Extreme value statistics for financial risk

Lecture 1: Probability theory

Lecture 2: Block maxima + PoT

Lecture 3: PoT + program packages

Lecture 4: Programming + multivariate block maxima

Lecture 5: Multivariate PoT

- *The challenge for extreme value statistics right now:* to go from 1 or 2 dimensions to 50 or more
- *The challenge for computation:* To maximize fifty-dimensional likelihoods with hundreds of parameters
- *The challenge for probability*: To construct parametric models which makes this possible – and to understand the models (and then model validation, asymptotics, …)

Extreme rainfall at many catchments, flooding of many dykes, storm insurance, landslide risk assessment, consistent risk estimation for financial portfolios, extreme influenza epidemics, …

Notation

Bold symbols are d-variate vectors. For instance, $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_d)$ and $\mathbf{0} = (0, \ldots, 0) \in R^d$. Operations and relations are componentwise, with shorter vectors recycled.

For instance $\boldsymbol{a}\boldsymbol{x} + \boldsymbol{b} = (a_1 x_1 + b_1, \ldots, a_d x_d + b_d)$, $\boldsymbol{x} \leq \boldsymbol{y}$ if $x_j \leq y_j$ for $j = 1, \ldots, d$, and $t^{\boldsymbol{\gamma}} = (t^{\gamma_1}, \ldots, t^{\gamma_d})$

Let $\boldsymbol{\eta} \in [-\infty, \infty)^d$ be the vector of lower endpoints of the marginal distributions of G

The multivariate GEV distributions are the limit distributions of maxima:
Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be an i.i.d. sequence of d-dimensional vectors with cdf F. If for some scaling and location sequences $\boldsymbol{a}_n > \mathbf{0}$ and $\boldsymbol{b}_n$

$$Pr\left(\boldsymbol{a}_n^{-1}\left(\vee_{i=1}^{n} \boldsymbol{X}_i - \mathbf{b}_{\mathrm{n}}\right) \vee \boldsymbol{\eta} \leq \boldsymbol{x}\right) \rightarrow_d G(\boldsymbol{x}), \qquad \text{as} \quad n \rightarrow \infty$$

where G has non-degenerate margins, then G is a GEV distribution. Conversely all GEV distributions can be obtained in this way.

This is written as $F \in D(G)$ or $\mathrm{X} \in D(G)$

The multivariate GEV distributions are the max-stable distributions: Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be an i.i.d. sequence with nondegenerate marginal cdf-s G_i. If for some scaling and location sequences $\boldsymbol{\alpha}_n > 0$ and $\boldsymbol{\beta}_n$

$$\Pr\left(\boldsymbol{\alpha}_n^{-1} \left(\vee_{i=1}^{n} \boldsymbol{X}_i - \boldsymbol{\beta}_n\right) \leq \boldsymbol{x}\right) = G(\boldsymbol{x}), \qquad \text{for} \quad n \to 1, 2, \ldots$$

then G is a GEV distribution. Conversely all GEV distributions can be obtained in this way.

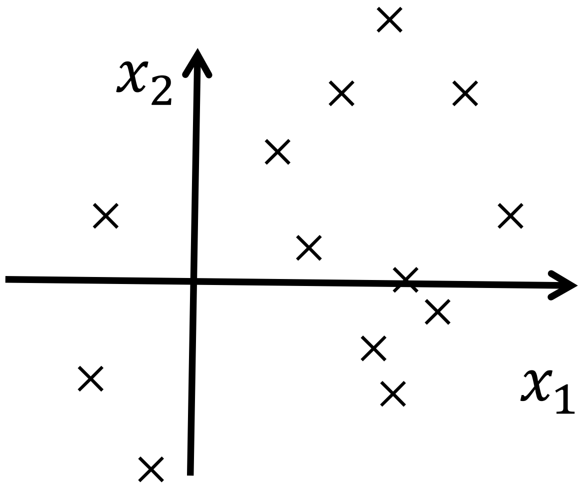
Think of $\boldsymbol{X}_1, \boldsymbol{X}_2, \dots, \boldsymbol{X}_\mathrm{n}$ as points in $\boldsymbol{R}^d$. Then there exist $\boldsymbol{b}_n > 0, \boldsymbol{a}_n$ such that the point process with points

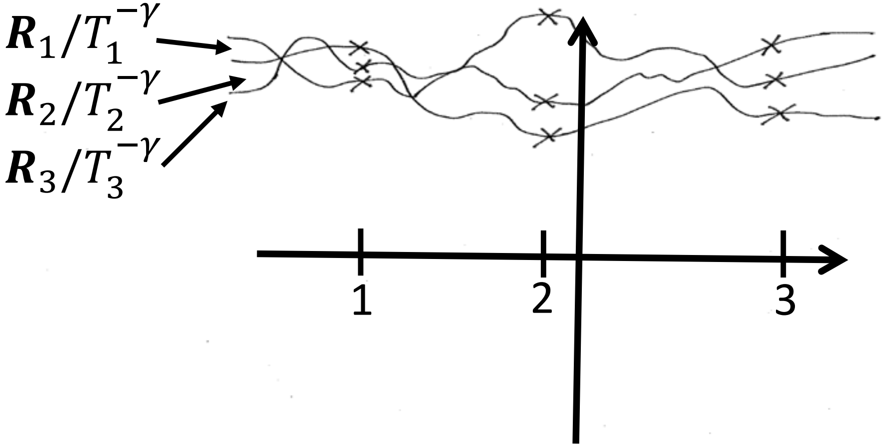
$$\left\{\frac{\boldsymbol{X}_i - \boldsymbol{a}_n}{\boldsymbol{b}_n} \vee \boldsymbol{\eta};\ \ i = 1, \dots n\right\}$$

converges as $n \to \infty$ to a point process ***iff*** $\boldsymbol{X} \in D(G)$. The limit may be written as

$$\{\boldsymbol{R}_i \times T_i^{-\gamma}\}$$

where (in sequel think of $\boldsymbol{\gamma} > \boldsymbol{0}$) $\boldsymbol{R}_i$ are i.i.d. copies of some ("any") positive stochastic d-dimensional vector and the T_i are points of a unit rate Poisson process on R_+





All d-dimensional GEV distributions may be obtained as

$$\mathrm{G}(\boldsymbol{x}) = Pr(\max_i\{\boldsymbol{R}_i \times T_i^{-\gamma}\} \leq \boldsymbol{x})$$

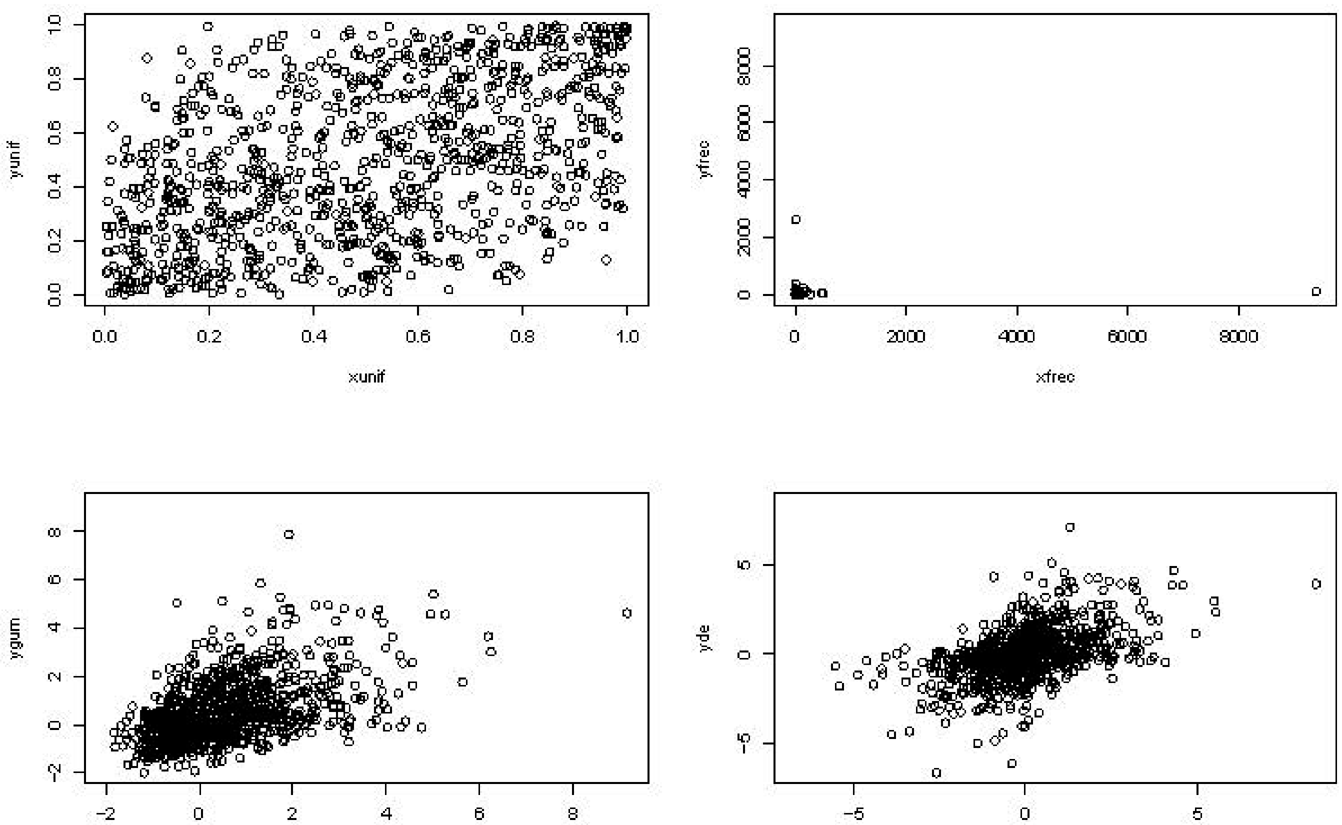
- Often the vector $\boldsymbol{R}$ is thought of as obtained by sampling from a process $W(\boldsymbol{s})$ with $\boldsymbol{s} \in R^d$ (as in figure on previous slide)
- Often G is defined on a standardized scale, say $\boldsymbol{\gamma} = \mathbf{1}$**,** and model then is transformed to the real scale
- *Smith process*: W is a non-random Gaussian density function
- *Brown-Resnick process:* $W(\boldsymbol{s}) = \exp\{\boldsymbol{X}(\boldsymbol{s}) - \boldsymbol{\sigma}^2/2\}$, with $\boldsymbol{X}$ a Gaussian process with variance $\boldsymbol{\sigma}^2$
- *Schlater model*
- …

The block maxima method

- *As in one dimension:* Get observations $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ of block maxima; assume the observations are i.i.d and follow a GEV distribution; use $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ to estimate the parameters of the GEV distribution
- *Likelihood estimation:* Distribution functions often are possible to calculate, but differentiation with respect to the d component variables often lead to combinatorial explosion of number of terms (because often $G(\boldsymbol{x}) = \exp\{-\ell(\boldsymbol{x})\}$)
- *Composite likelihood:* Use sum of likelihoods of pairs, or triplets, or …
- *Stephenson-Tawn likelihoods:* in between block maxima and PoT

- A. C. Davison, S. A. Padoan and M. Ribatet "Statistical Modeling of Spatial Extremes", *Statistical Science* 2012
- R. Huser, A. C. Davison, M. G. Genton "Likelihood estimators for multivariate extremes", *Extremes* 2016

- Copula modelling (i.e. modeling after transforming marginal distributions to uniformity) is often used
- Popular copulas: logistic = Gumbel; asymmetric logistic; inverted logistic; Gaussian; Husler-Reiss; bilogistic; Dirichlet; …



distributions with same copula, but different margins: Uniform; Fréchet; Gumbel; and Laplace pointwise 25-year return levels for max rainfall (mm) obtained from latent variable and max-stable models.

Top and bottom rows: lower and upper bounds of 95% pointwise credible/confidence intervals. *Middle row:* predictive pointwise posterior mean and pointwise estimates.

Left column: latent variable model. *Middle column:* Another latent variable model. *Right column:* Extremal t copula model.

Copied from Davison, Padoan & Ribatet paper