}$$

and since

$$\begin{aligned} P(M_{n'}>u_n) = {} & P(M_{n'}>u_{n'} \max\{Z_{n'+1},\cdots,Z_{2n'}\}\leqq u_n) \\ & + P(M_{n'}>u_n, \max\{Z_{n'+1},\cdots,Z_{2n'}\}>u_n), \end{aligned}$$

(4.5) now follows from (4.6)–(4.8).

(ii) It follows from [11], Theorems 4.1, 4.2, and 5.1 that we only have to show

(4.4) holds if and only if

(4.9) $$\limsup_{n\to\infty} |P(N_n((0,\varepsilon])\geqq k \mid N_n((0,\varepsilon])\geqq 1)-\theta_k/\theta|\to 0, \quad \text{as} \quad \varepsilon\downarrow 0.$$

Now, by part (i),

$$\lim_n P(N_n((0,\varepsilon])\geqq 1)=1-\exp(-\varepsilon\tau\theta)=\varepsilon\tau\theta+O(\varepsilon^2), \quad \text{as} \quad \varepsilon\downarrow 0,$$

and hence, for $\tau>0$,

$$P(N_n((0,\varepsilon])\geqq k \mid N_n((0,\varepsilon])\geqq 1)\sim P(N_n((0,\varepsilon])\geqq k)/(\varepsilon\tau\theta)+o(\varepsilon).$$

Thus, to show that (4.4) and (4.9) are equivalent, it suffices to show that

$$\limsup_{n\to\infty} |P(N_n((0,\varepsilon])=k-1|\, Z_0>u_n)-P(N_n((0,\varepsilon])\geqq k)/(\varepsilon\tau)|\to 0, \quad \text{as} \quad \varepsilon\downarrow 0.$$

However, this follows by similar computations as in part (i), after redefining v to be the last time before n' for which the interval $[v, n']$ contains k exceedances of u_n by $\{Z_t\}$, i.e. with $v=\max\{t\leqq n'; N_n([t/n,\varepsilon])=k\}$.

Corollary 4.2. Suppose $\{Z_t; t=0,1,\cdots\}$ is a stationary and regenerative sequence such that, with the notation of Sections 1, 2, $Y_1=S_1-S_0$ is aperiodic and satisfies $EY_1^{2+\delta}<\infty$, for some $\delta>0$, and assume further that for $\tau>0$ there is a sequence $u_n(\tau)$ such that $n(1-F(u_n(\tau)))\to\tau$.

(i) $\{Z_t\}$ has extremal index $\theta>0$ if and only if

(4.10) $$P(\sup\{Z_t; 1\leqq t<Y_0\}\leqq u_n \mid Z_0>u_n)\to\theta, \quad \text{as} \quad n\to\infty,$$

for $u_n=u_n(\tau_0)$ for some $\tau_0>0$, and then $N_n'\xrightarrow{d} N'$, as in part (i) of the theorem.

(ii) Suppose $\{Z_t\}$ has extremal index $\theta>0$. Then $N_n\xrightarrow{d} N$, as $n\to\infty$, for some point process N, if and only if there are constants $\theta_2\geqq\theta_3\geqq\cdots$, such that

(4.11) $$P(\#\{t\in[1,Y_0); Z_t>u_n\}=k-1 \mid Z_0>u_n)\to\theta_k, \quad \text{as} \quad n\to\infty,$$

for $k=2,3,\cdots$, and with N as in part (ii) of the theorem.

Proof. The proofs of the two parts are similar, so we only consider (i). Since $\{Z_t\}$ is regenerative, it satisfies $D(u_n(\tau))$ for each $\tau>0$, and it thus is sufficient to show that (4.3) and (4.10) are equivalent. Now, writing $\zeta=\max\{Z_t; 1\leqq t<Y_0\}$,

(4.12) $$P(M_{[n\varepsilon]}\leqq u_n \,|Z_0>u_n)\leqq P(\zeta\leqq u_n|\, Z_0>u_n)+P(Y_0>[n\varepsilon] \mid Z_0>u_n),$$

and similarly

(4.13) $$\begin{aligned}&P(M_{[n\varepsilon]}\leqq u_n \,|Z_0>u_n)\\ &\qquad\geqq P(\zeta<u_n|\, Z_0>u_n)-P(\max\{Z_{Y_0},\cdots,Z_{[n\varepsilon]}\}>u_n \mid Z_0>u_n).\end{aligned}$$

It is known, see e.g. [2], Theorem IV, 3.1 that $EY_1^{2+\delta}<\infty$ implies that $EY_0^{1+\delta}<\infty$, and hence

$$(4.14)\quad \begin{aligned} P(Y_0>[n\varepsilon] \mid Z_0>u_n) &\leqq P(Y_0>[n\varepsilon])/P(Z_0>u_n)\\ &\leqq EY_0^{1+\delta}[n\varepsilon]^{-1-\delta}P(Z_0>u_n)^{-1}\to 0, \quad \text{as} \quad n\to\infty, \end{aligned}$$

since $nP(Z_0>u_n)\to\tau$. Further, using that Z_0 and $Z_{Y_0}, Z_{Y_0+1}, \cdots$ are independent and, in the third step, (2.2) with $E=E_\pi$ and $f(Z_1, Z_2, \cdots)=1\{M_{[n\varepsilon]}>u_n\}$, we have that

$$(4.15)\quad \begin{aligned} P(\max\{Z_{Y_0}, \cdots, Z_{[n\varepsilon]}\}>u_n \mid Z_0>u_n) &\leqq P(\max\{Z_{Y_0}, \cdots, Z_{[n\varepsilon]+Y_0-1}\}>u_n \mid Z_0>u_n)\\ &= P_0(M_{[n\varepsilon]}>u_n)\leqq \mu P(M_{[n\varepsilon]}>u_n)\\ &\leqq \mu n\varepsilon P(Z_1>u_n)\sim \mu\tau_0\varepsilon, \quad \text{as} \quad n\to\infty. \end{aligned}$$

The equivalence of (4.3) and (4.10) follows, since (4.12)–(4.15) imply that

$$\limsup_{n\to\infty} |P(M_{[n\varepsilon]}\leqq u_n \mid Z_0>u_n)-P(\zeta\leqq u_n \mid Z_0>u_n)|\leqq \mu\tau_0\varepsilon\to 0, \quad \text{as} \quad \varepsilon\downarrow 0.$$

5. A criterion for $\theta=1$

If the extremal index exists and is equal to 1, so that $N_n'\xrightarrow{\text{d}}N'$ where N' is a Poisson process with intensity τ then, without further assumptions, also $N_n\xrightarrow{\text{d}}N'$. This is readily seen to follow, for example, from $N_n((a,b])\leqq N_n'((a,b])$ and $EN_n((a,b])\to EN'((a,b])$, for any $b>a\geqq 0$.

We use the results discussed at the beginning of Section 4 to prove a further criterion for $\theta=1$ which applies to instantaneous functions $Z_t=f(X_t)$ of a Markov chain $\{X_t\}$. The result is of the same type as [6], Theorem 2.2, but requires substantially weaker conditions. In addition to the notation of Sections 2 and 3 we write $P_n(x)=P_x(f(X_1)>u_n)=P_x(Z_1>u_n)$.

Theorem 5.1. Let $Z_1=f(X_t)$ where $\{X_t\}$ is an aperiodic Markov chain with a regeneration set, for which $E_\lambda(Y_0^\alpha)<\infty$ for some $\alpha>1$. If $nP_\pi(Z_0>u_n)\to\tau$ and (3.1) holds, for some $\tau>0$, and

$$(5.1)\qquad E_\pi(P_n(X_0)^s)n^{1+s/\alpha}\to 0, \quad \text{as} \quad n\to\infty,$$

for some $s>1$ with $1/\alpha+1/s<1$, then $\theta=1$ so that in particular $N_n\xrightarrow{\text{d}}N$ as $n\to\infty$, in $[0,\infty)$, where N is a Poisson process with intensity τ.

Proof. By similar arguments as in the proof of Theorem 3.2 we may without loss of generality assume that $\{Z_t\}$ is stationary. Further it follows from the assumptions that $\{Z_t\}$ is 1-dependent regenerative and hence it is sufficient to prove that

$$\frac{P_\lambda(\zeta_0\leqq u_n, \zeta_1>u_n)}{E_\lambda\left(\sum_{k=0}^{Y_0-1} 1_{\{Z_t>u_n\}}\right)}\to 1$$

as discussed at the beginning of Section 4, since $P_\lambda = P_0$ and $E_\lambda = E_0$ in this case. Further, this follows if

$$(5.2)\qquad \frac{P_\lambda(\zeta_1 > u_n)}{E_\lambda\left(\sum_{k=0}^{Y_0-1} 1_{\{Z_k > u_n\}}\right)} \to 1 \quad \text{and} \quad \frac{P_\lambda(\zeta_0 > u_n,\ \zeta_1 > u_n)}{E_\lambda\left(\sum_{k=0}^{Y_0-1} 1_{\{Z_k > u_n\}}\right)} \to 0, \quad \text{as} \quad n \to \infty.$$

Here $P_\lambda(\zeta_1 > u_n) = P_\lambda(\zeta_0 > u_n)$ and

$$P_\lambda(\zeta_0 > u_n) = E_\lambda(1\{Z_0 > u_n\}) + E_\lambda\left(\sum_{k=1}^{Y_0-1} 1\{\max\{Z_0, \cdots, Z_{k-1}\} \leqq u_n,\ Z_k > u_n\}\right)$$

$$= E_\lambda\left(\sum_{k=1}^{Y_0-1} 1\{Z_k > u_n\}\right) - E_\lambda\left(\sum_{k=1}^{Y_0-1} 1\{\max\{Z_0, \cdots, Z_{k-1}\} > u_n,\ Z_k > u_n\}\right).$$

Thus, since

$$E_\lambda\left(\sum_{k=1}^{Y_0-1} 1\{Z_k > u_n\}\right) = \mu P_\pi(Z_0 > u_n) \sim \mu\tau/n,$$

by (2.2) the first part of (5.4) holds if

$$(5.3)\qquad nE_\lambda\left(\sum_{k=1}^{Y_0-1} 1\{\max\{Z_0, \cdots, Z_{k-1}\} > u_n,\ Z_k > u_n\}\right) \to 0, \quad \text{as} \quad n \to \infty.$$

Using that Y_0 is an extended stopping time (see [2], p. 150) it follows from Hölder's inequality with $1/\alpha + 1/s + 1/r = 1$ that

$$E_\lambda\left(\sum_{k=1}^{Y_0} 1\{\max\{Z_0, \cdots. Z_{k-1}\} > u_n,\ Z_k > u_n\}\right)$$

$$= \sum_{k=1}^{\infty} E_\lambda(E_\lambda(1\{\max\{Z_0, \cdots, Z_{k-1}\} > u_n\}$$
$$\times 1\{Z_k > u_n\} 1\{Y_0 \geqq k\} \mid X_0, \cdots, X_{k-1}))$$

$$(5.4)\qquad = E_\lambda\left(\sum_{k=1}^{Y_0} 1\{\max\{Z_0, \cdots, Z_{k-1}\} > u_n\} P_\lambda(Z_k > u_n \mid X_0, \cdots, X_{k-1})\right)$$

$$= E_\lambda\left(\sum_{k=1}^{Y_0} 1\{\max\{Z_0, \cdots, Z_{k-1}\} > u_n\} P_n(X_{k-1})\right)$$

$$\leqq E_\lambda\left(Y_0 1\{\zeta_0 > u_n\} \max_{0 \leqq k \leqq Y_0-1} P_n(X_k)\right)$$

$$\leqq E_\lambda(Y_0^\alpha)^{1/\alpha} P_\lambda(\zeta_0 > u_n)^{1/r} E_\lambda\left(\max_{0 \leqq k \leqq Y_0-1} P_n(X_k)^s\right)^{1/s}.$$

Here $E(Y_0^\alpha) < \infty$ by assumption, $P_\lambda(\zeta_0 > u_n) \sim \mu\tau/n$, as above, and by (2.2)

$$E_\lambda\left(\max_{0 \leqq k \leqq Y_0-1} P_n(X_k)^s\right) \leqq E_\lambda\left(\sum_{k=0}^{Y_0-1} P_n(X_k)^s\right) = E_\lambda(P_n(X_0)^s),$$

and thus (5.3) follows from (5.4), since $s(1 - 1/r) = 1 + s/\alpha$.

For the second part of (5.4) we write

$$P_\lambda(\zeta_0 > u_n, \zeta_1 > u_n) \leqq E_\lambda\Big(1\{\zeta_0 > u_n\} \sum_{k=Y_0}^{Y_0+Y_1-1} 1\{Z_k > u_n\}\Big)$$

and proceed as in (5.4), to obtain the same bound, with Y_0 replaced by Y_1. Since $E_\lambda Y_0^\alpha = E_\lambda Y_1^\alpha$ the second part of (5.2) then follows as above.

6. Applications

(i) *Waiting times in the GI/G/1 queue.* In this customers arrive according to a renewal process with general interarrival distribution, and experience i.i.d. service times, again with some general distribution. Iglehart [12] finds the asymptotic distribution of the maximal waiting time. We will show that this also follows easily from Corollary 4.2, and obtain a more general point process convergence result. Iglehart further discusses the virtual waiting time, and Anderson [1] studies discrete quantities like the queue length. It is easy to see that if the nth customer has to wait a time W_n until he is served, then the waiting time of customer $n+1$ is W_n plus the difference, say D_{n+1}, between the service time of customer n and the interarrival time between customer n and $n+1$, if the quantity is positive, and 0 otherwise. Thus $\{W_n\}$ is succinctly described as a 'Lindley process', i.e. by

(6.1) $$W_{n+1} = (W_n + D_{n+1})^+, \quad n = 0, 1, \cdots,$$

where $^+$ denotes positive part, $\{D_n\}_{n=1}^\infty$ is an i.i.d. sequence, and W_0, the waiting time of the possibly fictitious zeroth customer, is independent of $\{D_n\}$. Clearly the process (6.1) is a Markov chain with stationary transition probabilities, and if $ED_1 < 0$ then $\{0\}$ is a regeneration set, by the strong law of large numbers. It is known, see e.g. [2], Section XII.5, that if in addition D_0 is non-lattice and there is a $\gamma > 0$ with $E\ \exp(\gamma D_1) = 1$, $E\,|D_1| \exp(\gamma D_1) < \infty$, then $\{W_n\}$ has a stationary distribution P_π and $P_\pi(W_0 > u) \sim C \exp(-\gamma u)$, as $u \to \infty$, with various expressions for the constant C given in [2]. Thus, $nP_\pi(W_0 > u_n) \to \tau$ if u_n is defined as, for example, $u_n = (\log n + \log C - \log \tau)/\gamma$. Further, the time $S_0 = Y_0$ to the first renewal (i.e. visit to $\{0\}$) has finite moments of all orders. Let $M = \sup\{D_1, D_1 + D_2, \cdots\}$ and let $N(x) = \#\{t \geqq 1; D_1 + \cdots + D_t > -x\}$. We now show that (4.10) and (4.11) hold, with

(6.2)
$$\theta = \int_0^\infty P(M \leqq -x)\gamma \exp(-\gamma x)\, dx,$$
$$\theta_k = \int_0^\infty P(N(x) = k-1)\gamma \exp(-\gamma x)\, dx, \ k = 2, 3, \cdots,$$

so that by Corollary 4.2 the point process N_n of exceedances of u_n by $\{W_t; t = 0, 1, \cdots\}$ converges, $N_n \xrightarrow{d} N$, under the stationary distribution P, where N is the compound Poisson process described in Theorem 4.1(ii). The same arguments as in

Theorems 3.1 and 3.2 show that this then also holds for an arbitrary initial distribution. In particular it follows that, for any initial distribution $P(\gamma(M_n - (\log nC\theta)/\gamma) \leqq x)$, $\exp(-e^{-x})$, as $n \to \infty$.

To prove (4.10), let $h(x) = P(\sup\{D_1, D_1 + D_2, \cdots\} \leqq -x)$. A straightforward argument shows that $P(\sup\{D_1 + \cdots + D_{S_0}, D_1 + \cdots + D_{S_0+1}, \cdots\} > u_n - W_0 \mid W_0 > u_n) \to 0$ as $n \to \infty$, and hence, by (6.1),

$$
\begin{aligned}
&P_\pi(\sup\{W_t; 1 \leqq t < S_0\} \leqq u_n \mid W_0 > u_n) \\
&= P_\pi(\sup\{W_0 + D_1, \cdots, W_0 + D_1 + \cdots + D_{S_0-1}\} \leqq u_n \mid W_0 > u_n) \\
&= P_\pi(\sup\{D_1, \cdots, D_1 + \cdots + D_{S_0-1}\} \leqq u_n - W_0 \mid W_0 > u_n) \\
&= E_\pi(h(W_0 - u_n) \mid W_0 > u_n) + o(1).
\end{aligned} \tag{6.3}
$$

Since $P_\pi(W_0 > u_n) \sim C \exp(-\gamma u)$, the conditional distribution of $W_0 - u_n$ given $W_0 > u_n$ tends to an exponential distribution with mean $1/\gamma$. Further, h is bounded and monotone decreasing and thus its set of discontinuity points is countable, and hence has probability 0 under the limiting exponential distribution, and it follows from (6.3) that

$$P_\pi(\sup\{W_t; 1 \leqq t < S_0\} \leqq u_n \mid W_0 > u_n) \to \int_0^\infty h(x)\gamma \exp(-\gamma x)\, dx$$

so that (4.10) holds, with θ given by (6.2). The proof of (4.11) is entirely similar.

(ii) *Autoregressive processes.* The sequence Z_t; $t = 0, 1, \cdots$ is an autoregressive process with i.i.d. innovations V_t; $t = 0, 1, \cdots$ if it satisfies the difference equation $Z_t + a_1 Z_{t-1} + \cdots + a_p Z_{t-p} = V_t$, for some constants $a_1, \cdots, a_p$, with (random) initial values $Z_{-1}, \cdots, Z_{-p}$. Let X_t be a p-dimensional random vector with components $X_t^{(k)} = Z_{t+1-k}$, $k = 1, \cdots, p$, so that $Z_t = X_t^{(1)}$ and

$$X_{t+1} = AX_t + BV_{t+1}, \quad t = 0, 1, \cdots, \tag{6.4}$$

for $B = (1, 0, \cdots, 0)'$ and

$$A = \begin{pmatrix} -a_1, & -a_2, & \cdots, & -a_p \\ 1 & & & \\ & 1 & 0 & \\ & 0 & \ddots & \\ & & 1 & 0 \end{pmatrix}.$$

Since the V_t's are i.i.d. it follows at once from (6.4) that $\{X_t\}$ is a Markov chain in $\mathbb{R}^p$ with stationary transition probabilities. We shall assume that all the zeros of the polynomial $1 + a_1 z + \cdots + a_p z^p$ are strictly outside of the unit circle, or equivalently that all the eigenvalues of A have absolute values less than some $\rho < 1$, and that

$E(\log \max(1, |V_0|)^\alpha) = C < \infty$, for some $\alpha > 1$. Then, for any $\eta > 1$, $P(|V_k| > \eta^k) \leqq C(\log n)^{-\alpha} k^{-\alpha}$, so that $P(|V_k| > \eta^k \text{ i.o.}) = 0$, by the Borel–Cantelli lemma. Since $\|A^k B V_k\| \leqq \rho^k |V_k|$, for $\|x\| = (\sum_{j=1}^p x^2)^{\frac{1}{2}}$ if $x = (x_1, \cdots, x_p)'$, it follows that $\sum_0^\infty A^k B V_k$ almost surely converges absolutely. Iteration of (6.4) gives that

$$X_{t+1} = \sum_{k=0}^{t} A^{t-k} B V_{k+1} + A^{t+1} X_0 \tag{6.5}$$

and since $\|A^{t+1} X_0\| \leqq \rho^{t+1} \|X_0\| \to 0$, a.s., it follows that the distribution of X_t converges to a unique stationary distribution, viz. the distribution of $\sum_0^\infty A^k B V_k$.

Now suppose that V_0 has a density (with respect to Lebesgue measure) which is bounded away from 0 on some interval, say by a constant $\delta > 0$. Without loss of generality we assume that this interval is $[-1, 1]$—this just amounts to a change of location and scale for the V_t's and Z_t's. By (6.5) with $t = p - 1$, $X_p = U + A^p X_0$, for

$$U = (A^{p-1}B, \cdots, AB, B)\begin{pmatrix} V_1 \\ \vdots \\ V_p \end{pmatrix} = L\begin{pmatrix} V_1 \\ \vdots \\ V_p \end{pmatrix}, \quad \text{say.}$$

Here L is a triangular matrix with ones in the diagonal, so that the determinant of L equals 1. Since $V_1, \cdots, V_p$ are i.i.d. with density bounded below by δ on $[-1, 1]$, their p-dimensional joint density is bounded from below by δ^p on the rectangle $K = \{(x_1, \cdots, x_p); |x_j| \leqq 1,\ 1 \leqq j \leqq p\}$ and hence the density of U is also bounded from below by δ^k on the non-degenerate simplex $L(K)$. Let R be a sphere around 0 such that $R \subseteq \frac{1}{2}L(K)$. Then, since $A^p x \in R$, it follows that the p-step transition density $f_p(x, y) \geqq \delta^k$, for $x, y \in \mathbb{R}$, so that by (ii) below (2.1) R is a regeneration set, provided it is recurrent. However, using the estimates above it is straightforward to check that the stationary distribution of X_t (i.e. the distribution of $\sum_0^\infty A^k B V_k = \sum_0^{p-1} A^k B V_k + \sum_p^\infty A^k B V_k$) has a positive density on R. Further, clearly $P_\pi(X_t \leqq x, X_k \leqq x) - P_\pi(X_t \leqq x) P_\pi(X_k \leqq x) \to 0$, as $t \to \infty$, for $k = 0, 1, \cdots$ and any continuity point x of the stationary d.f. of X_t. Hence $\{x_t\}$ is Rényi-mixing, so that by [17], Theorem 2.2 the set of limit points of $\{X_t\}$ is almost surely dense in $\mathbb{R}$. Thus R is recurrent, and hence is a regeneration set. It then follows from the equivalence of (i) and (ii) just above (2.2) and the fact that X_t has a unique stationary distribution, and from Theorem 3.1(i) that the only possible limit laws of $a_n(\max\{Z_1, \cdots, Z_n\} - b_n)$ are the extreme value distributions.

In this situation, it does not seem easy to get a handle on the cycle maxima needed to apply Theorem 3.2 directly; also, Theorem 4.1 would involve quite difficult computations, cf. [19]. Instead we find a simple criterion for $\theta = 1$, using Theorem 5.1. For this we assume that V_1 has a continuous density which is non-zero in the entire real line, that $E|V_1| < \infty$, and that $|a_1| + \cdots + |a_p| < 1$. Let $0 < \delta < 1 - \rho$ and let R be a sphere around 0 with radius $(E|V_1| + \delta)/(1 - \rho - \delta)$. By the same argument as before, it follows that under the present hypothesis any sphere is a regeneration set, and thus in particular R is one. Moreover, for $x \notin R$

$$\begin{aligned} E\{\|X_1\|+1 \mid X_0=x_0\} &= E\,\|Ax_0+BV_1\|+1 \\ &\leqq \rho\,\|x_0\|+E\,|V_1|+1 \\ &\leqq \rho\,\|x_0\|+(1-\rho-\delta)\,\|x_0\|-\delta+1 \\ &= (1-\delta)(\|x_0\|+1). \end{aligned}$$

Since $\{X_t\}$ has a regeneration set, it is Harris recurrent, and it then follows from [2], Theorem 3 (ii), with $g(x)=\|x\|+1$, that there is an $\eta>1$, with $E_x\eta^\tau \leqq c(\|x\|+1)$, for $x \notin R$. Here and below c denotes a generic constant. It follows that for any x

$$\begin{aligned} E_x\eta^{\tau(R)} &\leqq c(E_x(\eta^{\tau(R)-1}1\{X_1 \notin R\})+1) \\ &\leqq c(E_x\,\|X_1\|+1) \\ &\leqq c(\rho\,\|x\|+E\,|V_1|+1). \end{aligned}$$

Hence, $E_x\tau(R)^\alpha$ is uniformly bounded on $x \in R$, for any α, and since λ is concentrated on R, also $E_\lambda\tau(R)^\alpha<\infty$. Thus, by (2.4), $E_\lambda Y_0^\alpha<\infty$ for any α.

Next, let $H(x)=P(V_1>x)$ and choose $\gamma'>1$ such that $\gamma''=\gamma'(|a_1|+\cdots+|a_p|)<1$ (this is possible since $|a_1|+\cdots+|a_p|<1$ by assumption). Then, with the notation of Theorem 5.1 and with $Z_t=X_t^{(1)}$ as before,

$$\begin{aligned} E_\pi(P_n(X_0)^s) &= E_\pi(H(u_n+a_1Z_0+\cdots+a_pZ_{-p+1})^s) \\ &\leqq P_\pi(\max\{|Z_0|,\cdots,|Z_{-p+1}|\}>\gamma'u_n)+H((1-\gamma'')u_n)^s \qquad (6.6) \\ &\leqq pP_\pi(|Z_0|>\gamma'u_n)+H((1-\gamma'')u_n)^s. \end{aligned}$$

It is readily seen that $c=P_\pi(-a_1Z_0-\cdots-a_pZ_{-p+1}>0)>0$, and hence, by independence,

$$cH(x)=P_\pi(-a_1Z_0)-\cdots-a_pZ_{-p+1}>0,\ V_1>x)\leqq P_\pi(Z_1>x)=P_\pi(Z_0>x).$$

Together with (6.6) this yields that

$$E_\pi(P_n(X_0)^s)\leqq pP_\pi(|Z_0|>\gamma'u_n)+c^{-s}P_\pi(Z_0>(1-\gamma'')u_n)^s. \qquad (6.7)$$

Now choose $\{u_n\}$ such that $nP_\pi(Z_0>u_n)\to\tau>0$, and make the crucial assumption that

$$\liminf_{u\to\infty}\frac{\log P_\pi(|Z_0|>\gamma u)}{\log P_\pi(Z_0>u)}>1, \quad \text{for} \quad \gamma>1, \quad \text{and} \quad >0 \quad \text{for} \quad \gamma>0. \qquad (6.8)$$

It follows from (6.8) that there are constants $\delta'>1$ and $\delta''>0$ such that

$$P_\pi(|Z_0|>\gamma'u_n)=O(n^{-\delta'}), \quad P_\pi(Z_0>(1-\gamma'')u_n)=O(n^{-\delta''})$$

and hence by (6.7),

$$E_\pi(P_n(X_0)^s)=O(n^{-\delta'}+n^{-s\delta''})=O(n^{-1-s/\alpha}),$$

provided s and α are choosen suitably large. Thus the hypothesis of Theorem 5.1 is satisfied so that $\theta = 1$ and $N_n \xrightarrow{d} N$, where N is a Poisson process with intensity τ.

Appendix

(i) *Proof of* (2.4). Let N be the number of visits in R up to and including the time when the first regeneration occurs, so that N is independent of $\{X_t\}$ and $P(N=n)=(1-\varepsilon)^{n-1}\varepsilon$, for $n=1, 2, \cdots$. Further, let $\tau_1 = \tau(R)$ and for $k \geqq 2$ let τ_k be the time between the $(k-1)$th and kth visit to R. Since $Y_0 = \sum_{k=1}^N \tau_k + r$, it is sufficient to show that $E_\lambda(\sum_{k=1}^N \tau_k)^\alpha < \infty$. Since N is independent of $\{X_t\}$,

$$E_\lambda\Big(\sum_{k=1}^{N} \tau_k\Big)^\alpha \leqq E_\lambda \sum_{k=1}^{N} \tau_k^\alpha N^\alpha = \sum_{n=1} \Big(\sum_{k=1}^{N} E_\lambda \tau_k^\alpha\Big) n^\alpha (1-\varepsilon)^{n-1}\varepsilon.$$

By assumption, $E_\lambda \tau_1^\alpha < \infty$ and for $k \geqq 2$, $E_\lambda \tau_k^\alpha \leqq \sup_{x\in R} E_x \tau(R) < \infty$, and hence the last sum is finite.

(ii) *Proof of* (2.5). The 'if' part follows at once, since $\tau(R) \leqq Y_0$. On the other hand, with the notation of (i) above, it is well known (see e.g. [2], Proposition VI.3.4) that for $k \geqq 2$ the P_{π_R}-distribution of $X_{\tau_1+\cdots+\tau_{k-1}}$ is just τ_R, and thus

$$\begin{aligned} E_{\pi_R}\tau_k^\alpha &= E_{\pi_R}E_{\pi_R}(\tau_k^\alpha \parallel X_{\tau_1+\cdots+\tau_{k-1}}) = E_{\pi_R}E_{X_{\tau_1+\cdots+\tau_{k-1}}}((\tau R)^\alpha) \\ &= E_{\pi_R}\tau(R)^\alpha. \end{aligned}$$

Hence, if $E_{\pi_R}\tau(R)^\alpha < \infty$ it follows by the argument in (i) that $E_{\pi_R} Y_0^\alpha < \infty$.

(iii) *Proof that if* $\{Z_t\}$ *is stationary and regenerative, with* Y_1 *aperiodic and* $\mu = EY_1 < \infty$, *then* (3.1) *holds.* Let $a < \infty$ be the right-hand endpoint of the (stationary) distribution of Z_0, i.e. $a = \sup\{x; P_\pi(Z_0 > x) > 0\}$. It follows from (2.2) with $f(Z_0, Z_1, \cdots) = 1(Z_0 > x)$ that $P_\pi(Z_0 > x) > 0$ if and only if $P_\pi(\zeta_1 > x) = P_0(\zeta_0 > x) > 0$, so that a also is the right-hand endpoint of the distribution of ζ_1. Since $\zeta_1, \zeta_2, \cdots$ are i.i.d., $\max\{\zeta_1, \cdots, \zeta_k\} \to a$, and since furthermore $P_\pi(\zeta_0 > a) \leqq P_\pi(\max\{Z_0, Z_1, \cdots\} > a) = 0$, this establishes (3.1).

Acknowledgement

I want to thank Søren Asmussen for very many discussions, which have been vital for this paper, and Sidney Resnick for a useful conversation. George O'Brien has independently and at the same time found a result which is closely related to Theorem 4.1(i).

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