Extreme value statistics for financial risk

Lecture 1: Probability theory

Lecture 2: Block maxima + PoT

Lecture 3: PoT + program packages

Lecture 4: Programming + multivariate block maxima

Lecture 5: Multivariate PoT

Extremes shape much of the world around us

The philosophy of EVS is simple: extreme events, perhaps extreme water levels or extreme financial losses, are often quite different from ordinary everyday behavior, and ordinary behavior then has little to say about extremes, so that only other extreme events give useful information about future extreme events.

- Credit risk
- **Market risk**
- Operational risk
- **Insurance risk**
- Liquidity risk
- Reputational risk
- Legal risk
- and so on …

Operational risk

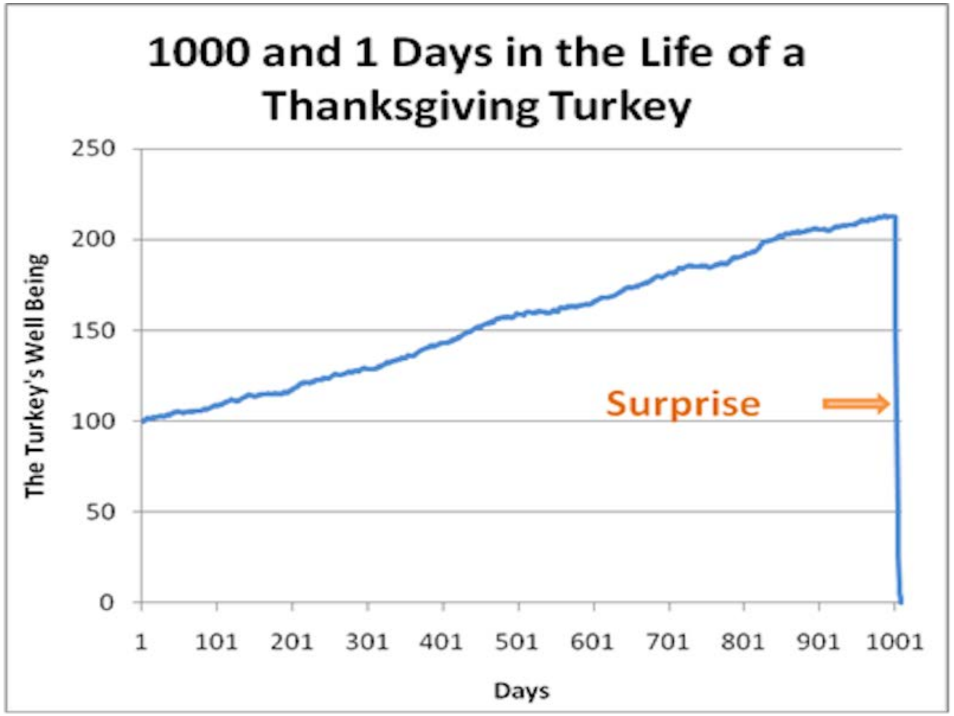
Market risk: risk that the value of a portfolio changes due to changes of market prices, exchange rates etc.

Credit risk: risk that the value of a portfolio changes because a debtor cannot meet his obligations.

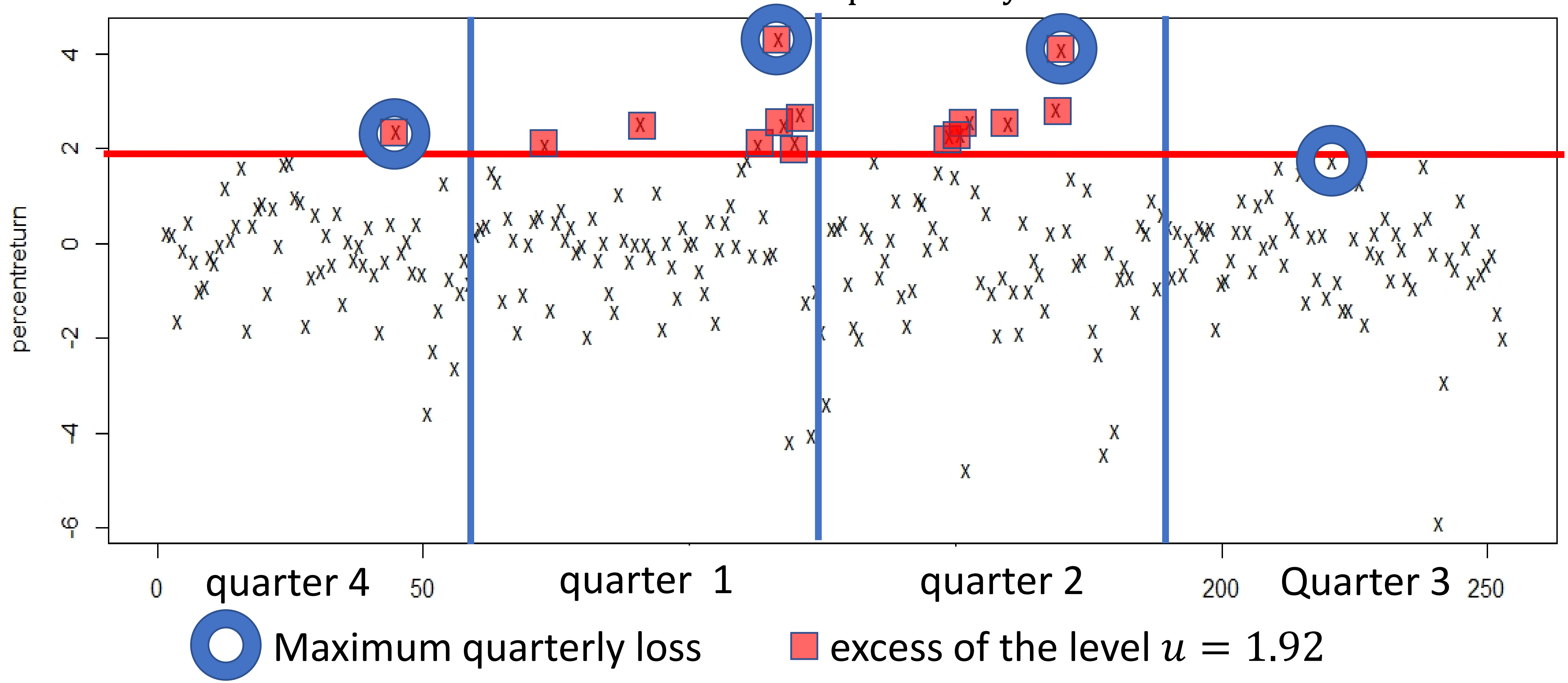
Operational risk: risk caused by problems in internal processes, people, systems

Risk: event or action which prevents an institution from meeting its obligations or reaching its goals.

If one does not understand the real-world situation well enough, the best quantitative tools will not help. Taleb’s Turkey example:



Apple losses (= - $100 \times \frac{\text{price tomorrow} - \text{price today}}{\text{price today}}$) one year back

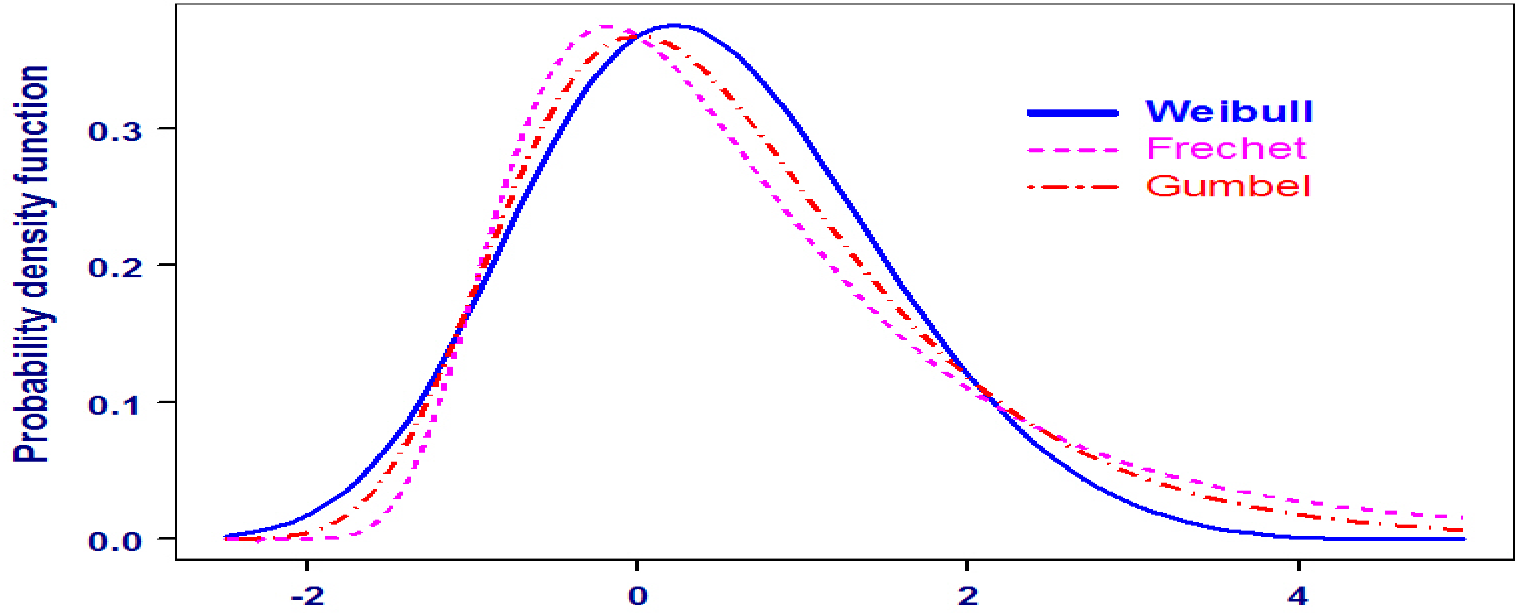


How large is the risk of a big quarterly loss?

How large is the risk of a big loss tomorrow?

Generalized extreme value (GEV) distributions

$$G(x) = e^{-\left(1+\gamma\frac{x-\mu}{\sigma}\right)_+^{-1/\gamma}}$$



$\gamma > 0$ Frechet distribution, finite left endpoint $x > \mu + \sigma/\gamma$, heavytailed

$\gamma = 0$ Gumbel distribution, $G(x) = \exp\{-e^{-(x-\gamma)/\sigma}\}$, unbounded

$\gamma < 0$ Weibull distribution, finite right endpoint $x < \mu + \sigma/|\gamma|$

The GEV cdf-s are the limit distributions of maxima: Let $X_1, X_2, \ldots$ be i.i.d. with cdf F. If for some scaling and location sequences $a_n > 0$ and b_n

$$Pr\left(a_n^{-1}\left(\vee_{i=1}^{n} X_i - b_n\right) \le x\right) \to_d G(x), \qquad \text{as} \quad n \to \infty,$$

where G is non-degenerate, then G is a GEV distribution. Conversely all GEV distributions can be obtained in this way.

Pf: If $Pr\left(a_n^{-1}\left(\vee_{i=1}^{n} X_i - b_n\right) \le x\right) = F(a_n x + b_n)^n \to_d G(x)$ then

$$F(a_n x + b_n)^{2n} \to_d G(x)^2 \quad \text{and} \quad F(a_{2n} x + b_{2n})^{2n} \to_d G(x)$$

By Kinchine's convergence of types theorem there hence exist $\alpha_2 > 0$ and β_2 with $G(\alpha_2 x + \beta_2)^2 = G(x)$. Generalize to conclude that to $t > 0$ there exist $\alpha_t > 0$ and β_t

$$G(\alpha_t x + \beta_t)^t = G(x).$$

This functional equation can be solved to give the GEV distributions. The converse is proved by straightforward checking. (*Exercise: do this*)

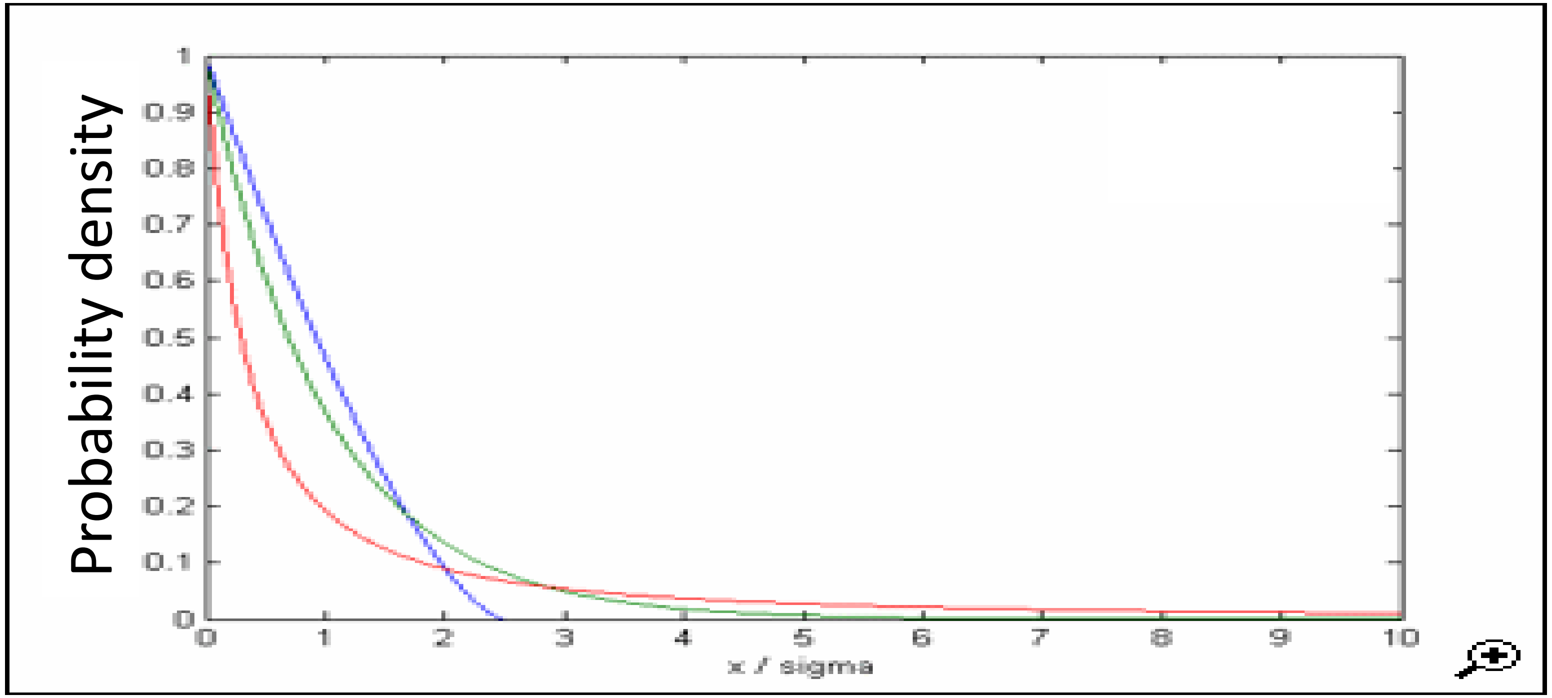
The GEV distributions are the max-stable distributions: Let $X_1, X_2, \ldots$ be i.i.d. with nondegenerate cdf G. If for some scaling and location sequences $\alpha_n > 0$ and β_n

$$\Pr\left(\alpha_n^{-1}\left(\vee_{i=1}^{n} X_i - \beta_n\right) \leq x\right) = G(x), \text{for } n = 1, 2, \ldots$$

then G is a GEV distribution. Conversely all GEV distributions can be obtained in this way.

Generalized Pareto(GP) distributions

$$H(x) = 1 - \left(1 + \gamma \frac{x}{\sigma}\right)_{+}^{-1/\gamma}$$



$\gamma > 0$ left endpoint 0, right endpoint ∞, heavytailed

$\gamma = 0$ cdf $H(x) = 1 - e^{-x/\sigma}$, exponential

$\gamma < 0$ left endpoint 0, right endpoint $\sigma/|\gamma|$

The GP distributions are the limit distributions threshold excesses: Let X have cdf F. If there exist continuous threshold and scaling functions u_t and $s_t > 0$ with $F(u_t) < 1$ and $F(u_t) \to 1$ as $t \to \infty$, such that

$$Pr\left(s_t^{-1}(X - u_t) \leq \mathrm{x} \,|\, X > u_t\right) \to_d H(\mathrm{x}), \qquad \text{as} \quad n \to \infty,$$

where H is non-degenerate, then H is a GP distribution. Conversely all GP distributions can be obtained in this way.

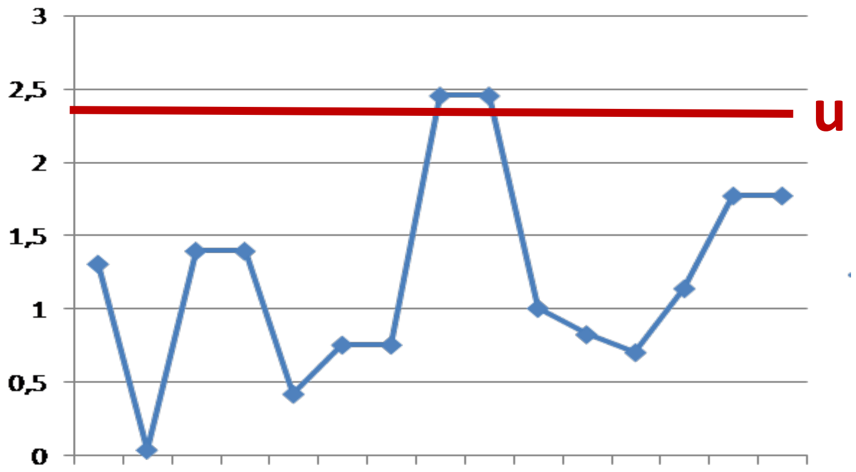
- maxima of F are in "the domain of attraction of a GEV cdf" if and only if threshold excesses of F are in "the domain of attraction of a GP cdf H"
- if normalization is chosen appropriately,

$$H(x) = \log \frac{G(x)}{G(0)}$$

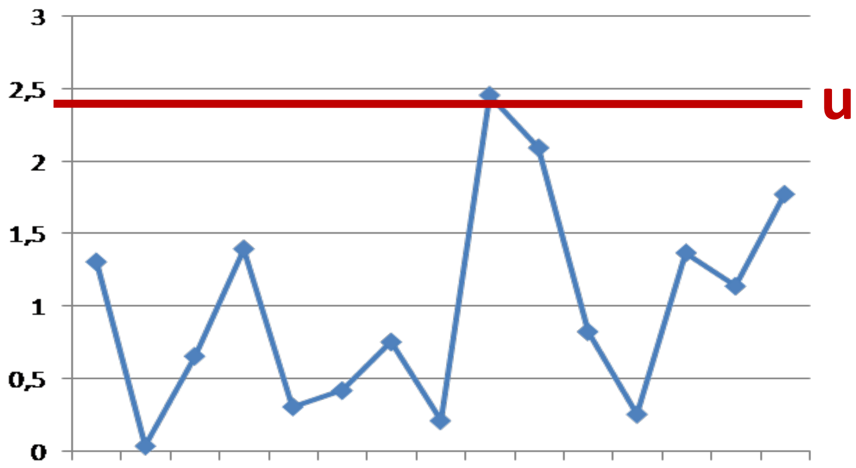
For i.i.d. sequences such that excesses are asymptotically GP distributed the times of exceedances are asymptotically a Poisson process

This follows at once from the elementary Poisson process limit theorem for Bernoulli variables

Dependence→ extremes typically come in clusters



stationary, cdf $F(x)$



i.i.d, cdf $F(x)$

- θ = "Extremal index" = 1/asymptotic mean cluster length $\in [0.1]$
- typically $Pr\left(\vee_{i=1}^{n} X_i \leq x\right) \approx F(x)^{n\theta}$
- typically clusters asymptotically i.i.d., dependence within clusters
- typically tail of cluster maxima asymptotically same as tail of F**!!**
- typically the GEV distributions are the only possible limit distributions

These results are proved for a stationary sequence $X_1, \mathrm{X}_2, \ldots$ by using Leadbetter's $D(u_n)$ mixing condition:

for any integers $i_1 < \cdots < i_p$ and $j_1 < \cdots < j_{p'} < n$ for which $j_1 - i_p > \ell$ we have

$$\left| F_{i_1,\ldots,i_{p,}\, j_1,\ldots j_{p'}}(u_n) - F_{i_1,\ldots,i_p}(u_n) F_{\,j_1,\ldots j_{p'}}(u_n) \right| < \alpha_\ell$$

where $\alpha_{\ell_n} \to 0$ as $n \to \infty$, for some sequence $\ell_n = o(n)$

The results hold, e.g., for a stationary Gaussian sequence if

$$Cov\,(X_0, X_n) \log(n) \to 0 \text{ as } n \to 0$$

No clustering and extremal index $\theta = 1$ if Leadbetter's $D'(u_n)$ condition holds

"Langbein's formula", a connection between block maxima and threshold excesses

Assume values larger than u follow an intensity λ Poisson process which is independent of the excesses, and that excesses are i.i.d. and have a GP distribution $H(x)$, and let M_T be the maximum in the time interval $[1, T]$. Then

$$
\begin{aligned}
P\big(M_T \le u + x\big) &= \sum_{k=0}^{\infty} P\big(M_T \le u + x,\ \text{there are k exceedances in } [0,\ \mathrm{T}]\big) \\
&= \sum_{k=0}^{\infty} H(x)^k \frac{(\lambda T)^k}{k!} \exp\{-\lambda T\} \\
&= \sum_{k=0}^{\infty} (1 - (1 + \frac{\gamma}{\sigma} x)_+^{-1/\gamma})^k \frac{(\lambda T)^k}{k!} \exp\{-\lambda T\} \\
&= \exp\{(1 - (1 + \frac{\gamma}{\sigma} x)_+^{-1/\gamma}) \lambda T\} \exp\{-\lambda T\} \\
&= \exp\{-(1 + \frac{\gamma}{\sigma} x)_+^{-1/\gamma} \lambda T\} \\
&= \exp\{-(1 + \gamma \frac{x - ((\lambda T)^\gamma - 1)\sigma/\gamma}{\sigma(\lambda T)^\gamma})_+^{-1/\gamma}\}
\end{aligned}
$$